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Optimality and Duality for Nondifferentiable Multiobjective Variational Problems with Higher Order Derivatives

I. Husain[1*], A. Ahmed[2], and Ruman, G. Mattoo[2]

[1] *Department of Mathematics, Jaypee Institute of Engineering and Technology, Guna, MP, India. (A constituent centre of Jaypee University of Information Technology), Waknaghat, Solan, HP, India*

[2] *Department of Statistics, University of Kashmir, Srinagar, Kashmir, India*

Abstract. Wolfe and Mond-Weir type vector dual variational problems are formulated for a class of nondifferentiable multiobjective variational problems involving higher order derivatives. By using concept of efficiency, weak, strong and converse duality theorems are established under invexity and generalized invexity assumptions. Validation of some of our duality results can also be served as a correction for the results existing in the literature. Related problems for which our duality results can hold, are also pointed out.

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*Corresponding author.

1. Introduction

Many authors have studied optimality and duality for multiobjective variational problems. Bector and Husain [1] were probably the first to introduce multiobjective programming in calculus of variation which is a powerful technique for the solutions of various problems appearing in dynamics of rigid bodies, optimization of orbits, theory of variation and many other fields.

In [11], Mishra and Mukerjee discussed duality for multiobjective variational problems involving generalized (F, ρ)-convex functions. In [10], Liu proved only some weak duality theorem for nondifferentiable multiobjective variational problems involving generalized (F, ρ) convex functions.

Recently Husain et al [7] have studied optimality and duality for multiobjective variational problems involving higher order derivatives. Chandra, Craven and Husain [3] obtained necessary optimality conditions for a constrained continuous programming having term with a square root of a quadratic form in the objective function, and using these optimality conditions formulated Wolfe type dual and established weak, Strong and Huard [12] type converse duality theorems under convexity of functions. Subsequently, for the problems of [3], Bector, Chandra and Husain [2] constructed a Mond-Weir type dual which allows weakening of convexity hypotheses of [3] and derived various duality results under generalized convexity of functionals.

The popularity of this type of problems seem to originate from the fact that, even though the objective function and/ or constraint functions are non-smooth, a simple representation of the dual problem may be found. The theory of non-smooth mathematical programming deals with much more general types of functions by means of generalized subdifferentials [4] and quasi differentials [6]. However, the square root of a positive semidefinite quadratic form is one of the few cases of a nondifferentiable function for which one can write down the sub or quasi differentials explicitly.

In this paper, we study optimality and duality for a class of nondifferentiable variatonal problem containing higher order derivatives. We formulate Wolfe and Mond-Weir type dual problems for this class of variational problems and prove various duality results under invexity and generalized invexity. The result of this research also serves as correction to some of the results obtained by Kim and Kim [9].

2. Pre-requisites

Consider the real interval $I = [a, b]$ and the continuously differentiable function $\phi : I \times R^n \times R^n \times R^n \to R$. In order to consider $\phi(t, x, \dot{x}, \ddot{x})$, where $x : I \to R^n$ is twice differentiable with its first and second order derivatives $\dot{x}$ and $\ddot{x}$ respectively, we denote the partial derivative of ϕ with respect to t by ϕ_t.

$$\phi_x = \left[\frac{\partial \phi}{\partial x^1}, \dots, \frac{\partial \phi}{\partial x^n}\right]^T, \ \ \phi_{\dot{x}} = \left[\frac{\partial \phi}{\partial \dot{x}^1}, \dots, \frac{\partial \phi}{\partial \dot{x}^n}\right]^T, \ \ \phi_{\ddot{x}} = \left[\frac{\partial \phi}{\partial \ddot{x}^1}, \dots, \frac{\partial \phi}{\partial \ddot{x}^n}\right]^T.$$

The partial derivative of other function will be written similarly. Let K designate the space of piecewise smooth functions $x : I \to R^n$ possessing derivatives $\dot{x}$ and $\ddot{x}$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)ds,$$

where α is given boundary value; thus $D \equiv \frac{d}{dt}$ except at discontinuities.

In the results to follow, we use $C(I, R^m)$ to denote the space of continuous functions $\phi : I \to R^m$ with the uniform norm; superscript T denotes matrix transpose.

Before stating our variational problem and deriving its necessary optimality conditions, we mention the following conventions for vectors x and y in n-dimensional Euclidian space R^n to be used throughout the analysis of this research.

$$x < y, \iff x_i < y_i, \ i = 1, 2, \dots, n.$$

$$x \leqq y, \iff x_i \leq y_i, \ i = 1, 2, \ldots, n.$$

$$x \leq y, \iff x \leqq y, \ i = 1, 2, \ldots, n, \text{ but } x \neq y$$

$$x \nleq y, \text{ is the negation of } x \leq y$$

For $x, y \in R$, $x \leq y$ and $x < y$ have the usual meaning.

We present the following nondifferentiable multiobjective variational problem with higher order derivatives as:

$$\text{(VP)} \quad \text{Minimize} \left(\int_I \left(f^1(t, x, \dot{x}, \ddot{x}) dt + \left(x(t)^T B^1(t) x(t) \right)^{\frac{1}{2}} \right) dt \right.$$
$$\left. , \ldots, \int_I \left(f^p(t, x, \dot{x}, \ddot{x}) dt + \left(x(t)^T B^p(t) x(t) \right)^{\frac{1}{2}} \right) dt \right)$$

Subject to

$$x(a) = 0 = x(b) \tag{2.1}$$

$$\dot{x}(a) = 0 = \dot{x}(b) \tag{2.2}$$

$$g(t, x, \dot{x}, \ddot{x}) \leqq 0, \quad t \in I, \tag{2.3}$$

where, $f^i : I \times R^n \times R^n \times R^n \to R$, $(i = 1, 2, \ldots, p)$, $g : I \times R^n \times R^n \times R^n \to R^m$, are assumed to be continuously differentiable functions, for each $i \in P, \{i = 1, 2, \ldots, p\}, B^i(t)$ is an $n \times n$ positive semidefinite symmetric matrix with $B^i(\cdot)$ continuous on I.

