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On Dimension of Hypervector Spaces

R. Ameri*, O.R. Dehghan

Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, IRAN.

Abstract. The purpose of this paper is the study of dimension of hypervector spaces. In this regard first we introduce the notions of linear independent (resp. dependent) and basis of hypervector spaces. Then we study the properties of hypervector spaces and prove that under certain conditions dimension for such spaces there exist. Finally, we use the fundamental relation on hypervector spaces to construct a functor from the category of hypervector spaces over a fixed field K and the category of classical vector spaces over K, and we will prove that this functor preserves dimension.

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Key words: hypervector spaces, linearly independent, basis, dimension, transformation, fundamental relation.

1. Introduction

Hyperstructures theory was born in 1934, when Marty [5] defined hypergroups, began to analysis their properties and applied them to groups, rational algebraic functions. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties. Since then many researchers have worked on hyperalgebraic structures and developed this theory (ref. [3], [4] and [10]) . M.S. Tallini introduced the notion of hypervector spaces ([6], [7]) and studied basic properties of them. In [8] the notion of Matroidal hypervector space was introduced and the basic properties of such space studied. In this paper we study the properties of dimension of hypervector spaces. In §3 we introduce the notions of linearly independent (resp. dependent), generator, and basis of a hypervector space. We show that in contrast of the classical vector spaces a hypervector space has not necessarily a basis. We will prove that under the certain conditions a hypervector space has a basis and then we investigate the basic properties of dimension of such spaces.

In §4 for a given hypervector space V over a classical field K, the fundamental relation on V, ε^*, is defined as the smallest equivalence relation on V such that V/ε^* is a classical vector spaces over K. Then it is proved that $dim_K V = dim_K V/\varepsilon^*$. In §5 we form the category of hypervector spaces, and then we use the fundamental relation to construct a functor between the category of hypervector spaces over K and the category of vector spaces over K, and prove that this functor preserves dimension.

*Corresponding author. *Email addresses:* `ameri@umz.ac.ir` (R. Ameri),
`dehghan@umz.ac.ir` (O.R. Dehghan)

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2. Preliminaries

A map $\circ : H \times H \longrightarrow P_*(H)$ is called *hyperoperation* or join operation, where $P_*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b \ : a \in A \text{ and } b \in B \}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.1. *[6] Let K be a field and $(V, +)$ be an abelian group. We define a hypervector space over K to be the quadrupled $(V, +, \circ, K)$, where " $\circ$ " is a mapping*

$$\circ : K \times V \longrightarrow P_*(V),$$

such that the following conditions hold:

(H_1) $\forall a \in K, \forall x, y \in V, a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law,
(H_2) $\forall a, b \in K, \forall x \in V, (a + b) \circ x \subseteq a \circ x + b \circ x$, left distributive law,
(H_3) $\forall a, b \in K, \forall x \in V, a \circ (b \circ x) = (ab) \circ x$, associative law,
(H_4) $\forall a \in K, \forall x \in V, a \circ (-x) = (-a) \circ x = -(a \circ x)$,
(H_5) $\forall x \in V, x \in 1 \circ x$.

Remark 2.1. *(i) In the right hand side of (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have in (H_2).*

(ii) We say that $(V, +, \circ, K)$ is anti-left distributive, if

$$\forall a, b \in K, \ \forall x \in V, \ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive, if

$$\forall a, b \in K, \ \forall x \in V, \ (a + b) \circ x = a \circ x + b \circ x,$$

In a similar way we define the anti-right distributive and strongly right distributive hypervector spaces, respectively. V is called strongly distributive if it is both strongly left and strongly right distributive.

(iii) The left hand side of (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. In [6] it is shown if V is either strongly right or left distributive, then Ω is a subgroup of $(V, +)$.

Example 2.1. *In $(\mathbb{R}^n, +)$ we define, $\forall a \in \mathbb{R}$ and $\forall x \in \mathbb{R}^n$, $a \circ x$ as the set of vectors in $\mathbb{R}^n$ belonging to the closed segment whose vertices are the origin, $\underline{0}$, and the point ax in $\mathbb{R}^n$. Then $(\mathbb{R}^n, +, \circ, \mathbb{R})$ is a hypervector space.*

Example 2.2. *In* $(\mathbb{R}^2,+)$ *we define*

$$\left\{\begin{array}{ll}\circ: & \mathbb{R}\times\mathbb{R}^2 \longrightarrow \;\; P_*(\mathbb{R}^2) \\ & a\circ(x,y) = ax\times\mathbb{R}\end{array}\right. \quad \mathit{or} \quad \left\{\begin{array}{ll}\circ: & \mathbb{R}\times\mathbb{R}^2 \longrightarrow \;\; P_*(\mathbb{R}^2) \\ & a\circ(x,y) = \mathbb{R}\times ay\end{array}\right.$$

Then $(\mathbb{R}^2,+,\circ,\mathbb{R})$ *is a strongly distributive hypervector space.*

Example 2.3. *Let* $(V,+,.,K)$ *be a classical vector space and* P *be a subspace of* V *and*

$$\left\{\begin{array}{ll}\circ: & K\times V \;\; \longrightarrow \;\; P_*(V) \\ & a\circ x = a.x + P\end{array}\right.$$

Then $(V,+,\circ,K)$ *is a strongly distributive hypervector space.*

Theorem 2.1. *[6] Every strongly right distributive hypervector space is strongly left distributive hypervector space. Let* $(V,+)$ *be an abelian group,* Ω *a subgroup of* V *and* K *a field such that* $W = V/\Omega$ *is a classical vector space over* K*. If* $p : V \longrightarrow W$ *is the canonical projection of* $(V,+)$ *onto* $(W,+)$ *and we set:*

$$\left\{\begin{array}{ll}\circ: & K\times V \longrightarrow \;\; P_*(V) \\ & a\circ x = p^{-1}\left(a.p\left(x\right)\right)\end{array}\right.$$

Then $(V,+,\circ,K)$ *is a strongly distributive hypervector space over* K*. Moreover every strongly distributive hypervector space can be obtained in such a way.*

Example 2.4. *[6] (i) In* $(\mathbb{R}^2,+)$ *we define the product times a scalar in* $\mathbb{R}$ *by setting:*

$$\forall a\in\mathbb{R},\ \forall x\in\mathbb{R}^2:\ a\circ x = \left\{\begin{array}{ll}\textit{line } ox & \textit{if } x\neq\underline{0} \\ \{\underline{0}\} & \textit{if } x\neq\underline{0},\end{array}\right.$$

where $\underline{0} = (0,0)$*. Then* $(\mathbb{R}^2,+,\circ,\mathbb{R})$ *is a strongly left, but not right, distributive hypervector space.*

