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On Hausdorff Spaces Via Ideals and Semi-I-irresolute Functions

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Abstract. We introduce the notion of semi-I-Hausdorff spaces which is weaker than Hausdorff spaces and independent both I-Hausdorff and quasi-I-Hausdorff.

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1. Introduction

In [4], Dontchev has introduced and studied I-Hausdorff spaces. In [13], Nasef has improved I-Hausdorff spaces and defined quasi-I-Hausdorff spaces. In [5], the

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present authors defined the notion of semi-open sets via ideals to obtain decomposition of continuity. In the present paper, we introduce the notion of semi-I-Hausdorff spaces which is weaker than Hausdorff spaces and independent both I-Hausdorff and quasi-I-Hausdorff spaces. Using semi-I-irresolute [6] functions, we also investigate its relation with semi-I-Hausdorff spaces.

2. Preliminaries

Throughout this paper, (X,τ) (simply X) denotes a topological space on which no separation axiom is assumed unless explicitly stated. For a subset A of a topological space X, the closure and the interior of A in X are denoted by $Cl(A)$ and $Int(A)$, respectively. A nonempty collection I of subsets on a topological space (X,τ) is called a topological ideal on (X,τ) if it satisfies the following two conditions: *(1)* if $A \in I$ and $B \subset A$, then $B \in I$ (heredity); *(2)* if $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity). If I is a proper ideal, that is, $X \notin I$, then $\{A : X - A \in I\}$ is a filter, hence proper ideals are sometimes called dual filters. By (X,τ,I), we will denote an ideal topological space which means a topological space (X,τ) with an ideal I on X. No separation property is assumed on X. For a space (X,τ,I) and a subset A of X, $A^*(I) = \{x \in X : U \cap A \notin I$ for each neighborhood U of $x\}$ is called the local function of A with respect to I and τ [7]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. The simplest ideals are $\{\varnothing\}$ and $\wp(X)$ which satisfy $\{\varnothing\} \subset I \subset \wp(X)$, for any ideal I on X. Note that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(I)$ (also denoted by τ^* when there is no chance for confusion) finer than τ.

Definition 2.1. *A subset A of an ideal topological space (X,τ,I) is said to be semi-open [10] (resp. $\beta-open$ [1], semi-I-open [5], I-open [9], quasi-I-open [2]) if $A \subset Cl(Int(A))$ (resp. $A \subset Cl(Int(Cl(A)))$, $A \subset Cl^*(Int(A))$, $A \subset Int(A^*)$, $A \subset$*

$Cl(Int(A^*)))$.

For a subsets defined above, the following diagram holds:

DIAGRAM I

$$\begin{array}{ccccc} \text{open} & \longrightarrow & \text{semi-}I\text{-open} & \longrightarrow & \text{semi-open} \\ & & & & \downarrow \\ I-\text{open} & \longrightarrow & \text{quasi-}I\text{-open} & \longrightarrow & \beta-\text{open} \end{array}$$

Definition 2.2. *A space (X,τ) is said to be semi-Hausdorff [11] (resp. β−Hausdorff [12]) if for every two different points x, y of X, there exist disjoint semi-open (resp. $\beta - open$) sets U, V of X such that $x \in U$ and $y \in V$.*

Definition 2.3. *An ideal topological space (X,τ,I) is called I-Hausdorff [4] (resp. quasi-I-Hausdorff [13]) if for every two different points x, y of X, there exist disjoint I-open sets (resp. quasi-I-open) U, V of X such that $x \in U$ and $y \in V$.*

3. Semi-*I*-Hausdorff Spaces

Definition 3.1. *An ideal topological space (X,τ,I) is called semi-I-Hausdorff if for each two distinct points $x \neq y$, there exist semi-I-open sets U and V containig x and y, respectively such that $U \cap V = \varnothing$. Then the points x and y are said to be $semi - I - separated$.*

Theorem 3.1. *For an ideal topological space (X,τ,I), the following statements hold:*

1. Every Hausdorff space is semi-*I*-Hausdorff.

2. Every semi-*I*-Hausdorff space is semi-Hausdorff.

Proof. This follows from the definition of semi-I-open sets.

For ideal topological spaces, the following diagram holds:

DIAGRAM II

Hausdorff	$\longrightarrow$	semi-I-Hausdorff	$\longrightarrow$	semi-Hausdorff
				$\downarrow$
I-Hausdorff	$\longrightarrow$	quasi-I-Hausdorff	$\longrightarrow$	$\beta-$Hausdorff

Remark 3.1. *(1) It is shown in Example 2.3 and 2.4 of* [4] *that Hausdorffness and I-Hausdorffness are independent of each other.*

(2) In the following examples, it will be shown that semi-I-Hausdorffness is independent to quasi-I-Hausdorffness and to I-Hausdorffness.