The following generalized Schwarz inequality [15] is required in the sequel.

$$(x(t)^T B^i(t) z(t)) \leq (x(t)^T B^i(t) x(t))^{\frac{1}{2}} (z(t)^T B^i(t) z(t))^{\frac{1}{2}}$$
$$\forall \ x(t) \in R^n, \ z(t) \in R^n, \ t \in I$$

Definition 2.1 (Invexity). *If there exists vector function $\eta(t, x, u) \in R^n$ with $\eta = 0$ and $x(t) = u(t)$, $t \in I$ and $D\eta = 0$ for $\dot{x}(t) = \dot{u}(t), t \in I$ such that for a scalar function*

$\phi(t,x,\dot{x},\ddot{x})$, the functional $\Phi(x,\dot{x},\ddot{x}) = \int_f \phi(t,x,\dot{x},\ddot{x})dt$ satisfies

$$\begin{aligned}
&\Phi(x,\dot{u},\ddot{u}) - \Phi(x,\dot{x},\ddot{x}) \\
&\quad \geqq \int_I \{\eta^T\phi_x(t,x,\dot{x},\ddot{x}) + (D\eta)^T\phi_{\dot{x}}(t,x,\dot{x},\ddot{x}) + (D^2\eta)^T\phi_{\ddot{x}}(t,x,\dot{x},\ddot{x})\}dt,
\end{aligned}$$

Φ is said to be invex in $x,\dot{x}$ and $\ddot{x}$ on I with respect to η.

Definition 2.2 (Pseudoinvexity). *Φ is said to be pseudoinvex in $x,\dot{x}$ and $\ddot{x}$ with respect to η if*

$$\int_I \{\eta^T\phi_x(t,x,\dot{x},\ddot{x}) + (D\eta)^T\phi_{\dot{x}}(t,x,\dot{x},\ddot{x}) + (D^2\eta)^T\phi_{\ddot{x}}(t,x,\dot{x},\ddot{x})\}dt \geqq 0$$

implies

$$\Phi(x,\dot{u},\ddot{u}) \geqq \Phi(x,\dot{x},\ddot{x})$$

Definition 2.3 (Quasi-invex). *The functional Φ is said to quasi-invex in $x,\dot{x}$ and $\ddot{x}$ with respect to η if*

$$\begin{aligned}
\Phi(x,\dot{u},\ddot{u}) &\leqq \Phi(x,\dot{x},\ddot{x}) \\
&\Rightarrow \int_I \{\eta^T\phi_x(t,x,\dot{x},\ddot{x}) + (D\eta)^T\phi_{\dot{x}}(t,x,\dot{x},\ddot{x}) + (D^2\eta)^T\phi_{\ddot{x}}(t,x,\dot{x},\ddot{x})\}dt \leqq 0
\end{aligned}$$

We require the following definition of efficient solution for our further analysis. In this definition we denote by X, the set of feasible solution for (VP).

Definition 2.4. *A point $\bar{x} \in X$ is said to be efficient solution of (VP) if for all feasible $x \in X$,*

$$\begin{aligned}
&\int_I \left(f^i(t,x(t),\dot{x}(t),\ddot{x}(t))dt + (x(t)^TB^i(t)x(t))^{\frac{1}{2}}\right)dt \\
&\quad \nleq \int_I \left(f^i\left(t,\bar{x}(t),\dot{\bar{x}}(t),\ddot{\bar{x}}(t)\right)dt + \left(\bar{x}(t)^TB^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\right)dt
\end{aligned}$$

for all $i \in P$.

In order to prove the strong duality theorem, we will invoke the following lemma due to Changkong and Haimes [5].

Lemma 2.1 ([5]). *A function $\bar{x} \in X$ be an efficient solution of (VP) if and only if $\bar{x} \in X$ is an optimal solution of the following problem $(P_k(\bar{x}))$ for all k.*

$$(P_k(\bar{x})): \qquad \text{Minimize} \int_I \left(f^k(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^k(t)x(t)\right)^{\frac{1}{2}} \right) dt$$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$g(t,x,\dot{x},\dot{x}) \leqq 0, \quad t \in I,$$

$$\int_I \left(f^i(t,x(t),\dot{x}(t),\ddot{x}(t))\,dt + \left(x(t)^T B(t)x(t)\right)^{\frac{1}{2}} \right) dt$$

$$\leq \int_I \left(f^i\left(t,\bar{x}(t),\dot{\bar{x}}(t),\ddot{\bar{x}}(t)\right) dt + \left(\bar{x}(t)^T B(t)\bar{x}(t)\right)^{\frac{1}{2}} \right) dt, \quad i \neq k$$

3. Optimality

In this section, we give necessary optimality conditions for the problem $(P_k(\bar{x}))$ which are required to establish strong duality theorem for Wolfe and Mond-Weir type vector dual.

In order to derive optimality conditions for $(P_k, (\bar{x}))$, we require the following Lemma 3.1.

Lemma 3.1 ([9]). *Define a function $h: R^n \to R$ by $h(x(t)) = \left(\bar{x}(t)^T B(t)\bar{x}(t)\right)^{\frac{1}{2}}$, where B is a symmetric positive semidefinite $n \times n$ matrix and continuous on I, then h is* *convex, and*

$$\partial h(x(t)) = \left\{ B(t)z(t) : z(t)^T B(t)z(t) \leq 1 \right\},$$

where $\partial h(x(t))$ *is subgradient of* h *at* $x(t)$.

Using the analysis in [10] and [6], the Fritz-John optimality conditions for $(P_k(\bar{x}))$ can be given by the following theorem.

Theorem 3.1 (Fritz-John Optimality Conditions)**.** *If* $\bar{x}$ *is optimal solution of* $(P_k(\bar{x}))$ *there exist scalars* $\tau^1, \tau^2, \ldots, \tau^p$*, piecewise smooth* $z^i : I \rightarrow R, i \in P$*, such that*

$$\sum_{i=1}^{p} \bar{\tau}^i \left(f_x^i \left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}} \right) - D f_{\dot{x}}^i \left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}} \right) + D^2 f_{\ddot{x}}^i \left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}} \right) + B^i(t) \bar{z}^i(t) \right)$$

$$+ \sum_{j=1}^{m} \bar{y}^j(t) \left(g_x^j \left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}} \right) - D g_{\dot{x}}^j \left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}} \right) + D^2 g_{\ddot{x}}^j \left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}} \right) \right) = 0, \quad t \in I$$

$$y(t)^T g \left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}} \right) = 0, \quad t \in I$$

$$\bar{x}(t)^T B^i(t) \bar{z}^i(t) = \left(\bar{x}(t)^T B^i(t) \bar{x}(t) \right)^{\frac{1}{2}}, \quad i \in P, t \in I$$

$$\bar{z}^i(t)^T B^i(t) \bar{z}^i(t) \leq 1,$$

$$\left(\bar{\tau}, \bar{y}(t) \right) \geqq 0, \quad \left(\bar{\tau}, \bar{y}(t) \right) \neq 0, \quad t \in I$$

Proof. The proof of the theorem easily follows on the line of analysis in [7] and [3]. Hence it is omitted for brevity.

4. Wolfe Type Vector Duality

In this section, we present Wolfe type vector dual to (VP) and establish various duality results.