From now on, in every strongly left distributive hypervector space we set:

$$\begin{array}{rcl} T & = & \{x\in V : x\in 0\circ x\} \\ & = & \{x\in V : 1\circ x = 0\circ x\} \\ & = & \{x\in V : \forall a\in K, a\circ x = 0\circ x\} \end{array}$$

Example 2.5. *Let* $(V,+)$ *be an abelian group and* Ω *a proper subgroup of* $(V,+)$*. for any field* K *set:*

$$\left\{\begin{array}{ll}\circ: & K\times V \longrightarrow \;\; P_*(V) \\ & a\circ x = \langle x,\Omega\rangle\end{array}\right.$$

where $\langle x,\Omega\rangle$ *is the subgroup of* $(V,+)$ *spanned by* x *and* Ω*. Then* $(V,+,\circ,K)$ *is a strongly left, but not right, distributive hypervector space such that* $T = V$*.*

3. Basis of Hypervector Spaces

In [8], the notions of generator, dependent (resp. independent) set, and basis was defined, in the sense of universal algebra. In the following we introduce and characterize these notions based on the theory of linear algebra and study the basic properties of them. In the sequel by V we mean a hypervector space over the field K.

Definition 3.1. *A nonempty subset W of V is called a subhyperspace if W is itself a hypervector space with the hyperoperation on V, i.e.*

$$\begin{cases} W \neq \varnothing, \\ \forall x, y \in W \Longrightarrow x - y \in W, \\ \forall a \in K,\ \forall x \in W \Longrightarrow a \circ x \subseteq W. \end{cases}$$

In this case we write $W \leqslant V$.

Lemma 3.1. *A nonempty subset W of V is a subhyperspace if and only if $a \circ u + b \circ v \subseteq W$, $\forall a, b \in K$, $\forall u, v \in W$.*

Proof. Let W be a subhyperspace of V. Then for every $a, b \in K$ and $u, v \in W$ we have $a \circ u \subseteq W$ and $b \circ v \subseteq W$. Thus $a \circ u + b \circ v \subseteq W$. Conversely if $u, v \in W$, then $u + v \in 1 \circ u + 1 \circ v \subseteq W$, hence $u + v \in W$. Also $\underline{0} \in 1 \circ \underline{0}$, implies that $a \circ u \subseteq a \circ u + 1 \circ \underline{0}$. Thus $a \circ u \subseteq W$ and hence W is a subhyperspace.

Remark 3.1. *(i)We denote by $\mathcal{S}$ the family of all subhyperspaces of V. We easily prove that:*

$$\begin{cases} V \in \mathcal{S}, \\ \{W_i\}_{i \in I}\ ,\ W_i \in \mathcal{S} \Longrightarrow \bigcap\limits_{i \in I} W_i \in \mathcal{S}. \end{cases}$$

It follows that $\mathcal{S}$ is a closure system in V.

(ii) If W_1 and W_2 are any two subhyperspaces of V, then $W_1 \cup W_2$ is a subhyperspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Definition 3.2. *A subset S of V is called linearly independent if for every vectors $v_1, v_2, \ldots, v_n$ in S, and $c_1, \ldots, c_n \in K$, $\underline{0} \in c_1 \circ v_1 + \cdots + c_n \circ v_n$, implies that $c_1 = c_2 = \cdots = c_n = 0$. A subset S of V is called linearly dependent if it is not linearly independent.*

Lemma 3.2. *If V is strongly left distributive, then a subset S of V is linearly independent if and only if for every vectors $v_1, v_2, \ldots, v_n$ in S, and $c_1, \ldots, c_n \in K$, $\Omega \cap c_1 \circ v_1 + \cdots + c_n \circ v_n \neq \emptyset$, implies that $c_1 = c_2 = \cdots = c_n = 0$.*

Proof. Let V be linearly independent and for vectors $v_1, v_2, \ldots, v_n$ in S, and $c_1, \ldots, c_n \in K$, $x \in \Omega \cap c_1 \circ v_1 + \cdots + c_n \circ v_n$. Then

$$\begin{aligned} \underline{0} \quad &= \quad x - x \in 0 \circ \underline{0} - c_1 \circ v_1 + \cdots - c_n \circ v_n \\ &\Longrightarrow \quad c_1 = c_2 = \cdots = c_n = 0. \end{aligned}$$

Conversely, if for vectors $v_1, v_2, \ldots, v_n$ in S, and $c_1, \ldots, c_n \in K$, $\underline{0} \in c_1 \circ v_1 + \cdots + c_n \circ v_n$, then by Remark 2.1

$$\underline{0} \in \Omega \cap c_1 \circ v_1 + \cdots + c_n \circ v_n.$$

Thus $c_1 = c_2 = \cdots = c_n = 0$.

Definition 3.3. *A basis for V is a linearly independent subset of V such that span V. We say that V has finite dimensional if it has a finite basis.*

Remark 3.2. *Note that some hypervector spaces V (some set W of vectors) may not have any collection of linearly independent vectors. Such hypervector space (set) is called independentless. Clearly if V is independentless, then V has not any basis and for such hypervector spaces dimension is not defined. In this case we say that V is dimensionless.*

Recall that

$$\begin{aligned} T &= \{x \in V : x \in 0 \circ x\} \\ &= \{x \in V : 1 \circ x = 0 \circ x\} \\ &= \{x \in V : \forall a \in K, a \circ x = 0 \circ x\} \end{aligned}$$

Corollary 3.1. *Every strongly left distributive hypervector space with $T = V$ is independentless.*

Example 3.1. *The hypervector space $(\mathbb{R}^2, +, \circ, \mathbb{R})$ in Example 2.4 is a nontrivial example of an independentless hypervector space, since $\underline{0}$ belongs to every line through the $\underline{0}$.*

Definition 3.4. *If S is a nonempty subset of V, then the linear span of S is defined by:*

$$\begin{aligned} L(S) &= \left\{ t \in V : \; t \in \sum_{i=1}^{n} a_i \circ s_i \,, a_i \in K \,, s_i \in S \,, n \in \mathbb{N} \right\} \\ &= \{t_1 + t_2 + \cdots + t_n : t_i \in a_i \circ s_i \,, \, a_i \in K \,, s_i \in S \,, n \in \mathbb{N}\}. \end{aligned} \tag{3.1}$$

