Quasilinearization and Multiple Solutions of the Second Order Nonlinear Boundary Value Problem

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Abstract - **We consider a two-point nonlinear boundary value problem for the second order differential equation. The conditions are provided for existence of multiple solutions. First we show that a quasi-linear problem has a solution of definite type which corresponds to the type of the principal linear part. Multiple solutions are obtained by repeatedly reducing the original equation to quasi-linear ones and proving the appropriate estimates for solutions of modified problems.**

Keywords- Types of Solutions; Multiplicity Results; Nonlinear Boundary Value Problems; Quasi-Linear Equations; Quasilinearization

I. INTRODUCTION

It is well known that the boundary value problem

$$\begin{cases} x'' = f(t,x,x'),\ t\in I:=[0,1],\ f\in C(I\times\mathbf{R}^2,\mathbf{R}), \\ x(0)=A, \quad x'(1)=B \end{cases} \tag{1}$$

is solvable if a function f is bounded. If f is not bounded, the existence of a solution is not guaranteed.

We consider the second order equation

$$x'' = f(t,x,x'), \tag{2}$$

together with the mixed boundary conditions

$$x(0)=0, \quad x'(1)=0, \tag{3}$$

provided that f is continuous along with the partial derivatives f_x and $f_{x'}$. Our intent in this paper is to show that the problem (2), (3) may have multiple solutions.

Our method was elaborated in [1], [2], [3]. We try to represent (2) in the quasi-linear form

$$\frac{d}{dt}\big(p(t)x'\big)+q(t)x = F(t,x,x'), \tag{4}$$

where $p(t)>0$, $p,q\in C(I)$, $F, F_x, F_{x'}\in C(I\times\mathbf{R}^2,\mathbf{R})$ and a function $F(t,x,x')$ is bounded. If equation (4) is equivalent to (2) in a bounded domain $\Omega\subset I\times\mathbf{R}^2$ and if the problem (4), (3) has a solution $x(t)$ with the graph $\{(t,x(t),x'(t)),\ \ t\in[0,1]\}\subset\Omega$ then $x(t)$ solves also the original nonlinear problem (2), (3). Besides we can show that the problem (4), (3) has a solution $x(t)$ such that it has the same (locally) oscillatory behavior as the linear part $\frac{d}{dt}\big(p(t)x'\big)+q(t)x$ has.

Next we try to represent (2) in the form (4) with another essentially different linear part and repeat the arguments. If this is possible we get another solution $\tilde{x}(t)$ of the problem (2), (3). It differs from $x(t)$ since properties of the linear parts (and therefore properties of solutions $x(t)$ and $\tilde{x}(t)$) in both cases are different).

In the sequel consider the problem (2), (3) and show that quasilinearization process can be applied repeatedly thus producing multiple solutions of this problem.

Equation in (1) may be thought as a control system, which is required to behave in accordance with certain specifications. For example, the requirement may be for an equation to have a solution with prescribed local oscillatory behavior (described by the respective equation of variations) and subject to given boundary conditions. Suppose the equation's natural behavior does not in itself meet the requirements. Then it may be necessary to use some control inputs (to change equation) to induce the desired behavior. Let us mention some relevant sources [4], [5], [6].

The paper is organized as follows. In the second section conventions on properties of linear parts in quasi-linear equations are made and definition of the type of a solution to the quasi-linear problem is given. The third section contains main result about existence of a solution of definite type for quasi-linear problem. The fourth section contains results on nonlinear boundary value problems. In the fifth section we apply the quasilinearization method as described to the study of the Emden-Fowler type equation. The results of numerical analysis are provided and the example is discussed.

II. QUASI-LINEAR BOUNDARY VALUE PROBLEMS

Consider the quasi-linear problem (4), (3). Several definitions will be used in the sequel.

Definition1. The linear part $(L_2x)(t) := \frac{d}{dt}\big(p(t)x'\big)+q(t)x$ is called by nonresonant with respect to the boundary conditions (3), if the homogeneous problem

$$(L_2x)(t)=0, \quad x(0)=0, \quad x'(1)=0 \tag{5}$$

has only the trivial solution.

The result below is well known [7], [8].

Theorem 1. If the linear part $(L_2x)(t)$ is nonresonant with respect to the boundary conditions (3) and a function $F(t,x,x')$ is bounded then the problem (4), (3) is solvable.

Lemma 1. The Green's function of the problem (5) is given by

$$G(t,s)=\begin{cases}\dfrac{u(t)v(s)}{W(s)p(s)}, & 0\le t<s\le 1,\\ \dfrac{u(s)v(t)}{W(s)p(s)}, & 0\le s<t\le 1,\end{cases} \tag{6}$$

where $u(t)$ and $v(t)$ are linearly independent solutions of $(L_2x)(t)=0,$ which satisfy the conditions $x(0)=0$ and $x'(1)=0$ respectively, $W(s)=u(s)v'(s)-v(s)u'(s).$

Proof. By using [9] (Ch. 3).