(MWD) : Maximize $\left(\int_I \left(f^1(t, u, \dot{u}, \ddot{u}) + u(t)^T B^1(t) z^1(t) + y(t)^T g(t, u, \dot{u}, \ddot{u}) \right) dt \right.$

$$,\dots,\int_I \left(f^p(t,u,\dot u,\ddot u)+u(t)^T B^p(t)z^p(t)+y(t)^T(t,u,\dot u,\ddot u)\right)dt\Big)$$

Subject to

$$u(a)=0=u(b) \tag{4.1}$$

$$\dot u(a)=0=\dot u(b) \tag{4.2}$$

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i(t,u,\dot u,\ddot u)dt+B^i(t)z^i(t)+y(t)^T g_u(t,u,\dot u,\ddot u)\right)$$

$$-D\left(\lambda^T f_{\dot u}(t,u,\dot u,\ddot u)+y(t)^T g_{\dot u}(t,u,\dot u,\ddot u)\right)$$

$$+D^2\left(\lambda^T f_{\ddot u}(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)=0,\quad t\in I \tag{4.3}$$

$$\bar z^i(t)^T B^i(t)\bar z^i(t)\leqq 1,\quad t\in I,\ i\in P \tag{4.4}$$

$$y(t)\geqq 0,\quad t\in I \tag{4.5}$$

$$\lambda>0,\quad \lambda^T e=1 \tag{4.6}$$

Theorem 4.1 (Weak Duality). *Let $\bar x$ be feasible for* (VP) *and $(u,\lambda,z^1,\dots,z^p,y)$ be feasible for (MWD). If for all feasible $\left(x,u,\lambda,z^1,\dots,z^p,y\right)$, $\sum_{i=1}^{p}\lambda^i\int_I\left(f^i(t,.,.,.)+(\cdot)^T B^i(t)z^i(t)+y(t)^T g(t,.,.,.)\right)dt$ is pseudoinvex with respect to η,*

Then the following cannot hold:

$$\int_I\left(f^i(t,x,\dot x,\ddot x)+x(t)^T B^i(t)z^i(t)\right)dt$$

$$\leqq\int_I\left(f^i(t,u,\dot u,\ddot u)dt+u(t)^T B^i(t)z^i(t)+y(t)^T g(t,u,\dot u,\ddot u)\right)dt, \tag{4.7}$$

for all $i\in P$, and

$$\int_I\left(f^j(t,x,\dot x,\ddot x)dt+x(t)^T B^j(t)z^j(t)\right)dt$$

$$< \int_I \left(f^j(t,u,\dot{u},\ddot{u})dt + \left(u(t)^T B^j(t)z^j(t)\right) + y(t)^T g^j(t,u,\dot{u},\ddot{u})\right)dt \quad (4.8)$$

for some $j \in P$.

Proof. Suppose that (4.7) and (4.8) hold. Then, from (2.3) and (4.5), we have,

$$\int_I \left(f^i(t,x,\dot{x},\ddot{x}) + x(t)^T B^i(t)z^i(t) + y(t)^T g(t,x,\dot{x},\ddot{x})\right)dt$$

$$\leqq \int_I \left(f^i(t,u,\dot{u},\ddot{u})dt + u(t)^T B^i(t)z^i(t) + y(t)^T g^j(t,u,\dot{u},\ddot{u})\right)dt,$$

for all $i \in P$, and

$$\int_I \left(f^j(t,x,\dot{x},\ddot{x})dt + x(t)^T B^j(t)z^j(t) + y(t)^T g(t,x,\dot{x},\ddot{x})\right)dt$$

$$< \int_I \left(f^j(t,u,\dot{u},\ddot{u})dt + \left(u(t)^T B^j(t)z^j(t)\right) + y(t)^T g^j(t,u,\dot{u},\ddot{u})\right)dt$$

for some $j \in P$.

Now using $\lambda > 0$ and $\sum_{i=1}^{p} \lambda^i = 1$, these inequalities yield,

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x},\ddot{x})dt + \left(x(t)^T B^i(t)z^i(t)\right) + y(t)^T g(t,x,\dot{x},\ddot{x})\right)dt$$

$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u},\ddot{u})dt + \left(u(t)^T B^i(t)z^i(t)\right) + y(t)^T g(t,u,\dot{u},\ddot{u})\right)dt$$

This, because of the pseudoinvexity of

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + (\cdot)^T B^i(t)z^i(t) + y(t)^T g(t,.,.,.)\right)dt$$

implies

$$\sum_{i=1}^{p} \lambda^i \int_I \eta^T \left[\left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t)z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u})\right)\right.$$

$$-\left(D\eta\right)^T\left(f^i_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)$$

$$+\left(D^2\eta\right)^T\left(f^i_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)\Big]dt<0$$

Integrating by parts, we get

$$\begin{aligned}0 \;>\; & \sum_{i=1}^{p}\lambda^i\int_I \eta^T\Big[\left(f^i_u\left(t,u,\dot{u},\ddot{u}\right)+B^i\left(t\right)z^i\left(t\right)+y\left(t\right)^T g_u\left(t,u,\dot{u},\ddot{u}\right)\right)\\ & -D\left(f^i_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)\Big]dt\\ & -\sum_{i=1}^{p}\lambda^i\int_I\left(D\eta\right)^T\left(f^i_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)dt\\ & +\sum_{i=1}^{p}\lambda^i\eta^T\left(f^i_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)\Big|_{t=a}^{t=b}\\ & +\sum_{i=1}^{p}\lambda^i\left(D\eta\right)^T\left(f^i_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)\Big|_{t=a}^{t=b}\end{aligned}$$

Using the boundary conditions which at $t=a,t=b$ gives $D\eta=0=\eta$, we have

$$\begin{aligned}= \;& \sum_{i=1}^{p}\lambda^i\int_I \eta^T\Big[\left(f^i_u\left(t,u,\dot{u},\ddot{u}\right)+B^i\left(t\right)z^i\left(t\right)+y\left(t\right)^T g_u\left(t,u,\dot{u},\ddot{u}\right)\right)\\ & -D\left(f^i_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)\Big]dt\\ & -\sum_{i=1}^{p}\lambda^i\int_I\left(D\eta\right)^T\left(f^i_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)dt\end{aligned}$$