Lemma 3.3. *$L(S)$ is the smallest subhyperspace of V containing S.*

Proof. Let $t_1, t_2 \in L(S)$, then $t_1 \in \sum_{i=1}^{n} a_i \circ s_i$ and $t_2 \in \sum_{i=1}^{m} \acute{a}_i \circ \acute{s}_i$, where $a_i, \acute{a}_i \in K$ and $s_i, \acute{s}_i \in S$. Thus

$$\begin{aligned} t_1 - t_2 &\in \sum_{i=1}^{n} a_i \circ s_i - \sum_{i=1}^{m} \acute{a}_i \circ \acute{s}_i \\ &= \sum_{i=1}^{n} a_i \circ s_i + \sum_{i=1}^{m} (-\acute{a}_i) \circ \acute{s}_i, \qquad \text{(by } H_4\text{)} \end{aligned}$$

Then $t_1 - t_2 \in L(S)$. Also if $t \in L(S)$ and $k \in K$, then

$$\begin{aligned} k \circ t &\subseteq k \circ \sum_{i=1}^{n} a_i \circ s_i \\ &\subseteq \sum_{i=1}^{n} k \circ (a_i \circ s_i) \quad \text{(by } H_1) \\ &= \sum_{i=1}^{n} (ka_i) \circ s_i \qquad \text{(by } H_3) \end{aligned}$$

so $k \circ t \subseteq L(S)$, therefore $L(S) \leq V$. Now suppose that W is a subhyperspace of V containing S. Then for every $t \in L(S)$ we have $t \in \sum_{i=1}^{n} a_i \circ s_i$, for some $a_i \in K$, $s_i \in S$ and $n \in \mathbb{N}$. Since W is a subhyperspace and $S \subseteq W$, thus $s_i \in W$, so $\sum_{i=1}^{n} a_i \circ s_i \subseteq W$, therefore $t \in W$. Consequently $L(S) \subseteq W$. Thus $L(S)$ is the smallest subhyperspace of V. Also for all $s \in S$, $s \in 1_K \circ s$, so $s \in L(S)$, therefore $S \subseteq L(S)$.

Sometimes we denote $L(S)$, by $\langle S \rangle$, which is called the subhyperspace generated by S.

Proposition 3.1. *Let V be strongly left distributive. Then*

$$\forall x \in V,\ \langle x \rangle = \bigcup_{a \in K} a \circ x.$$

Proof. Let $t \in \langle x \rangle$. Then there exist $a_1, \ldots, a_n \in K$ such that $t \in a_1 \circ x + \cdots + a_n \circ x$. So

$$\begin{aligned} t &\in a_1 \circ x + \cdots + a_n \circ x \\ &= (a_1 + \cdots + a_n) \circ x, \end{aligned}$$

thus $t \in \bigcup_{a \in K} a \circ x$. Conversely, if $t \in \bigcup_{a \in K} a \circ x$, then

$$\exists a \in K,\ t \in a \circ x \Longrightarrow t \in \langle x \rangle .$$

Proposition 3.2. *If W_1 and W_2 are two subhyperspaces of V, Then*

$$L(W_1 \cup W_2) = W_1 + W_2.$$

Proof. Obvious.

Lemma 3.4. *Let V be anti-left distributive and $v_1, v_2, ..., v_n$ be linearly independent in V. Then every element in their linear span belongs to a unique sum in the form $c_1 \circ v_1 + c_2 \circ v_2 + ... + c_n \circ v_n$ with $c_i \in K$.*

Proof. By Definition 3.4, every element in the linear span is belong to a set of the form $c_1 \circ v_1 + c_2 \circ v_2 + ... + c_n \circ v_n$. To prove uniqueness we must demonstrate if for some $u \in V, u \in c_1 \circ v_1 + c_2 \circ v_2 + ... + c_n \circ v_n$ and $u \in d_1 \circ v_1 + d_2 \circ v_2 + ... + d_n \circ v_n$, then $c_1 = d_1, ..., d_n = c_n$. But by (H_4) we obtain that

$$\begin{aligned}
\underline{0} &= u - u \\
&\in c_1 \circ v_1 + c_2 \circ v_2 + \cdots + c_n \circ v_n - \\
&\quad - (d_1 \circ v_1 + d_2 \circ v_2 + \cdots + d_n \circ v_n) \\
&= c_1 \circ v_1 + c_2 \circ v_2 + \cdots + c_n \circ v_n - \\
&\quad - (d_1 \circ v_1) - (d_2 \circ v_2) - \cdots - (d_n \circ v_n) \\
&= c_1 \circ v_1 + c_2 \circ v_2 + \cdots + c_n \circ v_n + \\
&\quad + (-d_1) \circ v_1 + (-d_2) \circ v_2 + \cdots + (-d_n) \circ v_n,
\end{aligned}$$

now since V is anti-left distributive, then we have

$$\underline{0} \in (c_1 - d_1) \circ v_1 + \cdots + (c_n - d_n) \circ v_n,$$

which implies that $c_1 = d_1, \ldots, c_n = d_n$, by linearly independently of $v_1, v_2, \ldots, v_n$.

Remark 3.3. *Clearly Lemma 3.4 is satisfied for every strongly distributive hypervector space.*

Definition 3.5. *A hypervector space V over K is said to be K-invertible or shortly invertible if and only if $u \in a \circ v$ implies that $v \in a^{-1} \circ u$.*

Theorem 3.1. *Let V be invertible. Then for every $v_1, ..., v_n$ in V, either $v_1, ..., v_n$ are linearly independent or for some $1 \leq j \leq n$, v_j is in a linear combination of the others.*

Proof. Suppose that $v_1, \ldots, v_n$ are not linearly independent. Then $\underline{0} \in c_1 \circ v_1 + \cdots + c_n \circ v_n$ for some $c_1, \ldots, c_n$, such that $c_j \neq 0$ for some j, $1 \leq j \leq n$. Thus $\underline{0} = t_1 + \cdots + t_n$ for some $t_i \in c_i \circ v_i$. By invertibility of V, it conclude that $v_j \in {c_j}^{-1} \circ t_j$ and hence

$$\begin{aligned}
v_j &\in {c_j}^{-1} \circ (-t_1 - \cdots - t_{j-1} - t_{j+1} - \cdots - t_n) \\
&\subseteq {c_j}^{-1} \circ (-t_1) + \cdots + {c_j}^{-1} \circ (-t_{j-1}) + \\
&\quad + {c_j}^{-1} \circ (-t_{j+1}) + \cdots + {c_j}^{-1} \circ (-t_n) \\
&= -{c_j}^{-1} \circ t_1 - \cdots - {c_j}^{-1} \circ t_{j-1} - {c_j}^{-1} \circ t_{j+1} - \cdots - {c_j}^{-1} \circ t_n \\
&\subseteq -{c_j}^{-1} \circ (c_1 \circ v_1) - \cdots - {c_j}^{-1} \circ (c_{j-1} \circ v_{j-1}) - \\
&\quad - {c_j}^{-1} \circ (c_{j+1} \circ v_{j+1}) - \cdots - {c_j}^{-1} \circ (c_n \circ v_n) \\
&= (-{c_j}^{-1} c_1) \circ v_1 + \cdots + (-{c_j}^{-1} c_{j-1}) \circ v_{j-1} + \\
&\quad + (-{c_j}^{-1} c_{j+1}) \circ v_{j+1} + \cdots + (-{c_j}^{-1} c_n) \circ v_n,
\end{aligned}$$

thus v_j is in a linear combination of $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$, as desired.