Lemma 2. If the linear part $(L_2x)(t)$ is nonresonant with respect to the boundary conditions (3) then a set S of all solutions of the boundary value problem (4), (3) is non-empty and compact in $C^1(I\times\mathbf{R}^2,\mathbf{R}).$

Proof. The non-emptyness of S (the solvability of the problem (4), (3)) follows from Theorem 1. Compactness follows from the integral representation of a solution of the problem (4), (3) via the Green's function:

$$x(t)=\int_0^1 G(t,s)\,F(s,x(s),x'(s))\,ds \tag{7}$$

and standard evaluations in order to show that the Arzela-Ascoli criterium is satisfied.

Lemma 3. There exists an element $x^*\in\mathsf{S}$ with the property that $x^{*\prime}(0)=\max\{x'(0):x\in\mathsf{S}\}$. Similarly there exists an element $x_*\in\mathsf{S}$ with the property that $x_*'(0)=\min\{x'(0):x\in\mathsf{S}\}$.

Proof. First let us prove that the set $\mathsf{S}_0:=\{x'(0):x\in\mathsf{S}\}$ is compact in $\mathbf{R}$. We have to show that the set above is bounded and closed. Boundedness follows from Lemma 2. Let show that this set is closed. Suppose that $x_n(0)\to r$, where $x_n\in\mathsf{S}$. Then, by compactness of the set S, one may find a subsequence x_{n_k} which tends to some $x\in\mathsf{S}$ as $n_k\to+\infty$. Obviously $x'(0)=r$. Thus $r\in\mathsf{S}_0$. Since one-dimensional closed sets have the minimal and the maximal elements, the proof follows.

Lemma 4. All solutions of (4) are extendable to the interval $[0,1]$ and uniquely defined by the initial data.

Proof. The first assertion follows from boundedness of F . Notice that since the continuous partial derivatives F_x and $F_{x'}$ exist, equation (4) satisfies the Lipschitz condition in any compact in $I\times\mathbf{R}^2$ domain. Then solutions of (4) are uniquely defined by the initial data and continuously depend on the initial data.

Represent a solution $x(t)$ of the homogeneous equation $(L_2x)(t)=0$ in polar coordinates as

$$x(t)=\rho(t)\sin\theta(t),\qquad x'(t)=\rho(t)\cos\theta(t). \tag{8}$$

The boundary conditions (3) in polar coordinates take the form

$$\theta(0)=0,\qquad \theta(1)=\frac{\pi}{2}(mod\,\pi). \tag{9}$$

Equation for the angular function $\theta(t)$ is

$$p(t)\theta'(t)=$$
$$p'(t)\sin\theta(t)\cos\theta(t)+p(t)\cos^2\theta(t)+q(t)\sin^2\theta(t). \tag{10}$$

If the angular function $\theta(t)$ is monotone for all $t\in I$ then zeros of $x(t)$ and zeros of $x'(t)$ alternate in the interval $[0,1]$. In this case it is possible to define different types of nonresonance of the linear part $(L_2x)(t)$.

Suppose that $\theta(t)$ is monotonically increasing.

Definition 2. We say that a linear part $(L_2x)(t)$ is *i*-non-resonant with respect to the boundary conditions (3), if the angular function $\theta(t)$ of a solution of the equation $(L_2x)(t)=0$, defined by the initial condition $\theta(0)=0$ satisfies the inequalities

$$\frac{(2i-1)\pi}{2}<\theta(1)<\frac{(2i+1)\pi}{2},\qquad i=1,2,3,\ldots. \tag{11}$$

In a case $0<\theta(1)<\dfrac{\pi}{2}$ we say that a linear part $(L_2x)(t)$ is 0 - nonresonant with respect to the boundary conditions (3).

Denote by $x(t;\gamma)$ a solution of the Cauchy problem (4),

$$x(0)=0,\quad x'(0)=\gamma. \tag{12}$$

Consider the difference $u(t;\gamma)=x(t;\gamma)-\xi(t)$, where $\xi(t)$ is a certain solution of the quasi-linear problem (4), (3); introduce local polar coordinates as $u(t,\gamma)=\rho(t;\gamma)\sin\phi(t;\gamma),\quad u'(t,\gamma)=\rho(t;\gamma)\cos\phi(t;\gamma),$ (13)

Lemma 5. The angular function $\phi(t;\gamma)$ defined by (13) tends to $\theta(t)$ uniformly in $t\in[0,1]$, as $\gamma\to+\infty$.