Again, integrating by parts we obtain

$$\begin{aligned}= \;& \sum_{i=1}^{p}\lambda^i\int_I \eta^T\Big[\left(f^i_u\left(t,u,\dot{u},\ddot{u}\right)+B^i\left(t\right)z^i\left(t\right)+y\left(t\right)^T g_u\left(t,u,\dot{u},\ddot{u}\right)\right)\\ & -D\left(f^i_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)\\ & +D^2\left(f^i_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)\Big]dt\end{aligned}$$

$$+\sum_{i=1}^{p}\lambda^{i}\eta^{T}\left(f_{\ddot{u}}^{i}(t,u,\dot{u},\ddot{u})+y(t)^{T}g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)\Big|_{t=a}^{t=b}$$

again using boundary conditions which at $t=a, t=b$ gives $D\eta = 0 = \eta$,

$$\int_{I}\eta^{T}\sum_{i=1}^{p}\lambda^{i}\Big[\left(f_{u}^{i}(t,u,\dot{u},\ddot{u})+B^{i}(t)z^{i}(t)+y(t)^{T}g_{u}(t,u,\dot{u},\ddot{u})\right)$$

$$-D\left(f_{\dot{u}}^{i}(t,u,\dot{u},\ddot{u})+y(t)^{T}g_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) \tag{4.9}$$

$$+D^{2}\left(f_{\ddot{u}}^{i}(t,u,\dot{u},\ddot{u})+y(t)^{T}g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)\Big]\,dt<0$$

From the equality constraint (4.3), we have

$$\int_{I}\eta^{T}\sum_{i=1}^{p}\lambda^{i}\Big[\left(f_{u}^{i}(t,u,\dot{u},\ddot{u})+B^{i}(t)z^{i}(t)+y(t)^{T}g_{u}(t,u,\dot{u},\ddot{u})\right)$$

$$-D\left(f_{\dot{u}}^{i}(t,u,\dot{u},\ddot{u})+y(t)^{T}g_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) \tag{4.10}$$

$$+D^{2}\left(f_{\ddot{u}}^{i}(t,u,\dot{u},\ddot{u})+y(t)^{T}g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)\Big]\,dt=0$$

The inequality (4.9) contradicts (4.10). Hence our assumption is invalid and the theorem follows.

Theorem 4.2 (Strong Duality). *Let $\bar{x} \in X$ be an efficient solution of (VP) and for at least one $k \in P$, $\bar{x}$ satisfies the regularity condition [3] for the problem $(P_k(\bar{x}))$. Then there exist multipliers $\lambda \in R^p$, piecewise smooth $\bar{y} \in R^m, z^i(t) \in R^n, i = \{1,2,\ldots,p\}$ such that $(\bar{x},\bar{u},\bar{y},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\lambda)$ is feasible for (MWD) and the objectives of (VP) and (MWD) are equal.*

Further, if the hypothesis of Theorem 4.1 is met, then $(x,u,y,z^1,\ldots,z^p,\lambda)$ is an efficient solution of (MWD).

Proof. By Lemma 2.1 $\bar{x}$ is an optimal solution of $(P_k(\bar{x}))$. This implies that there exist $\bar{\xi} \in R^p$ with $\bar{\xi}^1,\ldots,\bar{\xi}^p,\bar{z}^i(t) \in R^n, i = \{1,2,\ldots,p\}$ and piecewise smooth $\bar{v} \in R^m$

such that, the following optimality conditions (4.7) and (4.3) hold:

$$\bar{\xi}^k\left(f_x^k(t,x,\dot{x},\ddot{x})+B^k(t)\bar{z}^k(t)-Df_{\dot{x}}^k(t,x,\dot{x},\ddot{x})+D^2f_{\ddot{x}}^k(t,x,\dot{x},\ddot{x})\right)$$

$$+\sum_{\substack{i=1\\i\neq k}}^{p}\bar{\xi}^i\left(f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)\bar{z}^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})+D^2f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)$$

$$+\bar{v}(t)^T g_x(t,x,\dot{x},\ddot{x})-D\left(\bar{v}(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\right)$$

$$+D^2\left(\bar{v}(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right)=0 \tag{4.11}$$

$$\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}=\left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right),\quad i=1,\ldots,p \tag{4.12}$$

$$\bar{v}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt=0 \tag{4.13}$$

$$\left(\bar{z}(t)^T B^i(t)\bar{z}^i(t)\right)\leqq 1,\quad t\in I,\ i=1,2,\ldots,p \tag{4.14}$$

$$\bar{\xi}>0,\quad \bar{v}(t)\geqq 0,\quad t\in I \tag{4.15}$$

From (4.11) we obtain

$$\sum_{i=1}^{p}\bar{\xi}^i\left(f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)z^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})+D^2f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)$$

$$+\bar{v}(t)^T g_x(t,x,\dot{x},\ddot{x})-D\left(\bar{v}(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\right)$$

$$+D^2\left(\bar{v}(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right)=0 \tag{4.16}$$

Dividing (4.13) (4.15) and (4.16) by $\bar{\xi}^T e\,(\neq 0)$, and setting $\lambda^i=\left(\frac{\bar{\xi}}{\bar{\xi}^T e}\right)$, $i=1,\ldots,p$ and $\bar{y}(t)=\left(\frac{\bar{v}(t)}{\bar{\xi}^T e}\right)$, we have,

$$\sum_{i=1}^{p}\bar{\lambda}^i\left(f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)\bar{z}^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})+D^2f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)$$

$$+\bar{y}(t)^T g_x(t,x,\dot{x},\ddot{x})-D\left(\bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\right)$$

$$+D^2\left(\bar{y}(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right)=0 \tag{4.17}$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt = 0 \tag{4.18}$$

$$\bar{\lambda} > 0, \quad \lambda^T e = 1 \tag{4.19}$$

$$\bar{y}(t) \geqq 0, \quad t \in I \tag{4.20}$$

Consequently (4.14), (4.17), (4.19) and (4.20) implies that $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda})$ is feasible for (WD). Because of (4.18), the two objectives of the problem (VP)and (MWD) are equal. Hence by Theorem 4.1 $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda})$ is efficient solution for (MWD). This completes the proof.

For validating converse duality theorem, we regard (MWD) in term of function x for convenience instead of the function u.