Lemma 3.5. *Let V be strongly left distributive and invertible. If W is a subhyperspace of V span by $\beta = \{v_1, \ldots, v_n\}$, such that β is not independentless, Then β has a linearly independent subset such that spanning W.*

Proof. Since β is not independentless, so β has a linearly independent subset $\{v_1, \ldots, v_k\}$. Now if $k = n$, we are done. If not, weed out from this set the first v_j, which is in a linear combination of others. It is clearly that $j > k$. Let

$$v_j \in b_1 \circ v_1 + \cdots + b_{j-1} \circ v_{j-1} + b_{j+1} \circ v_{j+1} + \cdots + b_n \circ v_n. \tag{3.2}$$

The subset so constructed, $v_1, \ldots, v_k, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$ has $n - 1$ elements and its linear span is contained in W. Because if

$$v \in \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i \circ v_i,$$

then since $\underline{0} \in 0 \circ v_j$, it follows that $v \in W$. However, we claim that it is actually equal to W:

For every $w \in W$, there exist $a_1, a_2, \ldots, a_n$ in K such that $w \in \sum\limits_{i=1}^{n} a_i \circ v_i$, thus by (2) we have

$$\begin{aligned} w &\in a_1 \circ v_1 + \cdots + a_j \circ (b_1 \circ v_1 + \cdots + b_{j-1} \circ v_{j-1} + \\ &\quad + b_{j+1} \circ v_{j+1} + \cdots + b_n \circ v_n) + \cdots + a_n \circ v_n \\ &= a_1 \circ v_1 + \cdots + a_j b_1 \circ v_1 + \cdots + a_j b_{j-1} \circ v_{j-1} + \\ &\quad + a_j b_{j+1} \circ v_{j+1} + \cdots + a_j b_n \circ v_n + \cdots + a_n \circ v_n \\ &= (a_1 + a_j b_1) \circ v_1 + \cdots + (a_{j-1} + a_j b_{j-1}) \circ v_{j-1} + \\ &\quad + (a_{j+1} + a_j b_{j+1}) \circ v_{j+1} + \cdots + (a_n + a_j b_n) \circ v_n, \end{aligned}$$

that is, w is in a linear combination of $v_1, \ldots, v_k, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$. Continuing this weeding out process, we reach a subset $v_1, \ldots, v_k, v_{i_1}, \ldots, v_{i_r}$ whose linear span is still W but in which no element is in a linear combination of others. By Theorem 3.1 the elements $v_1, \ldots, v_k, v_{i_1}, \ldots, v_{i_r}$ must be linearly independent.

Corollary 3.2. *Let V be strongly left distributive and invertible. Then every non independentless spanning subset of V containing a basis.*

Corollary 3.3. *Let V be strongly left distributive and invertible. If V containing a finite spanning set which is not independentless, then V is finite dimensional.*

Proof. Suppose $V = \langle v_1, v_2, ..., v_n \rangle$ such that $v_1, ..., v_k$ are not independentless. Then by Lemma 3.5 there exist linearly independent vectors $v_1, \ldots, v_k, v_{i_1}, \ldots, v_{i_r}$ such that $V = \langle v_1, \ldots, v_k, v_{i_1}, \ldots, v_{i_r} \rangle$.

Theorem 3.2. *Let V be strongly left distributive and invertible. If V has a finite basis with n elements, then every linearly independent subset of V has no more than n elements.*

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{w_1, \ldots, w_m\}$ be a linearly independent subset of V. We show that $m \leq n$:

Every vector in V, so in particular w_m, is in a linear combination of $v_1, \ldots, v_n$. Therefore the vectors $w_m, v_1, \ldots, v_n$ are linearly dependent. Moreover they span V, since $v_1, \ldots, v_n$ already do so and $\underline{0} \in 0 \circ w_m$. Thus by Lemma 3.5 there exists some proper subset $\{w_m, v_{i_1}, \ldots, v_{i_r}\}$ of $\{w_m, v_1, \ldots, v_n\}$ with $k \leq n-1$, such that forms a basis for V. We have traded off one w, in forming this new basis, for at least one v_i. Repeat this procedure with the set $\{w_{m-1}, w_m, v_{i_1}, \ldots, v_{i_r}\}$. From this linearly dependent set, by Lemma 3.5 we can extract a basis of the form $\{w_{m-1}, w_m, v_{i_1}, \ldots, v_{i_s}\}$ such that $s \leq n-2$. Keeping up this procedure we eventually get down to a basis of V of the form $\{w_2, \ldots, w_{m-1}, w_m, v_\alpha, v_\beta, \ldots\}$. Since w_1 is not in a linear combination of $w_2, \ldots, w_{m-1}$, the above basis must actually include some v. To get to this basis we have introduced $m-1$ w's, each such introduction having cost us at least one v, and yet there is a v left. Thus $m-1 \leq n-1$ and so $m \leq n$.

Corollary 3.4. *Let V be strongly left distributive and invertible. If V is finite dimensional then every two basis of V have the same elements.*

Proof. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be two basis of V over K. Then by Theorem 3.2 we have $m \leq n$, since $w_1, \ldots, w_m$ are linearly independent. Now interchange the roles of the v's and w's and obtain that $n \leq m$. Together these say that $n = m$.

Lemma 3.6. *Let V be strongly left distributive and invertible. If V is finite dimensional, then every linearly independent subset of V is contained in a finite basis.*

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{u_1, \ldots, u_m\}$ be a linearly independent subset of V. Then vectors $u_1, \ldots, u_m, v_1, \ldots, v_n$ span V (since $v_1, \ldots, v_n$ span V and $\underline{0} \in \sum_{j=1}^{m} 0 \circ u_j$). By Lemma 3.5 there is a subset of these of the form $u_1, \ldots, u_m, v_{i_1}, \ldots, v_{i_r}$ which consist of linearly independent elements which span V.

Remark 3.4. *Let $(V, +, \circ, K)$ be a (resp. strongly left distributive) hypervector space and W be a subhyperspace of V. Consider the quotient abelian group $(V/W, +)$. Define the rule*

$$\begin{cases} * : K \times V/W \longrightarrow P_*(V/W) \\ (a, v + W) \longmapsto a \circ v + W \end{cases}$$

Then it is easy to verify that $(V/W, +, *, K)$ is a (resp. strongly left distributive) hypervector space over K and it is called the quotient hypervector space of V over W.