Example 3.1. *Let X be the real line with the "rigth-ray" topology τ that is the nontrivial open sets are the form (x,∞), where x is any real number. Let I be the ideal of all finite subsets of X. Then the ideal topological space (X,τ,I) is an I-Hausdorff space which is not Hausdorff* [4, *Example* 2.3]. *However, this space is not even semi-Hausdorff because, every nonempty semi-open set has the nonempty interior.*

Example 3.2. *Let $X=\{a,b\}$, τ be the discrete topology on X and $I=\wp(X)$. Then Dontchev* [4] *showed that the space is Hausdorff, but it is not I-Hausdorff. Moreover, Nasef* [13] *showed that the space is not even quasi-I-Hausdorff.*

Example 3.3. *Let $X=\{a,b,c\}$, $\tau=\{\varnothing, X, \{a\},\{b\},\{a,b\}\}$ and $I=\{\varnothing\}$. Then (X,τ,I) is a semi-I-Hausdorff space which is not Hausdorff. If we take $I=\wp(X)$, then (X,τ,I) is semi-Hausdorff, but it is neither semi-I-Hausdorff nor quasi-I-Hausdorff.*

Theorem 3.2. *Let (X,τ,I) be an ideal topological space.*

1. Let $I=\{\varnothing\}$. Then (X,τ,I) is semi-I-Hausdorff (resp. quasi-I-Hausdorff) if and only if it is semi-Hausdorff (resp. $\beta-$*Hausdorff*).

2. Let $I=\wp(X)$. Then (X,τ,I) is Hausdorff if and only if it is semi-I-Hausdorff.

Proof. *(1)* Let $I=\{\varnothing\}$. Then $A^*=Cl(A)$ and $Cl^*(A)=Cl(A)$ for every subset A of X. Therefore, we have $SIO(X,\tau)=SO(X,\tau)$ (resp. $QIO(X,\tau)=\beta(X,\tau)$) and hence (X,τ,I) is semi-I-Hausdorff (resp. quasi-I-Hausdorff) if and only if semi-Hausdorff (resp. $\beta-Hausdorff$), where $QIO(X,\tau)$ denotes the set of all quasi-I-open sets.

(2) Let $I=\wp(X)$. Then $A^*=\varnothing$ and $Cl^*(A)=A$ for every subset A of X. Let $A\in SIO(X,\tau)$, then $A\subset Cl^*(Int(A))=Int(A)$ and hence A is open in (X,τ). Therefore, (X,τ,I) is Hausdorff if and only if it is semi-I-Hausdorff.

Definition 3.2. *An ideal topological space (X,τ,I) is called semi-I-complete (resp. quasi-I-complete [13]) if $\tau^*=SIO(X,\tau)$ (resp. $\tau^*=QIO(X,\tau)$), that is, a subset A of X is τ^*-open if and only if it is semi-I-open (resp. quasi-I-open).*

Theorem 3.3. *Let (X,τ,I_n) be an ideal topological space, where I_n is the ideal of the nowhere dense sets of (X,τ).*

1. (X,τ,I_n) is semi-I-Hausdorff (resp. quasi-I-Hausdorff) if and only if it is semi-Hausdorff (resp. $\beta-Hausdorff$).

2. (X,τ,I_n) is semi-Hausdorff and semi-I-complete (resp. $\beta-Hausdorff$ and quasi-I-complete), then it is Hausdorff.

Proof. *(1)* Since I_n is the ideal of nowhere dense sets of (X,τ), we have $A^*=Cl(Int(Cl(A)))$ and hence by Example 2.10 of [8]

$Cl^*(A)=A\cup Cl(Int(Cl(A)))=\alpha Cl(A)$, where $\alpha Cl(A)$ denotes the $\alpha-closure$ of A.For every subset A of X,

$Cl^*(Int(A))=Int(A)\cup Cl(Int(Cl(Int(A))))=Int(A)\cup Cl(Int(A))=Cl(Int(A))$.