As in [13], by employing chain rule in calculus, it can be easily seen that the expression

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,x,\dot{x},\ddot{x})dt + B^i(t)z^i(t) + y(t)^T g_x(t,x,\dot{x},\ddot{x})\right)$$
$$-D\left(\lambda^T f_{\dot{x}}(t,x,\dot{x},\ddot{x}) + y(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\right)$$
$$+D^2\left(\lambda^T f_{\ddot{x}}(t,x,\dot{x},\ddot{x}) + y(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right) = 0, \quad t \in I,$$

may be regarded as a function θ of variables $t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}$ and λ, where $\dddot{x} = \frac{d^3}{dt^3}x = D^3 x$ and $\ddot{y} = D^2 y$. That is, we can write

$$\begin{aligned} &\theta\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda\right) \\ &= \sum_{i=1}^{p} \lambda^i \left(f_x^i(t,x,\dot{x},\ddot{x})dt + B^i(t)z^i(t) + y(t)^T g_x(t,x,\dot{x},\ddot{x})\right) \\ &\quad -D\left(\lambda^T f_{\dot{x}}(t,x,\dot{x},\ddot{x}) + y(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\right) \\ &\quad +D^2\left(\lambda^T f_{\ddot{x}}(t,x,\dot{x},\ddot{x}) + y(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right) = 0, \quad t \in I \end{aligned}$$

The problem (MWD) may now be briefly written as,

$$\text{Minimize}\left(\int_I -\left(f^1(t,x,\dot{x},\ddot{x})+\left(u(t)^T B^1(t) z^1(t)\right)+y^T(t) g^T(t,x,\dot{x},\ddot{x})\right) dt\right.$$

$$\left.,\ldots,\int_I -\left(f^p(t,x,\dot{x},\ddot{x})+\left(u(t)^T B^p(t) z^p(t)\right)+y^T(t) g^T(t,x,\dot{x},\ddot{x})\right) dt\right)$$

Subject to

$$x(a)=0=x(b)$$

$$\dot{x}(a)=0=\dot{x}(b)$$

$$\theta\left(t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},\lambda\right)=0$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1, \quad t\in I,\ i\in P$$

$$y(t)\geqq 0, \quad t\in I$$

$$\lambda>0, \quad \lambda^T e=1$$

Consider $\theta\left(t,x(\cdot),\dot{x}(\cdot),\ddot{x}(\cdot),\dddot{x}(\cdot),y(\cdot),\dot{y}(\cdot),\ddot{y}(\cdot),\lambda\right)=0$ as defining a mapping $\psi : X\times Y\times R^p \to Q$ where Y is a space of piecewise twice differentiable function and Q is the Banach Space. In order to apply Theorem 3.1[8] to the problem (MWD), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $\left[\psi_x(x,y,\lambda),\psi_y(x,y,\lambda),\psi_\lambda(x,y,\lambda)\right]$.

Theorem 4.3 (Converse Duality). *Let* $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{\lambda}\right)$ *be an efficient solution for (MWD) Assume that*

(H_1) *The Frèchèt derivative* ψ' *has a (weak*) closed range,*

(H_2) f *and* g *be twice continuously differentiable, and*

(H_3) $\left(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}}\right)\beta(t)=0, \Rightarrow \beta(t)=0,\ t\in I$

Further, if the assumptions of Theorem 4.1 *are satisfied, then* $\bar{x}$ *is an efficient solution of* (VP).

Proof. Since $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda})$ with ψ' having a (weak*) closed range, is an efficient solution of (MWD), then there exist $\alpha \in R^p$, $\eta \in R^p$, $\gamma \in R$, $\delta \in R$, $\xi \in R^m$ and piecewise smooth $\beta(t) : I \to R^n$ and $\mu(t) : I \to R^m$ such that the following Fritz-John optimality conditions[8] hold

$$-\sum_{i=1}^{p} \alpha^i \left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t) z^i(t) + y(t)^T g_x(t,x,\dot{x},\ddot{x}) \right)$$

$$+D\left(\alpha^T f_{\dot{x}}(t,x,\dot{x},\ddot{x}) + \left(\alpha^T e\right) y(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x}) \right)$$

$$-D^2\left(\alpha^T f_{\ddot{x}}(t,x,\dot{x},\ddot{x}) + \left(\alpha^T e\right) y(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x}) \right)$$

$$+\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2\beta(t)^T \theta_{\ddot{x}} - D^3\beta(t)^T \theta_{\dddot{x}} = 0, \quad t \in I \tag{4.21}$$

$$-(\alpha^T e) g^j(t,x,\dot{x},\ddot{x}) + \beta(t)^T \theta_{y^j} - D\beta(t)^T \theta_{\dot{y}^j} + D^2\beta(t)^T \theta_{\ddot{y}^j} - \mu^j(t) = 0, \quad t \in I \tag{4.22}$$

for $j = 1, 2, \ldots, m$

$$\left[f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t) z^i(t) - D f_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) \right.$$

$$\left. + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x}) \right] \beta(t) + \eta^i + \gamma = 0, \quad i = 1, \ldots, p \tag{4.23}$$

$$-\alpha^i x(t)^T B^i(t) + \beta(t) \lambda^i B^i(t) + \delta^i 2B^i(t) z^i(t) = 0 \tag{4.24}$$

$$\eta^T \bar{\lambda} = 0 \tag{4.25}$$

$$\mu(t)^T \bar{y}(t) = 0, \quad t \in I \tag{4.26}$$

$$\gamma \left(\sum_{i=1}^{p} \lambda^i - 1 \right) = 0 \tag{4.27}$$

$$\delta^i \left(z^i(t)^T B^i(t) z^i(t) \right) = 0, \quad t \in I \tag{4.28}$$

$$\left(\alpha, \lambda, \mu(t), \eta, \gamma, \delta, \right) \geqq 0, \quad t \in I \tag{4.29}$$

$$\left(\alpha, \beta(t), \lambda, \mu(t), \eta, \gamma, \delta, \right) \neq 0, \quad t \in I \tag{4.30}$$

Since $\lambda > 0$, (4.25) implies $\eta = 0$. Consequently (4.23) implies

$$\left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t) z^i(t) - D f_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x}) \right) \beta(t) = -\gamma = 0$$

From the equality constraint of (MWD), we have

$$\left(\bar{y}(t)^T g_x(t,x,\dot{x},\ddot{x}) - D\bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x}) + D^2 \bar{y}(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x}) \right)$$
$$= -\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t) \bar{z}^i(t) - D f_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x}) \right)$$

This, in view of (4.31), implies

$$\beta(t)^T \left(\bar{y}(t)^T g_x(t,x,\dot{x},\ddot{x}) - D\bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x}) + D^2 \bar{y}(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x}) \right)$$
$$= -\sum_{i=1}^{p} \lambda^i \beta(t)^T \left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t) \bar{z}^i(t) - D f_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x}) \right)$$
$$= -\sum_{i=1}^{p} \lambda^i (-\gamma) = \gamma \tag{4.31}$$

Postmultiplying (4.21) by $\beta(t)$ and then using (4.31) and (4.32), we obtain

$$\left(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}} = 0 \right) \beta(t) = 0, \quad t \in I$$

This, because of the hypothesis (H_3), gives

$$\beta(t) = 0, t \in I$$

Suppose $\alpha = 0$, then from (4.22) we have $\mu^j(t) = 0, j = 1, 2, \ldots, m$, and from (4.23) it follows that $\gamma = 0$.

Also from (4.24) we have $\delta^i B^i(t) z^i(t) = 0$ which together with (4.28) implies $\delta = 0$. Thus, $(\alpha, \beta(t), \lambda, \mu(t), \eta, \gamma, \delta,) = 0$, which is a contradiction to (4.30). Hence $\alpha > 0$.