Proof. Both functions $x(t;\gamma)$ and $\xi(t)$ are solutions of (4), therefore function $u(t;\gamma)$ satisfies such the initial value problem:

$$\frac{d}{dt}(p(t)u')+q(t)u=F(t,x(t;\gamma),x'(t;\gamma))-F(t,\xi(t),\xi'(t)),$$
$$u(0;\gamma)=0,\qquad u'(0;\gamma)=\gamma-\xi'(0). \tag{14}$$

Denote $v:=\dfrac{u}{\gamma-\xi'(0)}$. Then $v(t;\gamma)$ satisfies the equation:

$$\frac{d}{dt}\left(p(t)v'\right)+q(t)v=$$
$$\frac{F(t,x(t;\gamma),x'(t;\gamma))-F(t,\xi(t),\xi'(t))}{\gamma-\xi'(0)}=:\varepsilon(t,\gamma), \tag{15}$$

with

$$v(0;\gamma)=0, \qquad v'(0;\gamma)=1. \tag{16}$$

It follows from boundedness of F that $\varepsilon(t,\gamma)$ tends to zero as $\gamma\to+\infty$, uniformly in $t\in[0,1]$.

The local polar coordinates for $v(t;\gamma)$ are defined by

$$v(t,\gamma)=\rho(t;\gamma)\sin\phi(t;\gamma), \quad v'(t,\gamma)=\rho(t;\gamma)\cos\phi(t;\gamma),$$

Where

$$\rho^*(t;\gamma)=\frac{\rho\ (t;\gamma)}{\gamma-\xi'(0)}.$$

Therefore the angular function $\phi(t;\gamma)$ that satisfies

$$p(t)\phi'=-\frac{\varepsilon(t,\gamma)\sin\phi}{\rho^*(t,\gamma)}+$$
$$p'(t)\sin\phi\cos\phi+p(t)\cos^2\phi+q(t)\sin^2\phi \tag{17}$$

tends to the angular function $\theta(t)$ uniformly in $t\in[0,1]$ as $\gamma\to+\infty$.

Definition 3. We say that $x(t;\delta)$ is a neighboring solution of a solution $\xi(t)$, if $x(t;\delta)$ solves the same quasi-linear equation (4), satisfies the first boundary condition $x(0;\delta)=0$ and there exists $\varepsilon>0$ such that $\forall\delta\in(0,\varepsilon]$ $x'(0;\delta)=\xi'(0)+\delta$.

In order to classify solutions of the quasi-linear problem (4), (3) we consider the difference between neighboring solution $x(t;\delta)$ and the investigated solution $\xi(t)$, and make use of the local polar coordinates $\rho(t;\delta)$, $\phi(t;\delta)$, defined by (13). So

$$0=x(0;\delta)-\xi(0)=\rho(0;\delta)\sin\phi(0;\delta),$$
$$\delta=x'(0;\delta)-\xi'(0)=\rho(0;\delta)\cos\phi(0;\delta),$$

therefore $\phi(0;\delta)=0$ and $\rho(t;\delta)=\delta$.

Definition 4. We say that $\xi(t)$ is an i-type solution of the problem (4), (3) if there exists $\varepsilon>0$ such that $\forall\delta\in(0,\varepsilon]$ the angular function $\phi(t;\delta)$, defined by formulas (13) for the difference between neighboring solutions and $\xi(t)$, satisfies the inequalities

$$\begin{aligned} &0<\phi(1;\delta)<\frac{\pi}{2}, && \text{if}\quad i=0;\\ &\frac{(2i-1)\pi}{2}<\phi(1;\delta)<\frac{(2i+1)\pi}{2}, && \text{if}\quad i=1,2,3,... \end{aligned} \tag{18}$$

Main result for quasi-linear problem

Theorem 2. If the linear part $(L_2x)(t)$ in the quasi-linear equation (4) is i-nonresonant with respect to the boundary conditions (3), then the problem (4), (3) has an i-type solution.

Proof. Let $\xi(t)$ be a solution $x^*(t)$ with the maximum property, described in Lemma 3. Consider the difference $x(t;\delta)-\xi(t)$ for small enough positive δ, where $x(t;\delta)$ is a neighboring solution for $\xi(t)$.

Suppose that $\xi(t)$ is not an i-type solution. According to Definition 4 this means that there exists $\delta^*>0$, which satisfies the inequalities

$$\phi(1;\delta^*)\ge\frac{(2i+1)\pi}{2}, \qquad \text{if}\quad i=0,1,2,\dots,$$

Or

$$\phi(1;\delta^*)\le\frac{(2i-1)\pi}{2}, \qquad \text{if}\quad i=1,2,3,\dots$$

Case 1. There exists a small positive δ_0 such that either $\phi(1;\delta_0)=\frac{(2i+1)\pi}{2}$ (for $i=0,1,2,\dots$) or $\phi(1;\delta_0)=\frac{(2i-1)\pi}{2}$ (for $i=1,2,3,\dots$). In accordance with (13) we obtain

$$x(1;\delta_0)-\xi(1)=\rho(1;\delta_0)\sin\phi(1;\delta_0),$$
$$x'(0;\delta_0)-\xi'(1)=\rho(0;\delta_0)\cos\phi(1;\delta_0),$$

then

$$x'(1;\delta_0)=\rho(1;\delta_0)\cos\phi(1;\delta_0)$$

and therefore the neighboring solution $x(t;\delta_0)$ solves the quasi-linear problem under consideration. This contradicts the choice of $\xi(t)=x^*(t)$.