Therefore, $A\in SIO(X,\tau)$ if and only if $A\in SO(X,\tau)$.By this fact, it follows that (X,τ,I_n) is semi-I-Hausdorff if and only if it is semi-Hausdorff. On the other hand,

$Cl(Int(A^*)) = Cl(Int(Cl(Int(Cl(A))))) = Cl(Int(Cl(A)))$ for every subset A of X. Therefore, $A \in QIO(X,\tau)$ if and only if $A \in \beta(X,\tau)$. It follows that (X,τ,I_n) is quasi-I-Hausdorff if and only if it is $\beta - Hausdorff$.

(2) Let (X,τ,I_n) be semi-Hausdorff and semi-I-complete. Then $A \in SIO(X,\tau)$ if and only if $A \in \tau^*$ if and only if A is $\alpha - open$. By the proof of *(1)*, $SO(X,\tau) = SIO(X,\tau)$ and hence (X,τ,I) is Hausdorff. The another result is shown similarly.

Lemma 3.1. *Let I and J be two ideals on a topological space (X,τ). If $I \subset J$, then the following properties hold:*

1. $Cl_J^*(A) \subset Cl_I^*(A)$ for each subset A of X,

2. $SIO(X,\tau,J) \subset SIO(X,\tau,I)$.

Proof. *(1)* If $I \subset J$, then $A^*(J) \subset A^*(I)$ and $Cl_J^*(A) = A \cup A^*(J) \subset A \cup A^*(I) = Cl_I^*(A)$ for each subset A of X.

(2) Let $A \in SIO(X,\tau,J)$. Then $A \subset Cl_J^*(Int(A)) \subset Cl_I^*(Int(A))$ and hence $A \in SIO(X,\tau,I)$.

Theorem 3.4. *Let I and J be two ideals on a topological space (X,τ) and $I \subset J$. If (X,τ,J) is semi-I-Hausdorff, then (X,τ,I) is semi-I-Hausdorff.*

Proof. This is an immediate consequence of Lemma 1

A semi-I-open subspace of semi-I-Hausdorff space need not be semi-I-Hausdorff as shown in the following example.

Example 3.4. *Let $X = \{a,b,c\}$, $\tau = \{\varnothing, X, \{a\},\{b\},\{a,b\}\}$ and $I = \{\varnothing\}$. Then (X,τ,I) is semi-I-Hausdorff. But, take $A = \{a,c\} \in SIO(X,\tau)$, then $(A,\tau_{|A},I_{|A})$ is not semi-I-Hausdorff.*

Lemma 3.2. *(Hatir and Noiri [5]) Let (X,τ,I) be an ideal topological space. If $U \in \tau$ and $V \in SIO(X,\tau)$, then $U \cap V \in SIO(U,\tau_{|U},I_{|U})$.*

Theorem 3.5. *Let (X,τ,I) be a semi-I-Hausdorff space and $A\subset X$. Then if A is open, then $(A,\tau_{|A},I_{|A})$ semi-I-Hausdorff.*

Proof. This follows from Lemma 2

4. Semi-*I*-irresolute Functions

In this section, we investigate some properties of semi-*I*-irresolute functions. First, we shall recall some definition of functions.

Definition 4.1. *A function $f:(X,\tau,I)\longrightarrow(Y,\sigma,J)$ is said to be*

1. Semi-*I*-continuous [5] if for every $V\in\sigma$, $f^{-1}(V)$ is semi-*I*-open set,

2. Semi-*I*-irresolute if for every $V\in SJO(Y,\sigma)$, $f^{-1}(V)\in SIO(X,\tau)$,

3. Irresolute [3] if for every $V\in SO(Y,\sigma)$, $f^{-1}(V)\in SO(X,\tau)$.

Remark 4.1. *In* [6], *the present authors called semi-I-irresolute functions I-irresolute. However, Dontchev* [4] *defined a function $f:(X,\tau,I)\longrightarrow(Y,\sigma,J)$ to be I-irresolute if $f^{-1}(V)$ is I-open in (X,τ,I) for every I-open in (Y,σ,J).*

Theorem 4.1. *For a function $f:(X,\tau,I)\longrightarrow(Y,\sigma,J)$, the following properties are equivalent;*

1. f is semi-*I*-irresolute,

2. The inverse image of each semi-*I*-closed set in (Y,σ,J) is semi-*I*-closed in (X,τ,I),

3. For each $x\in X$ and each $V\in SIO(Y,\sigma)$ containing $f(x)$, there exists $U\in SIO(X,\tau)$ containing x such that $f(U)\subset V$.

Proof. The proof is obvious from the fact that the arbitrary union of semi-*I*-open sets is semi-*I*-open [6, $Theorem$ 3.4].