From the equation (4.22), we have

$$g^j(t, x, \dot{x}, \ddot{x}) = -\frac{\mu^j(t)}{(\alpha^T e)} \leqq 0, \quad t \in I$$

which implies $g^j(t, x, \dot{x}, \ddot{x}) \leqq 0, \quad t \in I$.

Therefore, $\bar{x}$ is feasible for (VP). Multiplying (4.23) by $y^j(t)$, and using (4.26), we have

$$y^j(t) g^j(t, x, \dot{x}, \ddot{x}) = 0, \quad t \in I$$

By generalized Schwarz inequality [15]

$$\left(\bar{x}(t)^T B^i(t) \bar{z}^i(t)\right) \leqq \left(\bar{x}(t)^T B^i(t) \bar{x}(t)\right)^{\frac{1}{2}} \left(\bar{z}^i(t) B^i(t) \bar{z}^i(t)\right)^{\frac{1}{2}} \tag{4.32}$$

Now let $\frac{2\delta^i}{\alpha^i} = \xi^i$. Then $\xi^i \geqq 0$ and from (4.24), we have

$$B^i(t) x(t) = \xi^i 2B^i(t) z^i(t), \quad i = 1, 2, \ldots, p$$

This is the condition for the equality in (4.33). Therefore, we have

$$\left(\bar{x}(t)^T B^i(t) z^i(t)\right) = \left(x(t)^T B^i(t) \bar{x}(t)\right)^{\frac{1}{2}} \left(z^i(t) B^i(t) z^i(t)\right)^{\frac{1}{2}}$$

From (4.28), either $\delta^i = 0$ or $z^i(t)^T B^i(t) z^i(t) = 1$ and hence $B^i(t) \bar{x}(t) = 0$. Therefore, in either case $\left(x(t)^T B^i(t) z^i(t)\right) = \left(x(t)^T B^i(t) z^i(t)\right)^{\frac{1}{2}}$, $i = 1, 2, \ldots, p$.

Hence

$$\int_I \left(f^i\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt + \left(x(t)^T B^i(t) z^i(t)\right)^{\frac{1}{2}} + y^j(t) g^j\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right)\right) dt$$

$$= \int_I \left(f^i\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt + \left(x(t)^T B^i(t) z^i(t)\right)^{\frac{1}{2}}\right) dt, \quad i = 1, 2, \ldots, p$$

The efficiency of $\bar{x}$ for (VP) is an immediate consequence of the application of Theorem 4.1.

REMARKS: Theorem 4.3 serves as a correction to Theorem 5 of Kim and Kim [9] as its hypothesis (iii) is not required to establish it.

5. Mond-weir Type Duality

In this section, we establish various duality theorems for the Mond-Weir type vector dual.

(M-WVD) : Maximize $\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t)\right) dt\right.$

$$\left., \ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + u(t)^T B^p(t) z^p(t)\right) dt\right)$$

Subject to

$$u(a) = 0 = u(b) \tag{5.1}$$

$$\dot{u}(a) = 0 = \dot{u}(b) \tag{5.2}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,u,\dot{u},\ddot{u})\,dt + B^i(t) z^i(t) + y(t)^T g_x(t,u,\dot{u},\ddot{u})\right)$$

$$-D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right) = 0, \quad t \in I \tag{5.3}$$

$$\sum_{j=1}^{m} \int_I y^j(t) g^j(t,u,\dot{u},\ddot{u})\,dt \geqq 0, \quad t \in I \tag{5.4}$$

$$\bar{z}^i(t)^T B^i(t) \bar{z}^i(t) \leqq 1, \quad t \in I,\ i \in P \tag{5.5}$$

$$\lambda > 0, y(t) \geqq 0, \quad t \in I \tag{5.6}$$

Theorem 5.1 (Weak Duality). *Let $\bar{x}$ be feasible for (VP) and $\left(u,\lambda,z^1,\ldots,z^p,y\right)$ be feasible for (M-WVD). If for feasible $\left(x,u,\lambda,z^1,\ldots,z^p,y\right)$, $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + (\cdot)^T B^i(t) z^i(t)\right) dt$ is pseudoinvex and $\int_I y(t)^T g(t,.,.,.)\,dt$ is quasi-invex with respect to same η, the following cannot hold:*

$$\int_I \left(f^i(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}}\right) dt$$

$$\leq \int_I \left(f^i(t,u,\dot{u},\ddot{u})\,dt + \left(u(t)^T B^i(t) z^i(t)\right)^{\frac{1}{2}}\right) dt, \textit{ for all } \; i \in P, \tag{5.7}$$

and

$$\int_I \left(f^j(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^j(t) x(t)\right)^{\frac{1}{2}}\right) dt$$

$$< \int_I \left(f^j(t,u,\dot{u},\ddot{u})\,dt + \left(u(t)^T B^j(t) z^j(t)\right)^{\frac{1}{2}}\right) dt, \textit{ for some } \; j \in P \tag{5.8}$$

Proof. Suppose that (5.7) and (5.8) hold. Using $\lambda > 0$ and $\sum_{i=1}^{p} \lambda^i = 1$, then in view of Schwartz inequality [15], this gives

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^i(t) z^i(t)\right)^{\frac{1}{2}}\right) dt$$

$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u},\ddot{u})\,dt + \left(u(t)^T B^i(t) z^i(t)\right)^{\frac{1}{2}}\right) dt$$

In view of $\left(x(t)^T B^i(t) z(t)\right) \leq \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}} \left(z(t)^T B^i(t) z(t)\right)^{\frac{1}{2}}$
and $\left(z(t)^T B^i(t) z(t)\right) \leq 1$, from this inequality we obtain,

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x},\ddot{x})\,dt + x(t)^T B^i(t) z^i(t)\right) dt$$

$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u},\ddot{u})\,dt + u(t)^T B^i(t) z^i(t)\right) dt$$

By pseudo invexity of $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + (\cdot)^T B^i(t) z^i(t)\right) dt$ with respect to η, this implies

$$0 > \sum_{i=1}^{p} \lambda^i \int_I \Big[\eta^T \left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t)\right)$$

$$+ (D\eta)^T \left(f_{\dot{u}}^i(t,u,\dot{u},\ddot{u})\right) + \left(D^2\eta\right)^T \left(f_{\ddot{u}}^i(t,u,\dot{u},\ddot{u})\right) \Big] dt$$

This, by integration by parts and using boundary conditions as earlier, yields

$$\sum_{i=1}^{p}\lambda^i \int_I \eta^T \left[\left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t)\right)\right.$$
$$\left. - D f_{\dot{u}}^i(t,u,\dot{u},\ddot{u}) + D^2 f_{\ddot{u}}^i(t,u,\dot{u},\ddot{u})\right] dt < 0 \tag{5.9}$$