Case 2. There exists a small positive δ_1 such that $\phi(1;\delta_1)>\frac{(2i+1)\pi}{2}$ (for $i=0,1,2,\dots$) or $\phi(1;\delta_1)<\frac{(2i-1)\pi}{2}$ (for $i=1,2,3,\dots$). Since the linear part $(L_2x)(t)$ is i-nonresonant with respect to the boundary conditions (3), then by Lemma 5 there exists such $\delta_2>0$ that $\forall\delta\ge\delta_2$ the angular function $\phi(t;\delta)$ of difference $x(t;\delta)-\xi(t)$ satisfies (18).

Denote $v(\delta):=\phi(1;\delta)$, if $\delta\in[\delta_1,\delta_2]$. Function $x(t;\delta)$ is a solution of quasi-linear equation (4) and is extendable (by Lemma 4) to the interval I and continuously depends on δ, therefore function $v(\delta)$ is continuous in the interval $[\delta_1,\delta_2]$. Then there exists at least one value $\delta_3\in(\delta_1,\delta_2)$ such that $\phi(1;\delta_3)=\frac{\pi}{2}(\mathrm{mod}\,\pi)$. Similarly as in the Case 1 we obtain that $x(t;\delta_3)$ is a solution of (4), (3) and $\xi(t)$ is not a solution with the maximum property.

Thus assumption is incorrect and therefore $\xi(t)$ is an i-type solution.

III. SOLVABILITY OF NONLINEAR BOUNDARY VALUE PROBLEM

Consider nonlinear problem (2), (3).

Definition 5. Let the nonlinear equation (2) and quasi-linear one (4) be equivalent in a domain

$$\Omega=\{(t,x,x'):0\le t\le 1,|x|\le N,|x'|\le N_1\} \qquad (19)$$

for some constants N and N_1 and a linear part $(L_2x)(t)$ is nonresonant with respect to the boundary conditions (3). If any solution $x(t)$ of the quasi-linear problem (4), (3) satisfies the estimates

$$|x(t)|\le N,\quad |x'(t)|\le N_1 \quad \forall t\in I$$

then we will say that the problem (2), (3) allows for quasilinearization with respect to the linear part $(L_2x)(t)$ and domain Ω.

Lemma 6. If the problem (2), (3) allows for quasilinearization with respect to some linear part $(L_2x)(t)$ and some domain Ω given by (1) then the problem (2), (3) has a solution, which satisfies the estimate above.

Proof. The proof is evident.

Corollary. If the problem (2), (3) allows for quasilinearization with respect to some i - nonresonant linear part $(L_2x)(t)$ and some domain Ω then there exists an i - type solution of the problem (2), (3).

Remark 1. An i - type solution of the nonlinear boundary problem (2), (3) can be defined by analogy with Definition 4.

Theorem 3. Suppose that the problem (2), (3) allows for quasilinearization with respect to Ω and i - nonresonant linear part $(L_2x)(t)$, and, at the same time, it allows for quasilinearization with respect to a domain D

$$D=\{(t,x,x'):\ 0\le t\le 1,\ |x|\le M,\ |x'|\le M_1\}$$

and j - nonresonant linear part $(\ell_2x)(t)$, where $i\ne j$. Then nonlinear problem (2), (3) has at least two different solutions.

Proof. Suppose that nonlinear equation (2) is equivalent to quasi-linear one

$$(L_2x)(t)=\Phi(t,x,x') \qquad \forall(t,x,x')\in\Omega \qquad (20)$$

and also is equivalent to another quasi-linear equation

$$(\ell_2x)(t)=\Psi(t,x,x') \qquad \forall(t,x,x')\in D. \qquad (21)$$

Let $\xi(t)$ be a maximal (in the sense of Lemma 3) solution of the quasi-linear problem (20), (3). Solution $\xi(t)$ satisfies

$$|\xi(t)|<N,\quad |\xi'(t)|<N_1 \quad \forall t\in I.$$

Let $\eta(t)$ be a maximal solution of the quasi-linear problem (21), (3). Solution $\eta(t)$ satisfies

$$|\eta(t)|<M,\quad |\eta'(t)|<M_1 \quad \forall t\in I.$$

Both functions $\xi(t)$ and $\eta(t)$ solve nonlinear problem (2), (3). Suppose that $\xi(t)\equiv\eta(t)$.

