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Sandwich Theorems for Some Analytic Functions Defined by Convolution

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Abstract. For certain analytic functions defined by convolution products, we obtain several applications of first order differential subordination and superordination, that generalize some previous results obtained by different authors.

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1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic functions in U, we say that f *is subordinate to* g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in U, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

Let $H(\mathrm{U})$ denote the class of analytic functions in U, and let $H[a,n]$ denote the subclass of the functions $f \in H(\mathrm{U})$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C},\ n \in \mathbb{N}).$$

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Supposing that h and g are two analytic functions in U, let

$$\varphi(r,s,t;z):\mathbb{C}^3\times \mathrm{U}\to\mathbb{C}.$$

If h and $\varphi(h(z),zh'(z),z^2h''(z);z)$ are univalent functions in U, and if h satisfies the second-order superordination

$$g(z)\prec\varphi\left(h(z),zh'(z),z^2h''(z);z\right), \tag{2}$$

a function $q\in H(\mathrm{U})$ is called a *subordinant of* (2), if $q(z)\prec h(z)$ for all the functions h satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q(z)\prec\widetilde{q}(z)$ for all of the subordinants q of (2), is said to be *the best subordinant*.

Recently, Miller and Mocanu [14] obtained sufficient conditions for the functions g, h and φ, such that the following implication holds:

$$g(z)\prec\varphi\left(h(z),zh'(z),z^2h''(z);z\right)\Rightarrow g(z)\prec h(z).$$

Using the results of [14], [4] investigated certain classes of first order differential superordinations, as well as superordination-preserving integral operators [5]. Ali et al. [1] used the results of [4] to obtain sufficient conditions for normalized analytic functions to satisfy

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U.

Very recently, Shanmugam et al. [21] obtained sufficient conditions for a normalized analytic function f to satisfy

$$q_1(z)\prec\frac{f(z)}{zf'(z)}\prec q_2(z)\quad\text{and}\quad q_1(z)\prec\frac{z^2f'(z)}{\left[f(z)\right]^2}\prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U, with $q_1(0)=q_2(0)=1$.

For the functions f given by (1), and $g\in\mathscr{A}$ given by $g(z)=z+\sum\limits_{k=2}^{\infty}b_kz^k$, the *Hadamard (or convolution) product* of f and g is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k,\ z\in\mathrm{U}.$$

In this paper we obtained several interesting subordination results for the function $\left(\dfrac{(f*g)(z)}{z}\right)^{\alpha}$, $\alpha\in\mathbb{C}^*$, that generalize some previous results obtained by different authors.

Remark 1. *(i) For different choices of the function g, the convolution product $f*g$ reduces to several interesting functions. For example, if*

$$g(z)=z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_{k-1}\cdot\ldots\cdot(\alpha_l)_{k-1}}{(\beta_1)_{k-1}\cdot\ldots\cdot(\beta_s)_{k-1}(1)_{k-1}}z^k,\ z\in\mathrm{U}, \tag{3}$$

where, $\alpha_i > 0$ $(i = 1,2,\dots l)$, $\beta_j > 0$ $(j = 1,2,\dots s)$, $l \le s+1$, $l,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1,2,\dots\}$, we see that $f * g = H_{l,s}(\alpha_1)f$, where $H_{l,s}(\alpha_1)$ is the Dziok-Srivastava operator, introduced and studied in [8] (see also [9], [10]).

The operator $H_{l,s}(\alpha_1)$, contains many interesting operators, such as Hohlov linear operator (see [11], [19]), the Bernardi-Libera-Livingston operator (see [12]), and Owa-Srivastava fractional derivative operator (see [17]).

(ii) Also, if

$$g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+l+\lambda(k-1)}{1+l} \right]^m z^k,\ z \in \mathrm{U}, \tag{4}$$

where $\lambda \ge 0$, $l \ge 0$, $m \in \mathbb{N}_0$, we see that $f * g = \mathrm{I}(m,\lambda,l)f$, where $\mathrm{I}(m,\lambda,l)$ is the generalized multiplier transformation introduced and studied by Cătaş et. al. [6].

The operator $\mathrm{I}(m,\lambda,l)$ contains, as special cases, the multiplier transformation (see [7]), the generalized Sălăgean operator introduced and studied by Al-Oboudi [2] (see also [20]).

2. Definitions and Preliminaries

To prove our results we shall need the following definition and lemmas.