Remark 4.2. *Irresolute functions are not in general semi-I-irresolute as shown by the following example.*

Example 4.1. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ *and*

$I = \{\emptyset, \{a\}\}$ and $J = \{\emptyset\}$. Then the identity function $f : (X, \tau, I) \longrightarrow (X, \sigma, J)$ is irresolute, but it is not semi-I-irresolute since $\{a, c\} \in SIO(X, \sigma, J)$ and $f^{-1}(\{a, c\}) = \{a, c\} \notin SIO(X, \tau, I)$.

Theorem 4.2. *Let* $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ *be a function, where I and J are ideals on Y, respectively. If* $I = J = \{\emptyset\}$ *or* I_n, *then semi-I-irresoluteness and irresoluteness are equivalent.*

Proof. This follows from the proofs of Theorems 2(1) and 3(1).

Theorem 4.3. *Let f be a semi-I-irresolute injection from a space* (X, τ, I) *into a space* (Y, σ, J). *If Y is semi-I-Hausdorff, then X is also semi-I-Hausdorff.*

Proof. Let $x, y \in X$ and $x \neq y$. Then $f(x) \neq f(y)$ thus $f(x)$ and $f(y)$ are semi-I-separated in Y by semi-I-open sets U and V, respectively. Since f is semi-I-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint semi-I-open sets containing x and y, respectively. This shows that X is semi-I-Hausdorff.

Theorem 4.4. *Let* (X, τ, I) *be an ideal topological space with the following property; if* $x \neq y$, *where* $x, y \in X$, *then there exist a Hausdorff space* (Y, σ) *and a semi-I-continuous function* $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ *such that* $f(x) \neq f(y)$. *Then X is semi-I-Hausdorff.*

Proof. The proof is straightforward.

Theorem 4.5. *Let* $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ *be a function and* $V \in \sigma$. *Then*

$f^{-1}(V^*) \subset (f^{-1}(V))^*$ implies $f^{-1}(Cl^*(V)) \subset Cl^*(f^{-1}(V))$.

Proof. $f^{-1}(Cl^*(V)) = f^{-1}(V \cup V^*) = f^{-1}(V) \cup f^{-1}(V^*)$

$$\subset f^{-1}(V) \cup (f^{-1}(V))^* = Cl^*(f^{-1}(V)).$$

Remark 4.3. *The converse of Theorem 10 is false as shown by the following example.*

Example 4.2. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ *and*

$\sigma = \{\emptyset, X, \{c\}, \{a, b\}\}$. Let us take $I = \wp(X)$ and $J = \{\emptyset, \{c\}\}$. The define the identity function $f : (X, \tau, I) \longrightarrow (X, \sigma, J)$. For the subset $\{a, b\} \in \sigma$, we have $(\{a, b\})^* = \{a, b\}$ and $(f^{-1}(\{a, b\}))^* = (\{a, b\})^* = \emptyset$ and thus

$f^{-1}(V^*) \subsetneq (f^{-1}(V))^*$ for $V = \{a, b\} \in \sigma$. But

$f^{-1}(Cl^*(V)) \subset Cl^*(f^{-1}(V))$ for every $V \in \sigma$.

Lemma 4.1. *(Hatir and Noiri [6]) Let A and B be subsets of an ideal topological space* (X, τ, I). *Then the following properties hold:*

1. $A \in SIO(X, \tau)$ if and only if there exists $U \in \tau$ such that $U \subset A \subset Cl^*(U)$,

2. If $A \in SIO(X, \tau)$ and $A \subset B \subset Cl^*(A)$, then $B \in SIO(X, \tau)$.

The following theorem slightly improve the Theorem 4.5 in [6] which states that if $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is semi-I-continuous and $f^{-1}(V^*) \subset (f^{-1}(V))^*$ for each $V \in \sigma$, then f is semi-I-irresolute.

Theorem 4.6. *If* $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ *is semi-I-continuous and*

$f^{-1}(Cl^*(V)) \subset Cl^*(f^{-1}(V))$ for each $V \in \sigma$, then f is semi-I-irresolute.

Proof. Let B be any semi-I-open set of (Y, σ, J). By Lemma 3, there exists $V \in \sigma$ such that $V \subset B \subset Cl^*(V)$. Therefore, we have

$f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(Cl^*(V)) \subset Cl^*(f^{-1}(V))$. Since f is semi-I-continuous and $V \in \sigma$, $f^{-1}(V) \in SIO(X, \tau)$ and hence by Lemma 3 , $f^{-1}(B)$ is semi-I-open in (X, τ, I). This shows that f is semi-I-irresolute.

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