Consider solutions $x(t;\delta)$ of (2), whish satisfy the conditions

$$x(0;\delta)=0,\quad x'(0;\delta)=\xi'(0)+\delta \quad \text{or} \quad x'(0;\delta)=\eta'(0)+\delta.$$

For small enough positive values of δ these solutions satisfy the inequalities

$$|x(t;\delta)|<K,\quad |x'(t;\delta)|<K_1,\quad \forall t\in I,$$

where $K=\min\{N,M\}$ and $K_1=\min\{N_1,M_1\}$, and thus they solve both quasi-linear equations (20) and (21). Therefore the angular function $\phi(t;\delta)$ of difference $x(t;\delta)-\xi(t)$ (or $x(t;\delta)-\eta(t)$) satisfies the inequalities:

$$\frac{(2i-1)\pi}{2}<\phi(1;\delta)<\frac{(2i+1)\pi}{2},\qquad i=1,2,\ldots,$$

and at the same time,

$$\frac{(2j-1)\pi}{2}\pi<\phi(1;\delta)<\frac{(2j+1)\pi}{2},\qquad j=1,2,\ldots,$$

and $i\ne j$. The obtained contradiction shows that the solutions $\xi(t)$ and $\eta(t)$ are different.

Remark. Theorem 3 is proved for a case $i\ne 0$, $j\ne 0$. Other cases can be treated similarly.

IV. QUASILINEARIZATION WITH RESPECT TO THE LINEAR PART

$$(\ell_2x)(t):=\frac{d}{dt}\left(e^{2mt}x'\right)+e^{2mt}k^2x$$

In this section we show that the quasilinearization scheme works for certain classes of equations.

Consider the second order Emden-Fowler type equation

$$x''=-2m\,x'-g(t)\cdot|x|^p\operatorname{sgn}x, \qquad (22)$$

where $m\ne 0,\quad g\in C(I;(0,+\infty)),\quad p>0,\quad p\ne 1$, and

$$\operatorname{sgn}x=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1, & x<0,\end{cases}$$

together with the boundary conditions under consideration (3). The given nonlinear equation (22) is equivalent to the equation

$$x''+2m\,x'+k^2x=k^2x-g(t)\cdot|x|^p\operatorname{sgn}x$$

or

$$\frac{d}{dt}\left(e^{2mt}x'\right)+e^{2mt}k^2x=e^{2mt}\left(k^2x-g(t)\cdot|x|^p\operatorname{sgn}x\right). \qquad (23)$$

Suppose that $k^2>m^2$, then the following propositions are valid.

Proposition 1. In a case $k^2>m^2$ the linear part $(\ell_2x)(t):=\frac{d}{dt}\left(e^{2mt}x'\right)+e^{2mt}k^2x$ is nonresonant with respect to the boundary conditions (3) if

$$m\tan\sqrt{k^2-m^2}\ne\sqrt{k^2-m^2}. \qquad (24)$$

Remark 2. Assumption $k^2>m^2$ ensures an existence of oscillatory solutions to the equation $(\ell_2x)(t)=0$.

Denote $r:=\sqrt{k^2-m^2}$.

Proposition 2. If (24) holds then the Green's function $G_{m,k}(t,s)$ for the problem $(\ell_2x)(t)=0$, (3) exists in the form

$$G_{m,k}(t,s)=\begin{cases}\dfrac{e^{-m(t+s)}\cdot v(s)\cdot u(t)}{W}, & 0\le t<s\le 1,\\ \dfrac{e^{-m(t+s)}\cdot u(s)\cdot v(t)}{W}, & 0\le s<t\le 1,\end{cases} \tag{25}$$

where

$$W=r\cdot(m\sin r-r\cos r),$$
$$u(t)=\sin(rt),$$
$$v(t)=\sqrt{m^2+r^2}\sin\left(r(t-1)+\varphi\right),$$

where

$$\varphi=\arcsin\frac{r}{\sqrt{m^2+r^2}}.$$

Notice that functions $u(t)$ and $v(t)$ can be estimated by

$$|u(t)|\le 1,\quad |v(t)|\le\sqrt{m^2+r^2}=|k|. \tag{26}$$

Proposition 3. If k and m are such that $r=\pi i$, $i=1,2,3,\ldots$, then the linear part $(\ell_2 x)(t):=\frac{d}{dt}\left(e^{2mt}x'\right)+e^{2mt}k^2x$ is i-nonresonant with respect to the boundary conditions (3).