Lemma 1. *[13] Let q be univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(\mathrm{U})$, with $\varphi(w) \neq 0$ when $w \in q(\mathrm{U})$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is a starlike function in U,*

(ii) $\mathrm{Re}\,\dfrac{zh'(z)}{Q(z)} > 0,\ z \in \mathrm{U}.$

If p is analytic in U, with $p(0) = q(0)$, $p(\mathrm{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{5}$$

then $p(z) \prec q(z)$, and q is the best dominant of (5).

Lemma 2. *[21] Let $\mu \in \mathbb{C}$, $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and let q be a convex function in U, with*

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\mu}{\gamma}\right) > 0,\ z \in \mathrm{U}.$$

If p is analytic in U and

$$\mu p(z) + \gamma z p'(z) \prec \mu q(z) + \gamma z q'(z), \tag{6}$$

then $p(z) \prec q(z)$, and q is the best dominant of (6).

Definition 1. *[14] Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial \mathrm{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \mathrm{U} \setminus E(f)$.

Lemma 3. *[5] Let q be univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(\mathrm{U})$. Suppose that*

(i) $\operatorname{Re} \dfrac{\theta'(q(z))}{\varphi(q(z))} > 0,\ z \in \mathrm{U},$

(ii) $h(z) = zq'(z)\varphi(q(z))$ *is starlike in* U.

If $p \in H[q(0), 1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset D$, the function $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{7}$$

then $q(z) \prec p(z)$, and q is the best subordinant of (7).

Lemma 4. *[18] The function $q(z) = (1 - z)^{-2ab}$ is univalent in U if and only if $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$.*

3. Main Results

Theorem 1. *Let q be convex in U, and let $\alpha, \eta \in \mathbb{C}^*$ such that*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\alpha}{\eta}\right) > 0,\ z \in \mathrm{U}. \tag{8}$$

Let $g \in \mathscr{A}$, and for all functions $f \in \mathscr{A}$ with $(f * g)(z) \neq 0$, $z \in \dot{\mathrm{U}} = \mathrm{U} \setminus \{0\}$, set

$$\chi_g(\alpha, \eta; f)(z) = (1 - \eta)\left(\frac{(f * g)(z)}{z}\right)^{\alpha} + \eta \frac{z(f * g)'(z)}{(f * g)(z)}\left(\frac{(f * g)(z)}{z}\right)^{\alpha}. \tag{9}$$

Then,

$$\chi_g(\alpha, \eta; f) \prec q(z) + \frac{\eta}{\alpha} zq'(z) \tag{10}$$

implies

$$\left(\frac{(f * g)(z)}{z}\right)^{\alpha} \prec q(z),$$

and q is the best dominant of (10). (All the powers are the principal ones)

Proof. If we define the function ψ by

$$\psi(z) = \left(\frac{(f*g)(z)}{z}\right)^{\alpha}, \ z \in \mathrm{U}, \tag{11}$$

then ψ is analytic in U and $\psi(0) = 1$. Therefore, by differentiating (11) logarithmically with respect to z, we have

$$\psi(z) + \frac{\eta}{\alpha} z\psi'(z) = (1-\eta)\left(\frac{(f*g)(z)}{z}\right)^{\alpha} + \eta \frac{z(f*g)'(z)}{(f*g)(z)}\left(\frac{(f*g)(z)}{z}\right)^{\alpha}.$$

From the assumption (10) and the above relation we deduce

$$\psi(z) + \frac{\eta}{\alpha} z\psi'(z) \prec q(z) + \frac{\eta}{\alpha} zq'(z),$$

hence, the assertion of our theorem follows by using Lemma 2 with $\mu = 1$ and $\gamma = \eta/\alpha$.

Taking $q(z) = (1+Az)/(1+Bz)$ $(-1 \le B < A \le 1)$ in Theorem 1, the condition (8) becomes

$$\mathrm{Re}\left(\frac{1-Bz}{1+Bz} + \frac{\eta}{\alpha}\right) > 0, \ z \in \mathrm{U}. \tag{12}$$

It is easy to check that the function $\phi(z) = (1-\zeta)/(1+\zeta)$, $|\zeta| < |B| \le 1$, is convex in U, and since $\phi(\overline{\zeta}) = \overline{\phi(\zeta)}$ for all $|\zeta| < |B|$, it follows that the image $\phi(\mathrm{U})$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\mathrm{Re}\,\frac{1-Bz}{1+Bz} : z \in \mathrm{U}\right\} = \frac{1-|B|}{1+|B|} \ge 0.$$