Theorem 3.3. *Let V be strongly left distributive and invertible. If V is finite dimensional and W is subhyperspace of V, then the following hold:*

(i) W is finite dimensional and $dimW \leq dimV$.

(ii) $dimV/W = dimV - dimW$.

Proof. (i) Let Y be a basis of W. Then by Lemma 3.6 there exists a basis X of V such that contains Y. Thus by Corollary 3.4 we have

$$\dim W = |Y| \leq |X| = \dim V.$$

(ii) Let $\{w_1, \ldots, w_m\}$ be a basis of W. By Lemma 3.6 we can fill this out to a basis, $\{w_1, \ldots, w_m, v_1, \ldots, v_r\}$ of V, where $m + r = \dim V$ and $m = \dim W$. Let $\bar{v}_1, \ldots, \bar{v}_r$ be the images in $\bar{V} = V/W$, of $v_1, \ldots, v_r$. Since any vector $v \in V$ is in a linear combination of $w_1, \ldots, w_m, v_1, \ldots, v_r$, so

$$v \in \alpha_1 \circ w_1 + \cdots + \alpha_m \circ w_m + \beta_1 \circ v_1 + \cdots + \beta_r \circ v_r,$$

then

$$\begin{aligned} v &\in \overline{\alpha_1 \circ w_1} + \cdots + \overline{\alpha_m \circ w_m} + \overline{\beta_1 \circ v_1} + \cdots + \overline{\beta_r \circ v_r} \\ &\subseteq \overline{\beta_1 \circ v_1} + \cdots + \overline{\beta_r \circ v_r} \\ &= \beta_1 * \bar{v}_1 + \cdots + \beta_r * \bar{v}_r, \end{aligned}$$

(since $\overline{\alpha_i \circ w_i} = \alpha_i \circ w_i + W \subseteq W$). Thus $\bar{v}_1, \ldots, \bar{v}_r$ span V/W. We claim that they are linearly independent, for if

$$\underline{0} \in \gamma_1 * \bar{v}_1 + \cdots + \gamma_r * \bar{v}_r,$$

then

$$\begin{aligned} \underline{0} &\in \gamma_1 \circ v_1 + \cdots + \gamma_r \circ v_r + W \\ &\subseteq \gamma_1 \circ v_1 + \cdots + \gamma_r \circ v_r + \lambda_1 \circ w_1 + \cdots + \lambda_m \circ w_m, \end{aligned}$$

which by the linear independence of the set $\{w_1, \ldots, w_m, v_1, \ldots, v_r\}$ forces $\gamma_1 = \cdots = \gamma_r = \lambda_1 = \cdots = \lambda_m = 0$. We have shown that V/W has a basis of r elements, and

$$dim V/W = dim V - m = \dim V - dim W.$$

Example 3.2. *Let $\beta = \{e_1, \ldots, e_n\}$ be a basis for a classical vector space $(V, +, ., K)$ and let $\acute{\beta} = \{e_1, \ldots, e_k\}$ be a basis for subspace P of V. Then $\beta^* = \{e_{k+1}, \ldots, e_n\}$ is a basis for hypervector space $(V, +, \circ, K)$ in Example 2.3, because for every $x \in V$ we have:*

$$\begin{aligned} x &= a_1.e_1 + \cdots + a_k.e_k + a_{k+1}.e_{k+1} + \cdots + a_n.e_n \\ &\in a_1.e_1 + \cdots + a_k.e_k + a_{k+1}.e_{k+1} + \cdots + a_n.e_n + P \\ &= a_{k+1}.e_{k+1} + \cdots + a_n.e_n + P \\ &= a_{k+1} \circ e_{k+1} + \cdots + a_n \circ e_n, \end{aligned}$$

thus β^* span $(V, +, \circ, K)$. Moreover β^* is linearly independent, because

$$\begin{aligned} & \underline{0} \in a_{k+1} \circ e_{k+1} + \cdots + a_n \circ e_n \\ \Longrightarrow\ & \underline{0} \in a_{k+1}.e_{k+1} + \cdots + a_n.e_n + P \\ \Longrightarrow\ & a_{k+1}.e_{k+1} + \cdots + a_n.e_n \in P \\ \Longrightarrow\ & a_{k+1}.e_{k+1} + \cdots + a_n.e_n = a_1.e_1 + \cdots + a_k.e_k \\ \Longrightarrow\ & a_1.e_1 + \cdots + a_k.e_k - a_{k+1}.e_{k+1} - \cdots - a_n.e_n = 0 \\ \Longrightarrow\ & a_1 = \cdots = a_k = a_{k+1} = \cdots = a_n = 0. \end{aligned}$$

Therefor β^* is a basis.

Example 3.3. *Let* $(K[x], +, \cdot, K)$ *be the vector space of all polynomials of degree less than* n *in* x*. Define the external operation "* $\circ$ *" on* $K[x]$ *by*

$$\begin{cases} \circ : K \times K[x] \longrightarrow P_*(K[x]) \\ a \circ f(x) = a \cdot f(x) + \langle x \rangle, \end{cases}$$

where $\langle x \rangle = \{kx : k \in K\}$ *.Then it is easy to verify that* $(K[x], +, \circ, K)$ *is a strongly distributive hypervector space and* $\beta = \{1, x^2, x^3, \ldots, x^{n-1}\}$ *is a basis for* $(K[x], +, \circ, K)$*.*

Definition 3.6. *Let* V *and* W *be hypervector spaces over* K*. A mapping* $T : V \longrightarrow W$ *is called*

(i) weak linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \cap a \circ T(x) \neq \varnothing, \ \forall x, y \in V, \ a \in K,$$

(ii) linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \subseteq a \circ T(x), \ \forall x, y \in V, \ a \in K,$$

(iii) good linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) = a \circ T(x), \ \forall x, y \in V, \ a \in K.$$

A (resp. weak, good) linear isomorphism is defined as usual. If $T : V \longrightarrow W$ *is a (resp. weak, good) linear isomorphism, then it is denoted by (resp.* $V \cong_w W$, $V \cong_g W$*)* $V \cong W$*.*

Definition 3.7. *Let* $T : V \longrightarrow W$ *be a linear transformation. The kernel of* T *is denoted by* $\ker T$ *and defined by*

$$\ker T = \{x \in V : T(x) \in \Omega\},$$

where $\Omega = 0 \circ \underline{0}_W$*.*

Proposition 3.3. *Let* W *be a subhyperspace of* V*. Then the mapping*

$$\begin{cases} \pi : V \longrightarrow V/W \\ x \longmapsto x + W \end{cases}$$

is an onto good linear transformation. The mapping π *is called canonical transformation.*

Proof. Obvious.