We wish to make the right side in (23) bounded. Denote $f_k(t,x):=k^2x-g(t)\cdot|x|^p\operatorname{sgn}x$. The function $f_k(t,x)$ is odd in x for fixed t. Let us consider it for nonnegative values of x. There exists a positive local extremal point x_0 (it is point of maximum for $p>1$ and point of minimum for $0<p<1$),

$$x_0=\left(\frac{k^2}{p\cdot g(t)}\right)^{\frac{1}{p-1}}.$$

We can calculate the value of the function at the point x_0.

Set

$$m_k(t)=\left|f_k(t,x_0)\right|=\left(\frac{k^2}{p}\right)^{\frac{p}{p-1}}\cdot|p-1|\cdot g(t)^{\frac{1}{1-p}} \tag{27}$$

and choose $n_k(t)$ so that

$$|x|\le n_k(t)\quad\Rightarrow\left|f_k(t,x)\right|\le m_k(t),\quad\forall t\in I.$$

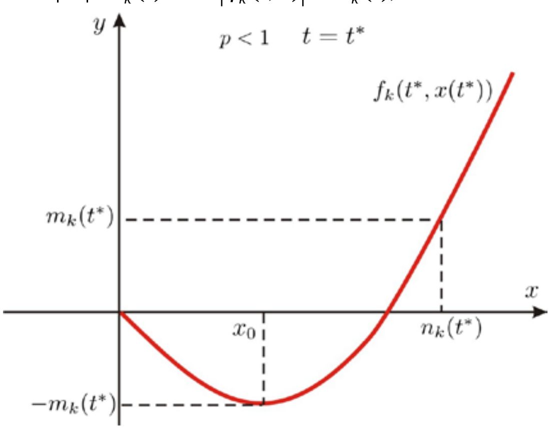


Figure 1. Existence of a number n_k.

Fig. 1 illustrates the case of $0<p<1$ for a fixed $t=t^*$.

The value of $n_k(t)$ can be computed solving the equation

$$f_k(t,x)=-f_k(t,x_0),$$

or, equivalently,

$$k^2x-g(t)x^p=\left(\frac{k^2}{p}\right)^{\frac{p}{p-1}}\cdot(1-p)\cdot g(t)^{\frac{1}{1-p}}$$

with respect to x for any fixed t. Computation gives that $n_k(t)$ can be represented as

$$n_k(t)=\left(\frac{k^2}{g(t)}\right)^{\frac{1}{p-1}}\cdot\beta, \tag{28}$$

where a constant β satisfies the equation

$$\beta^p=\beta+(p-1)\cdot p^{\frac{p}{1-p}}. \tag{29}$$

Equation (29) has a root $\beta>1$ for any positive p ($p\ne 1$).

Set

$$N_k=\min\left\{n_k(t):t\in[0,1]\right\},$$
$$M_k=\max\left\{m_k(t):t\in[0,1]\right\}.$$

One can consider the quasi-linear problem

$$\frac{d}{dt}\left(e^{2mt}x'\right)+e^{2mt}k^2x=e^{2mt}F_k(x), \tag{30}$$

(3), where

$$F_k(x):=f_k\left(\delta(-N_k,x,N_k)\right)$$

and

$$\max\left\{\left|F_k\right|:x\in\mathbf{R}\right\}=M_k. \tag{31}$$

The problem (23), (3) is equivalent to the quasi-linear problem (30), (3) in the domain Ω_k

$$\Omega_k=\left\{(t,x):\ 0\le t\le 1,\quad |x|\le N_k\right\}.$$

The quasi-linear problem (30), (3) can be rewritten in integral form, that is,

$$x(t)=\int_0^1 G_{m,k}(t,s)\,e^{2ms}F_k\left(s,x(s)\right)ds.$$

Then it follows from (25), (26), (31), that

$$|x(t)|\le\frac{|k|\cdot e^{|m|}\cdot M_k}{|W|},$$

or

$$|x(t)|\le\frac{|k|\cdot e^{|m|}\cdot M_k}{r\,|m\sin r-r\cos r|}. \tag{32}$$

If the inequality

$$\frac{|k|\cdot e^{|m|}\cdot M_k}{r\,|m\sin r-r\cos r|}<N_k$$

holds then a solution of the quasi-linear problem is a solution of the original problem too.

To simplify calculations, let $k>0$ and $|\cos r|=1$, (that is, $r=\pi n,\ \ n=1,2,\ldots$). Then the latter inequality reduces to

$$\frac{k\cdot e^{|m|}\cdot M_k}{r^2}<N_k. \qquad (33)$$

Theorem 4. Suppose that $0<g_1\le g(t)\le g_2 \quad \forall t\in I$. If the inequality

$$\frac{e^{|m|}\cdot(r^2+m^2)^{\frac{3}{2}}}{r^2}<\beta\cdot\frac{p^{\frac{p}{p-1}}}{|p-1|}\left(\frac{g_1}{g_2}\right)^{\frac{1}{|p-1|}} \qquad (34)$$

holds for some $r=\pi i$, $i=1,2,\ldots$ where $\beta>1$ is a root of the equation $\beta^p=\beta+(p-1)\cdot p^{\frac{p}{1-p}}$, then there exists an i-type solution of the problem (22), (3).