Now, from the feasibility of (VP) and (M-WVD), we have

$$\int_I y(t)^T g(t,x,\dot{x},\ddot{x})\, dt \leqq \int_I y(t)^T g(t,u,\dot{u},\ddot{u})\, dt$$

which, because of the quasi-invexity of $\int_I y(t)^T g(t,.,.,.)\, dt$ with respect to η implies

$$\int_I \eta^T y(t)^T g_u(t,u,\dot{u},\ddot{u}) + (D\eta)^T y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u})$$
$$+ \left(D^2\eta\right)^T y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\, dt \leqq 0$$

This, as earlier, implies

$$\int_I \eta^T \left[y(t)^T g_u(t,u,\dot{u},\ddot{u}) - D y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + D^2 y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right] dt \leqq 0 \tag{5.10}$$

Combining (5.9) and (5.10), we have the inequality as

$$\int_I \eta^T \left[\sum_{i=1}^{p}\lambda^i \left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u})\right)\right.$$
$$- D\left(f_{\dot{u}}^i(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u})\right)$$
$$\left. + D^2\left(f_{\ddot{u}}^i(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)\right] dt < 0$$

which contradicts the dual equality constraint. Hence the theorem is validated.

Theorem 5.2 (Strong Duality). *Let $\bar{x}$ be an efficient solution of* (VP) *and for at least one $k \in P$, $\bar{x}$ satisfies the regularity condition* [3] *for the problem $(P_k(\bar{x}))$. Then there exist multipliers $\bar{\lambda} \in R^p$, piecewise smooth $\bar{y} \in R^m$ and $\bar{z}^i(t) \in R^n, i = \{1,2,\ldots,p\}$, such that $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\lambda\right)$ is feasible for* (M-WVD) *and the objectives of* (VP) *and* (M-WVD) *are equal.*

Further, if the generalized invexity of hypothesis of Theorem 5.1 is met, then $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{\lambda}\right)$ is an efficient solution of (M-WVD).

Proof. Since $\bar{x}$ is an solution of the problem $(P_k(\bar{x}))$, by analysis of Theorem 3.1, it implies that there exists $\bar{\lambda} \in R^p$, piecewise smooth $\bar{y} \in R^m$ and $\bar{z}^i(t) \in R^n, i = \{1,2,\ldots,p\}$ such that, (4.17), (4.18), (4.19), (4.20) and (4.14) holds:

From (4.18), it implies

$$\int_I \bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt = 0 \tag{5.11}$$

Now from (4.17), (5.11), (4.14) and (4.20) together with $\bar{\lambda} > 0$, it follows that $(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{\lambda})$ is feasible. From the equality of the objectives of (VP) and (M-WVD), along with the hypotheses of Theorem 5.1, the efficiency of $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{\lambda}\right)$ follows. This completes the proof.

(M-WVD) may be rewritten in the following form:

$$\text{Minimize}\left(-\int_I\left(f^1(t,x,\dot{x},\ddot{x})+u(t)^T B^1(t)z^1(t)\right)dt\right.$$
$$\left.,\ldots,\int_I-\left(f^p(t,x,\dot{x},\ddot{x})+u(t)^T B^p(t)z^p(t)\right)dt\right)$$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$\theta\left(t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},\lambda\right) = 0$$

$$\sum_{j=1}^m\int_I y^j(t)g^j(t,x,\dot{x},\ddot{x})dt \geqq 0, \quad t\in I$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1, \quad t\in I, \quad i\in P$$

$$\lambda > 0, y(t) \geqq 0, \quad t \in I$$

Theorem 5.3 (Converse Duality). *Let $(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda})$ be an efficient solution for (M-WDP). Assume that*

(A_1) *The Frèchèt derivative ψ' has a (weak*) closed range,*

(A_2) *f and g are twice continuously differentiable,*

(A_3) $f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)z^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\dot{x}}^i(t,x,\dot{x},\ddot{x}), \quad i \in \{1,2,\ldots,p\}$ *are linearly independent and*

(A_4) $\left(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2\beta(t)^T \theta_{\ddot{x}}\right)\beta(t) = 0, \Rightarrow \beta(t) = 0,\ t \in I$

Further, if the hypotheses of Theorem 5.1 *are met, then $\bar{x}$ is an efficient solution of* (VP)

Proof. Since $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda}\right)$ with ψ' having a (weak*) closed range, is an efficient solution of (M-WDP), then there exist $\alpha \in R^p, \eta \in R^p, \gamma \in R, \delta \in R, \xi \in R^m$ and piecewise smooth $\beta(t) : I \to R^n$ and $\mu(t) : I \to R^m$ such that the following Fritz-John optimality conditions [8] holds

$$-\sum_{i=1}^{p} \alpha^i \left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)z^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)$$

$$-\gamma\left(y(t)^T g_x(t,x,\dot{x},\ddot{x}) - Dy(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x}) + D^2 y(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right)$$

$$+\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2\beta(t)^T \theta_{\ddot{x}} - D^3\beta(t)^T \theta_{\dddot{x}} = 0, \quad t \in I \tag{5.12}$$

$$-\gamma g^j(t,x,\dot{x},\ddot{x}) + \beta(t)^T \theta_{y^j} - D\beta(t)^T \theta_{\dot{y}^j} + D^2\beta(t)^T \theta_{\ddot{y}^j} - \mu^j(t) = 0, \quad t \in I \tag{5.13}$$

for $j = 1, 2, \ldots, m$.

$$-\alpha^i x(t)^T B^i(t) + \lambda^i \beta(t)^T B^i(t) + 2\delta^i B^i(t) z^i(t) = 0 \tag{5.14}$$

$$\left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)z^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})\right.$$

$$\left.+D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\beta(t)\right] - \eta^i = 0, \quad i = 1,\ldots,p \tag{5.15}$$

$$\gamma \int_I y(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt = 0 \tag{5.16}$$

$$\eta^T \lambda = 0 \tag{5.17}$$

$$\mu^T(t)\bar{y}(t) = 0, \quad t \in I \tag{5.18}$$

$$\delta^i\left(z^i(t)^T B^i(t)z^i(t) - 1\right) = 0, \quad t \in I \tag{5.19}$$

$$\left(\alpha,\mu(t),\delta,\eta,\gamma\right) \geqq 0 \tag{5.20}$$

$$\left(\alpha,\beta(t),\mu(t),\delta,\eta,\gamma\right) \geqq 0 \tag{5.21}$$

Since $\lambda > 0$, (5.17) implies $\eta = 0$. Consequently (5.15) implies

$$\left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)z^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})\right.$$

$$\left.+D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)\beta(t) = 0, \quad i = 1,\ldots,p \tag{5.22}$$