Proposition 3.4. *Let* $T : V \longrightarrow U$ *be a good linear transformation.*

(i) If W *is a subhyperspace of* V*, then the image of* W*,* $T(W)$ *is a subhyperspace of* U*.*

(ii) If L *be a subhyperspace of* U*, then the preimage of* L*,* $T^{-1}(L)$ *is a subhyperspace of* V *containing* $\ker T$*.*

Proof. (i) Let $a \in K$ and $\acute{x}, \acute{y} \in T(W)$, such that $\acute{x} = T(x)$, $\acute{y} = T(y)$ for some $x, y \in W$. Thus $x + y \in W$ and $a \circ x \subseteq W$. So

$$\acute{x} - \acute{y} = T(x) - T(y) = T(x - y) \in T(W),$$

and

$$a \circ \acute{x} = a \circ T(x) = T(a \circ x) \subseteq T(W),$$

therefore $T(W) \leqslant U$.

(ii) Let $a \in K$ and $x, y \in T^{-1}(L)$, such that $T(x) = \acute{x}$, $T(y) = \acute{y}$, for some $\acute{x}, \acute{y} \in L$. Thus $\acute{x} + \acute{y} \in L$ and $a \circ \acute{x} \subseteq L$. So

$$\acute{x} - \acute{y} = T(x) - T(y) = T(x - y),$$

$$\Longrightarrow x - y \in T^{-1}(L),$$

and

$$a \circ \acute{x} = a \circ T(x) = T(a \circ x),$$

$$\Longrightarrow a \circ x \subseteq T^{-1}(a \circ \acute{x}) \subseteq T^{-1}(L),$$

therefor $T^{-1}(L) \leqslant V$. Also if $x \in \ker T$, then

$$T(x) \in 0 \circ \underline{0}_U \subseteq 0 \circ L \subseteq L,$$

$$\Longrightarrow x \in T^{-1}(L),$$

thus $\ker T \subseteq T^{-1}(L)$.

Proposition 3.5. *Let V and U be strongly left distributive hypervector spaces over the field K, and $T : V \longrightarrow U$ be a linear transformation. Then $\ker T$ is a subhyperspace of V. Moreover, $\Omega \subseteq kerT$.*

Proof. Since

$$\begin{aligned} T(\Omega) &= T(0 \circ \underline{0}) \\ &\subseteq 0 \circ T(\underline{0}) \\ &= 0 \circ \underline{0} = \Omega, \end{aligned}$$

thus $\varnothing \neq \ker T \subseteq V$. Also $\forall a, b \in K$ and $\forall x, y \in \ker T$, $T(x) \in \Omega, T(y) \in \Omega$. so

$$\begin{aligned} T(a \circ x + b \circ y) &= T(a \circ x) + T(b \circ y) \\ &\subseteq a \circ T(x) + b \circ T(y) \\ &\subseteq a \circ \Omega + b \circ \Omega \\ &= \Omega + \Omega = \Omega, \end{aligned}$$

$$\Longrightarrow a \circ x + b \circ y \subseteq \ker T.$$

Therefore $\ker T \leqslant V$.

Proposition 3.6. *Let V and U be strongly left distributive hypervector spaces over the field K, and $T : V \longrightarrow U$ be a good linear transformation. Then there is a one-to-one correspondence between subhyperspaces of V containing $\ker T$ and subhyperspaces of U.*

Proof. Let $\mathcal{A} = \{W : W \leqslant V \text{ and } W \supseteq \ker T\}$, and $\mathcal{B} = \{L : L \leqslant U\}$, then we show that the mapping

$$\begin{cases} \varphi : \mathcal{A} \longrightarrow \mathcal{B} \\ W \longmapsto T(W) \end{cases}$$

is a bijection. By Proposition 3.4, φ is well defined. Now if W_1 and W_2 are two elements of $\mathcal{A}$ such that $W_1 \neq W_2$. Without loss of generality, suppose that $W_2 \nsubseteq W_1$ then

$$\exists w_1 \in W_1 - W_2 \text{ or } \exists w_2 \in W_2 - W_1.$$

If $w_1 \in W_1 - W_2$, then $T(w_1) \in T(W_1) - T(W_2)$, so $T(W_1) \neq T(W_2)$. If $w_2 \in W_2 - W_1$, then similarly $T(W_1) \neq T(W_2)$. Therefore φ is a welldefined one to one map. Now if $L \in \mathcal{B}$, let $W = T^{-1}(L)$. Then by Proposition 3.5, we obtain that $W \in \mathcal{A}$ and $T(W) = L$. Consequently φ is a bijection.

Corollary 3.5. *Every subhyperspace of V/W is of the form L/W, such that L is a subhyperspace of V containing W.*

Proof. By Proposition 3.3 the mapping $\pi : V \longrightarrow V/W$ is a good linear transformation. It is easy to verify that $\ker \pi = W$. Thus by Proposition 3.6, for an arbitrary subhyperspace $\bar{L}$ of V/W, there exists a subhyperspace L of V, such that $W \subseteq L$ and $\varphi(L) = \bar{L}$. Moreover, $\varphi(L) = \{l + W : l \in L\} = L/W$. This complete the proof.

Theorem 3.4. *Let V and U be strongly left distributive hypervector spaces over the field K, and $T : V \longrightarrow U$ be a linear transformation. Then*

$$V/\ker T \cong T(V)/\Omega,$$

Moreover if T is onto, then

$$V/\ker T \cong U/\Omega.$$

Proof. We show that the mapping

$$\begin{cases} \varphi : V/\ker T \longrightarrow T(V)/\Omega \\ \varphi(x + \ker T) = T(x) + \Omega \end{cases}$$

is an isomorphism. For this let $x + \ker T$ and $y + \ker T$ be two elements of $V/\ker T$. Then

$$\begin{aligned} x + \ker T = y + \ker T &\Longleftrightarrow x - y \in \ker T \\ &\Longleftrightarrow T(x - y) \in \Omega \\ &\Longleftrightarrow T(x) - T(y) \in \Omega \\ &\Longleftrightarrow T(x) + \Omega = T(y) + \Omega \\ &\Longleftrightarrow \varphi(x + \ker T) = \varphi(y + \ker T), \end{aligned}$$

thus φ is welldefined and one to one. It is clearly that φ is onto. Now for every $x + \ker T$ and $y + \ker T$ belong to $V/\ker T$ and $a \in K$ we have:

$$\begin{aligned}
\varphi\left((x+\ker T)+(y+\ker T)\right) &= \varphi\left(x+y+\ker T\right)\\
&= T(x+y)+\Omega\\
&= T(x)+T(y)+\Omega\\
&= T(x)+\Omega+T(y)+\Omega\\
&= \varphi(x+\ker T)+\varphi(y+\ker T),
\end{aligned}$$

and

$$\begin{aligned}
\varphi\left(a * (x+\ker T)\right) &= \varphi\left(a\circ x+\ker T\right)\\
&= \{\varphi\left(t+\ker T\right) : t\in a\circ x\}\\
&= \{T\left(t\right)+\Omega : t\in a\circ x\}\\
&= T\left(a\circ x\right)+\Omega\\
&\subseteq a\circ T\left(x\right)+\Omega\\
&= a\bullet\left(T\left(x\right)+\Omega\right)\\
&= a\bullet\varphi(x+\ker T).
\end{aligned}$$

Therefore φ is an isomorphism.