Proof. If $r=\pi i$, $i=1,2,3\ldots$ then accordingly to Proposition 3 and Theorem 2 the linear part $(\ell_2 x)(t):=\frac{d}{dt}\left(e^{2mt}x'\right)+e^{2mt}k^2x$ is i-nonresonant and quasi-linear problem (30), (3) has an i-type solution.

If the inequality (33) holds for some value $r=\pi i$, $i=1,2,\ldots$ then a solution $x(t)$ of the quasi-linear problem (30), (3) satisfies the estimate

$$|x(t)|<N_k, \quad \forall t\in[0,1]$$

and the original problem (22), (3) (or (23), (3)) allows for quasilinearization with respect to a domain Ω_k and a linear part $(\ell_2 x)(t)$ and therefore original problem has an i-type solution.

Consider the inequality (33) supposing that $g(t)$ satisfies $0<g_1\le g(t)\le g_2$.

If $p>1$ then

$$\max_{t\in[0,1]} m_k(t)=\left(\frac{k^2}{p}\right)^{\frac{p}{p-1}}\cdot|p-1|\cdot g_1^{\frac{1}{1-p}},$$
$$\min_{t\in[0,1]} n_k(t)=\left(\frac{k^2}{g_2}\right)^{\frac{1}{p-1}}\cdot\beta; \qquad (35)$$

but in the case $0<p<1$

$$\max_{t\in[0,1]} m_k(t)=\left(\frac{k^2}{p}\right)^{\frac{p}{p-1}}\cdot|p-1|\cdot g_2^{\frac{1}{1-p}},$$
$$\min_{t\in[0,1]} n_k(t)=\left(\frac{k^2}{g_1}\right)^{\frac{1}{p-1}}\cdot\beta. \qquad (36)$$

Hence the inequality (33) reduces to (34).

Corollary. If a function $g(t)$ is constant and $r=\pi i$, for some $i=1,2,\ldots$, then nonlinear problem (22), (3) is solvable if the inequality

$$\frac{e^{|m|}\cdot(r^2+m^2)^{\frac{3}{2}}}{r^2}<\beta\cdot\frac{p^{\frac{p}{p-1}}}{|p-1|} \qquad (37)$$

holds.

In the Table I and Table II below the results of calculations are provided. For certain values of p and m the values of r in the form πn, $n=1,2,3,\ldots$ are given which satisfy the inequality (37).

TABLE I. NUMERICAL RESULTS FOR $0<p<1$

	$\|m\|=\frac{3}{2}$	$\|m\|=1$	$\|m\|=\frac{1}{2}$	$\|m\|=\frac{1}{3}$
$p=\frac{3}{4}$	------	π	$\pi, 2\pi$	$\pi, 2\pi$
$p=\frac{4}{5}$	------	π	$\pi, 2\pi, 3\pi$	$\pi, 2\pi, 3\pi$
$p=\frac{5}{6}$	π	$\pi, 2\pi$	$\pi, 2\pi, 3\pi$	$\pi, 2\pi, 3\pi, 4\pi$
$p=\frac{6}{7}$	π	$\pi, 2\pi$	$\pi, 2\pi, 3\pi, 4\pi$	$\pi, 2\pi, 3\pi, 4\pi, 5\pi$
$p=\frac{7}{8}$	π	$\pi, 2\pi, 3\pi$	$\pi, 2\pi, 3\pi, 4\pi, 5\pi$	$\pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$
...	...	...	...	...

TABLE II. NUMERICAL RESULTS FOR $p>1$

	$\|m\|=\frac{3}{2}$	$\|m\|=1$	$\|m\|=\frac{1}{2}$	$\|m\|=\frac{1}{3}$
$p=\frac{8}{7}$	π	$\pi, 2\pi, 3\pi$	$\pi, 2\pi, 3\pi, 4\pi, 5\pi$	$\pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$
$p=\frac{7}{6}$	π	$\pi, 2\pi$	$\pi, 2\pi, 3\pi, 4\pi$	$\pi, 2\pi, 3\pi, 4\pi, 5\pi$
$p=\frac{6}{5}$	π	$\pi, 2\pi$	$\pi, 2\pi, 3\pi$	$\pi, 2\pi, 3\pi, 4\pi$
$p=\frac{5}{4}$	------	π	$\pi, 2\pi, 3\pi$	$\pi, 2\pi, 3\pi$
$p=\frac{4}{3}$	------	π	$\pi, 2\pi$	$\pi, 2\pi$