Using the duality constraint of (M-WVD) in (5.12), we have

$$-\sum_{i=1}^p \left(\alpha^i - \gamma\lambda^i\right)\left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)z^i(t)\right.$$

$$\left.-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)$$

$$+\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}} = 0, \quad t \in I \tag{5.23}$$

$$= \quad -\sum_{i=1}^p \left(\alpha^i - \gamma\lambda^i\right)\left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)z^i(t)\right.$$

$$\left.-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)\beta(t)$$

$$+\left(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}}\right)\beta(t) = 0, \quad t\in I$$

This in conjunction with (5.22) yields

$$\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} = 0, \quad t\in I$$

which because of the hypothesis (A_4) implies

$$\beta(t) = 0, t\in I \tag{5.24}$$

Using (5.24) in (5.23), we have

$$-\sum_{i=1}^{p}\left(\alpha^i - \gamma\lambda^i\right)\left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)z^i(t)\right.$$
$$\left. - Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right) = 0$$

This, due to the hypothesis (A_3) gives,

$$\alpha^i - \gamma\lambda^i = 0, \quad i = 1,2,\dots,p \tag{5.25}$$

Suppose $\gamma = 0$, then from (5.25) we have $\alpha = 0$. The relation (5.13) gives $\mu(t) = 0$, $t\in I$.

As earlier, (5.14) implies $\delta = 0$. Hence we get $(\alpha,\beta(t),\mu(t),\eta,\gamma,\delta) = 0$, which contradicts (5.20). Hence $\gamma > 0$. Consequently, (5.25) implies $\alpha > 0$.

From (5.14) we have

$$g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right) \leqq 0$$

This implies the feasibility of $\bar{x}$ for (VP).

In view of the explanations given in the proof of Theorem 4.2, (5.14) together with (5.19) readily yields

$$\left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right) = \left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}, \quad i = 1,2,\dots,p$$

Hence,

$$\int_I \left(f^i\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt + \left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right)\right)dt$$

$$= \int_I \left(f^i\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt + \left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\right)dt, \quad i=1,2,\ldots,p$$

This, in view of the hypothesis of Theorem 5.1, implies that $\bar{x}$ is efficient solution of (VP).

6. Related Problems

It is possible to extend the duality theorems established in the previous two sections to the corresponding variational problems with natural boundary values rather than fixed end points.

$$(VP)_0: \quad \text{Minimize}\left(\int_I \left(f^1(t,x,\dot{x},\ddot{x})dt + \left(x(t)^T B^1(t)x(t)\right)^{\frac{1}{2}}\right)dt\right.$$
$$\left.,\ldots,\int_I \left(f^p(t,x,\dot{x},\ddot{x})dt + \left(x(t)^T B^p(t)x(t)\right)^{\frac{1}{2}}\right)dt\right)$$

Subject to

$$g^j(t,x,\dot{x},\dot{x}) \leqq 0, \quad t\in I,\ j=1,\ldots,m$$

$$(MWD)_0: \quad \text{Maximize}\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t)z^1(t) + y^j(t)g^j(t,u,\dot{u},\ddot{u})\right)dt\right.$$
$$\left.,\ldots,\int_I \left(f^p(t,u,\dot{u},\ddot{u}) + u(t)^T B^p(t)z^p(t) + y^j(t)g^j(t,u,\dot{u},\ddot{u})\right)dt\right)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t)z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u})\right)$$
$$-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u})\right)$$
$$+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) = 0, \quad t\in I$$

$$\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) = 0 \text{ at } t = a, t = b,$$
$$\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) = 0, \text{ at } t = a, t = b,$$
$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1, \quad t \in I, \ i \in P$$
$$y(t) \geqq 0, \quad t \in I$$
$$\lambda > 0, \ \lambda^T e = 1$$

$$(M\text{-}WVD)_0: \quad \text{Maximize} \left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t) \right) dt, \ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + u(t)^T B^p(t) z^p(t) \right) dt \right)$$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right)$$
$$-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) = 0, \quad t \in I$$
$$\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) = 0 = y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}), \text{ at } t = a, t = b,$$
$$\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) = 0 = y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}), \text{ at } t = a, t = b,$$
$$\sum_{j=1}^{m} \int_I y^j(t) g^j(t,u,\dot{u},\ddot{u}) dt \geqq 0, \quad t \in I$$
$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1, \quad t \in I, \ i \in P$$
$$\lambda > 0, y(t) \geqq 0, \quad t \in I$$

If the function in the problem (WD) and (M-WD) are independent of t, then these problems reduce to those treated by Mond, Husain and Prasad [14].

$$(VP)_1: \quad \text{Minimize} \left(f^1(x) + \left(x^T B^1 x \right)^{\frac{1}{2}}, \ldots, f^p(x) + \left(x^T B^p x \right)^{\frac{1}{2}} \right)$$

Subject to

$$g(x) \leqq 0$$

$(MWD)_1$: Maximize $\left(f^1(u) + u^T B^1 z^1 + y^T g(u), \ldots, f^p(u) + u^T B^p z^p + y^T g(u)\right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_x(u) + B^i z^i\right) + y^T g_x(u) = 0$$

$$\bar{z}^i B^i \bar{z}^i \leqq 1, \quad i \in P$$

$$y \geqq 0, \quad \lambda \in \Lambda^+$$

where $\Lambda^+ = \left\{\lambda \in R^p \,|\lambda > 0,\, \lambda^T e = 1, e = (1, 1, \ldots, 1)^T \in R^p\right\}$

$(M\text{-}WVD)_1$: Maximize $\left(f^1(u) + u^T B^1 z^1, \ldots, f^p(u) + u^T B^p z^p\right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_x(u) + B^i z^i\right) + y^T g_x(u) = 0$$

$$y^T g(u) \geqq 0, \quad t \in I$$

$$\bar{z}^i B^i \bar{z}^i \leqq 1, \quad i \in P$$

$$\lambda > 0, y(t) \geqq 0, \quad t \in I$$

7. Conclusion

We have considered Wolfe and Mond-Weir type vector dual variational problems for a class of nondifferentiable multiobjective variational problem involving higher order derivatives. Making use of the concepts of efficiency, we obtain weak, strong and converse duality theorems under assumptions of invexity and generalized invexity. We have also established close relationship between these problems with corresponding nonlinear programming problem. One can replace the square roots of quadratic form by support function of a compact convex set that is somewhat more general and for which the subdifferential may be expressed. It is difficult to exhibit any practical application to our model as they are inherently very involved. There is a rich scope to study this problem in multiobjective setting. One can also formulate a fractional analogue of our model to study various duality results.

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