Corollary 3.6. *Let V be a strongly left distributive, and let $\beta = \{x_1, \ldots, x_n\}$ be a basis for V. Then $V/0\circ\omega \cong K^n$, where $\omega = \sum_{i=1}^{n} x_i$.*

Proof. Note that $(K^n, +, \circ_K, K)$ is a strongly distributive hypervector space with trivial external operation $\circ_K$, that is:

$$\begin{cases} \circ_K : K\times K^n \longrightarrow P_*\left(K^n\right) \\ a\circ_K\left(a_1,\ldots,a_n\right) = \left\{\left(aa_1,\ldots,aa_n\right)\right\}. \end{cases}$$

Define the mapping

$$\begin{cases} T : V \longrightarrow K^n \\ x\in \sum_{i=1}^{n} a_i\circ x_i \longmapsto \left(a_1,\ldots,a_n\right), \end{cases}$$

It is easy to see that T is an onto linear transformation, such that $\ker T == 0\circ\omega$, since V is strongly left distributive. Then by Theorem 3.4 it follows that:

$$V/0\circ\omega \cong V/\ker T \cong K^n/0 \cong K^n.$$

4. Fundamental Relation of Hypervector spaces

Let $(V,+,\circ,K)$ be a hypervector space over K. The smallest equivalence relation ε^* on V, such that the quotient V/ε^* is a vector space over K is called the *fundamental relation* of V. T. Vougiouklis in [10] introduced and studied the fundamental relation of H_v-vector space (a general class of hypervector spaces). In the following we characterize the fundamental relation on hypervector spaces (in the sense of Tallini) and study the relationship between V and V/ε^*.

Let $\mathbf{U}$ be the set of all finite linear combinations of elements of V with coefficient in K, that is

$$\mathbf{U}=\left\{\sum_{i=1}^{n} a_i\circ x_i : a_i\in K \text{ and } x_i\in V\ ,\ n\in\ \mathbb{N}\right\}.$$

Define the relation ε over V by

$$x\varepsilon y \Longleftrightarrow \exists \mathbf{u}\in\mathbf{U} : \{x,y\}\subseteq \mathbf{u},\ \ \forall x,y\in V.$$

Then ε^* is the *transitive closure* of ε. Define addition operation and scalar multiplication on V/ε^* by

$$\begin{cases} \oplus : V/\varepsilon^*\times V/\varepsilon^* \longrightarrow V/\varepsilon^* \\ \varepsilon^*(x)\oplus\varepsilon^*(y)=\{\varepsilon^*(t) : t\in\varepsilon^*(x)+\varepsilon^*(y)\}\,, \end{cases}$$

and

$$\begin{cases} \odot : K\times V/\varepsilon^* \longrightarrow V/\varepsilon^* \\ a\odot\varepsilon^*(x)=\{\varepsilon^*(z) : z\in a\circ\varepsilon^*(x)\}\,, \end{cases}$$

Lemma 4.1. *The following statement are satisfied:*

(i) $\varepsilon^*(a\circ x)=\varepsilon^*(y)$ *for all* $y\in a\circ x$, $\forall a\in K$, $\forall x\in V$, *where* $\varepsilon^*(a\circ x)=\bigcup\limits_{b\in a\circ x}\varepsilon^*(b)$.

(ii) $\varepsilon^*(x)\oplus\varepsilon^*(y)=\varepsilon^*(x+y)$.

(iii) $\varepsilon^*(\underline{0})$ *is the identity element of* $(V/\varepsilon^*,\oplus)$.

(iv) $(V/\varepsilon^*,\oplus,\odot,K)$ *is a vector space over* K.

The vector space $(V/\varepsilon^*,\oplus,\odot,K)$ *is called the fundamental vector space of* V.

Proof. The proof is similar to the proof of [11, Thm. 2.4].

Theorem 4.1. *Let* $(V,+,\circ,K)$ *be a hypervector space and* $(V/\varepsilon^*,\oplus,\odot,K)$ *be the fundamental vector space of* V. *Then*

$$\dim V=\dim V/\varepsilon^*.$$

Proof. Let $S=\{x_1,x_2,\ldots,x_n\}$ be a basis for V. Then we show that $S^*=\{\varepsilon^*(x_1),\varepsilon^*(x_2),\ldots,\varepsilon^*(x_n)\}$ is a basis for V/ε^*. For this let $\varepsilon^*(x)\in V/\varepsilon^*$, then:

$$x\in V \Longrightarrow \exists\ a_1,\ldots,a_n\in K;\ x\in\sum_{i=1}^{n} a_i\circ x_i,$$

$$\Longrightarrow x=t_1+\cdots+t_n;\ \ \text{for some } t_i\in a_i\circ x_i\ ,\ 1\leqslant i\leqslant n.$$

Then by Lemma 4.1, we obtain

$$\varepsilon^*(a_i \circ x_i) = \varepsilon^*(t_i).$$

and

$$\begin{aligned}
\varepsilon^*(x) &= \varepsilon^*(t_1 + \cdots + t_n) \\
&= \varepsilon^*(t_1) \oplus \cdots \oplus \varepsilon^*(t_n) \\
&= \varepsilon^*(a_1 \circ x_1) \oplus \cdots \oplus \varepsilon^*(a_n \circ x_n) \\
&= a_1 \odot \varepsilon^*(x_1) \oplus \cdots \oplus \ a_n \odot \varepsilon^*(x_n),
\end{aligned}$$

therefore V/ε^* is generated by S^*. Now we show that S^* is linearly independent.