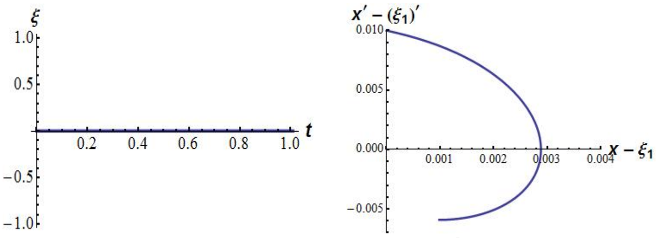
A. Example

Consider the second order nonlinear boundary value problem

$$x''=-x'-25\cdot|x|^{1.2}\operatorname{sgn}x,$$
$$x(0)=x'(1)=0. \qquad (38)$$

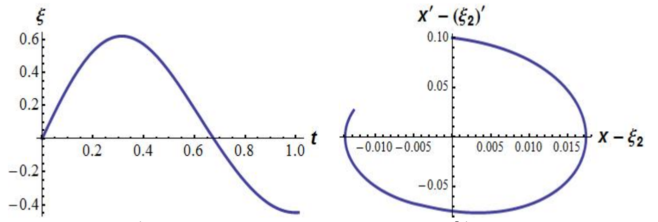
It is a special case of the problem (22), (3), when $p=1.2$, $m=0.5$ and $g(t)\equiv\text{const}=25$. In accordance with calculations (see Table II) there exist at least three solutions of different types to the given problem (38). We have computed them.

The problem (38) has the trivial solution $\xi_1(t)\equiv 0$, which is an 1-type solution, because $\forall\delta\in(0,3]$ an angular function $\phi(t;\delta)$ of difference between neighboring solution $x(t;\delta)$ and trivial one $\xi_1(t)$ satisfies the inequality $\frac{\pi}{2}<\phi(1;\delta)<\frac{3\pi}{2}$. This means that in the interval $t\in[0,1]$ a curve, which corresponds to a phase portrait of difference $x(t;\delta)-\xi_1(t)$, starting on a vertical axis crosses a horizontal axis exactly once (see Fig. 2 b).



a) b)

Figure 2. a) 1-type solution $\xi_1(t)\equiv 0$ of (38); b) phase portrait of the difference $x(t;\delta)-\xi_1(t)$ for $t\in[0,1]$ if $\delta=0.01$.

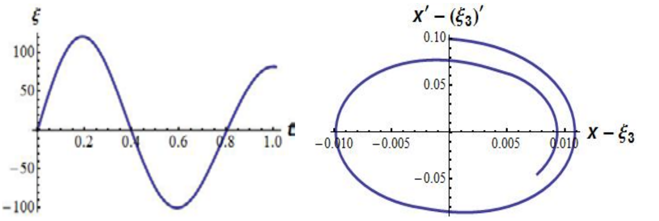


a) b)

Figure 3. a) 2-type solution $\xi_2(t)$ of (38); b) phase portrait of the difference $x(t;\delta)-\xi_2(t)$ for $t\in[0,1]$ if $\delta=0.1$.

Fig. 3 illustrates another solution $\xi_2(t)$ of the problem (38) with initial data $\xi_2(0)=0$, $\xi'_2(0)=3.31615$. This solution, actually, is a 2-type solution, because $\forall\delta\in(0,900]$ an angular function $\phi(t;\delta)$ of difference between neighboring solution $x(t;\delta)$ and this one $\xi_2(t)$ satisfies the inequality $\frac{3\pi}{2}<\phi(1;\delta)<\frac{5\pi}{2}$ (a curve of difference above crosses a horizontal axis exactly two times).

Fig. 4 shows a 3-type solution $\xi_3(t)$ of the problem (38) with initial data $\xi_3(0)=0$, $\xi'_3(0)=1027.5336$. A curve of difference $x(t;\delta)-\xi_3(t)$ crosses a horizontal axis exactly three times.



a) b)

Figure 4. a) 3-type solution $\xi_3(t)$ of (38); b) phase portrait of the difference $x(t;\delta)-\xi_3(t)$ for $t\in[0,1]$ if $\delta=0.1$.

V. CONCLUSIONS

The nonresonant quasi-linear problem (4) has a specific solution which reflects the properties of the linear part on the left. The norms of a solution and its derivative can be evaluated using the respective Green's function. The original problem (1) may be converted to a nonresonant quasi-linear problem of the form (4) so that both equations in (1) and (4) are equivalent in a compact region Ω_k. If the graph of a solution of modified problem (4) is located in Ω_k then this solution solves also the original problem. If this quasi-linearization technique can be successfully applied with different nonresonant linear parts then the original problem has multiple solutions. Additional information about the oscillatory types of solutions and their locations is obtained also. This is important for calculation purposes.

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