Suppose that $a_1 \odot \varepsilon^*(x_1) \oplus \cdots \oplus a_n \odot \varepsilon^*(x_n) = \varepsilon^*(\underline{0})$, then:

$$\begin{aligned}
& \varepsilon^*(a_1 \circ x_1) \oplus \cdots \oplus \varepsilon^*(a_n \circ x_n) = \varepsilon^*(\underline{0}) \\
\Longrightarrow \ & \varepsilon^*(a_1 \circ x_1 + \cdots + a_n \circ x_n) = \varepsilon^*(\underline{0}) \\
\Longrightarrow \ & \underline{0} \in a_1 \circ x_1 + \cdots + a_n \circ x_n.
\end{aligned}$$

Since S is linearly independent, so $a_1 = \cdots = a_n = 0$. Consequently, S^* is linearly independent.

5. Category of Hypervector Spaces

For hypervector spaces V and W on the field K, by $Hom_K^w(V,W)$, $Hom_K^i(V,W)$ and $Hom_K^g(V,W)$ we mean the set of all week linear transformation, linear transformation and good linear transformation, respectively.

Definition 5.1. *The category of hypervector spaces over K denoted by $\mathcal{HV}_K^w$ is defined as follows:*

(i) The objects of $\mathcal{HV}_K^w$ are all hypervector spaces over K;

(ii) For the objects V and W of $\mathcal{HV}_K^w$, the set of all morphisms from V to W is the set $Hom_K^w(V,W)$;

(iii) The composition ST of morphisms $T : V \longrightarrow L$ and $S : L \longrightarrow W$ is defined as usual;

(iv) For any object V, the morphism $1_V : V \longrightarrow V$, is the identity map from V to V.

Note that in the Definition 5.1 part (ii) if we replace $Hom_K^w(V,W)$ by $Hom_K^i(V,W)$ or $Hom_K^g(V,W)$, then we will obtain some new categories, which we denote them by $\mathcal{HV}_K^i$ and $\mathcal{HV}_K^g$, respectively. In fact, $\mathcal{HV}_K^g \preceq \mathcal{HV}_K^i \preceq \mathcal{HV}_K^w$ (by $A \preceq B$ read A is a subcategory of B). We denote the category of all vector spaces over the field K by $\mathcal{V}_K$. In fact, $\mathcal{V}_K \preceq \mathcal{HV}_K^g$. (see [2])

Lemma 5.1. *Let V and W be two hypervector spaces and $T : V \longrightarrow W$ be a good linear transformation. Then*

(i) $\forall x \in V, T(\varepsilon^*(x)) \subseteq \varepsilon^*(T(x))$;

(ii) The map

$$\begin{cases} T^* : V/\varepsilon^* \longrightarrow W/\varepsilon^* \\ T^*(\varepsilon^*(x)) = \varepsilon^*(T(x)) \end{cases}$$

is a linear transformation.

Proof. Straightforward.

In [7] the notions of pseudonorm and norm in hypervector spaces was introduced. Let $(V, +, \circ, K)$ be a hypervector space over a valued field K, (for $a \in K$, we denote by $/a/$ the valuation of a in K). A pseudonorm in V is a mapping $\| \,\| : V \longrightarrow \mathbb{R}$, such that the following conditions hold:

(i) $\|\underline{0}\| = 0$,

(ii) $\forall x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$,

(iii) $\forall a \in K, \forall x, y \in V$, $\sup \|a \circ x\| = /a/ \,\|x\|$.

A pseudonorm in V is called norm if:

(iv) $\|x\| = 0 \Longleftrightarrow x = \underline{0}$.

Now let $\| \,\|$ be a norm on the fundamental vector space V/ε^* of V. Define the mapping $\| \,\|^* : V \longrightarrow \mathbb{R}$ by $\|x\|^* = \|\varepsilon^*(x)\|$. The next result follows:

Theorem 5.1. *The mapping $\| \,\|^*$ is a pseudonorm on V.*

Proof. (i) $\|\underline{0}\|^* = \|\varepsilon^*(\underline{0})\| = 0$,

(ii) $\forall x, y \in V$,

$$\begin{aligned}
\|x + y\|^* &= \|\varepsilon^*(x + y)\| \\
&= \|\varepsilon^*(x) \oplus \varepsilon^*(y)\| \\
&\leq \|\varepsilon^*(x)\| + \|\varepsilon^*(y)\| \\
&= \|x\|^* + \|y\|^*,
\end{aligned}$$

(iii) $\forall a \in K, \forall x, y \in V$,

$$\begin{aligned}
\sup \|a \circ x\|^* &= \sup \|\varepsilon^*(a \circ x)\| \\
&= \|a \odot \varepsilon^*(x)\| \\
&= /a/ \,\|\varepsilon^*(x)\| \\
&= /a/ \,\|x\|^*.
\end{aligned}$$

Remark 5.1. *The $\| \,\|^*$ is called the fundamental pseudonorm associated to $\| \,\|$.*

Theorem 5.2. *The mapping $F : \mathcal{HV}_K^g \longrightarrow \mathcal{V}_K$ is defined by $F(V) = V/\varepsilon^*$ is a functor. Moreover, the functor F preserves the dimension.*

Proof. By Lemma 5.1, F is well-defined. Let $T : V \longrightarrow W$ and $S : W \longrightarrow L$ be good linear transformations. Then we have $F(SoT) = (SoT)^*$. Also

$$\begin{aligned}
(SoT)^* (\varepsilon^*(x)) &= \varepsilon^* ((SoT)(x)) \\
&= \varepsilon^* (S(T(x))) \\
&= S^*\varepsilon^*(T(x)) \\
&= S^*T^*(\varepsilon^*(x)) \\
&= F(S)F(T)(\varepsilon^*(x)),
\end{aligned}$$

for all $x \in V$. Hence $F(SoT) = F(S)F(T)$.Also, $F(1_V^*) : V/\varepsilon^* \longrightarrow V/\varepsilon^*$, is obtained by $1_V^*(\varepsilon^*(x)) = \varepsilon^*(x)$, is the identity morphism. Therefore, F is a functor. Also by Theorem 4.1,

$$\dim(F(V)) = \dim(V/\varepsilon^*) = \dim(V).$$

Corollary 5.1. *Let $T : V \longrightarrow W$ be a morphism in $\mathcal{HV}_K^g$. Then the following diagram is commutative:*

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \varphi_V \downarrow & & \downarrow \varphi_W \\ V/\varepsilon^* & \xrightarrow{T^*} & W/\varepsilon^* \end{array}$$

where φ_V and φ_W are the canonical projections of V and W, respectively.

Proof. Let $x \in V$. Then

$$\begin{aligned} \varphi_W(T(x)) &= \varepsilon^*(T(x)) \\ &= T^*(\varepsilon^*(x)) \\ &= T^*(\varphi_V(x)) \\ &= T^*\varphi_V(x). \end{aligned}$$

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