Then, the inequality (12) is equivalent to

$$\mathrm{Re}\,\frac{\alpha}{\eta} \ge \frac{|B|-1}{1+|B|}, \tag{13}$$

hence, we have the following corollary:

Corollary 1. *Let $-1 \le B < A \le 1$, let $\alpha, \eta \in \mathbb{C}^*$, and suppose that the condition (13) holds. Let $g \in \mathcal{A}$, and for all functions $f \in \mathcal{A}$ with $(f*g)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, suppose that*

$$\chi_g(\alpha, \eta; f) \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2}, \tag{14}$$

where $\chi_g(\alpha, \eta; f)$ is given by (9).

Then

$$\left(\frac{(f*g)(z)}{z}\right)^{\alpha} \prec \frac{1+Az}{1+Bz},$$

and $(1+Az)/(1+Bz)$ is the best dominant of (14). (All the powers are the principal ones)

Letting g be of the form (3), and using the identity [8]

$$z\left(H_{l,s}(\alpha_1)f(z)\right)' = \alpha_1 H_{l,s}(\alpha_1+1)f(z) - (\alpha_1-1)H_{l,s}(\alpha_1)f(z), \tag{15}$$

we obtain the next result:

Corollary 2. *Let q be convex in* U*, let $\alpha,\eta \in \mathbb{C}^*$, and suppose that q satisfies the condition (8). For all functions $f \in \mathscr{A}$ with $H_{l,s}(\alpha_1)f(z)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, set*

$$\chi_1(\alpha_1;\alpha,\eta;f)(z) = (1-\eta\alpha_1)\left(\frac{H_{l,s}(\alpha_1)f(z)}{z}\right)^{\alpha} + \eta\frac{\alpha_1 H_{l,s}(\alpha_1+1)f(z)}{H_{l,s}(\alpha_1)f(z)}\left(\frac{H_{l,s}(\alpha_1)f(z)}{z}\right)^{\alpha}. \tag{16}$$

Then,

$$\chi_1(\alpha_1;\alpha,\eta;f)(z) \prec q(z) + \frac{\eta}{\alpha}zq'(z), \tag{17}$$

implies

$$\left(\frac{H_{l,s}(\alpha_1)f(z)}{z}\right)^{\alpha} \prec q(z),$$

and q is the best dominant of (17). (All the powers are the principal ones)

Remark 2. *The Corollary 2 was also obtained by Murugusundaramoorthy and Magesh [15, Theorem 3.1].*

Letting g be of the form (4), and using the identity [6]

$$\lambda z\left(\mathrm{I}(m,\lambda,l)f(z)\right)' = (l+1)\,\mathrm{I}(m+1,\lambda,l)f(z) - (1+l-\lambda)\,\mathrm{I}(m,\lambda,l)f(z), \tag{18}$$

where $\lambda > 0$, $l \geq 0$, $m \in \mathbb{N}_0$, we deduce:

Corollary 3. *Let q be convex in* U*, let $\alpha,\eta \in \mathbb{C}^*$, and suppose that q satisfies the condition (8). For all functions $f \in \mathscr{A}$ with $\mathrm{I}(m,\lambda,l)f(z)(z) \neq 0$, $z \in \dot{\mathrm{U}}$ $\left(\lambda > 0,\ l \geq 0,\ m \in \mathbb{N}_0\right)$, set*

$$\chi_2(m,\lambda,l;\alpha,\eta;f)(z) = \left(1-\frac{\eta(l+1)}{\lambda}\right)\left(\frac{\mathrm{I}(m,\lambda,l)f(z)}{z}\right)^{\alpha} + \frac{\eta(l+1)}{\lambda}\frac{\mathrm{I}(m+1,\lambda,l)f(z)}{\mathrm{I}(m,\lambda,l)f(z)}\left(\frac{\mathrm{I}(m,\lambda,l)f(z)}{z}\right)^{\alpha}, \tag{19}$$

Then,

$$\chi_2(m,\lambda,l;\alpha,\eta;f)(z) \prec q(z) + \frac{\eta}{\alpha}zq'(z), \tag{20}$$

implies

$$\left(\frac{\mathrm{I}(m,\lambda,l)f(z)}{z}\right)^{\alpha} \prec q(z),$$

and q is the best dominant of (20). (All the powers are the principal ones)

Theorem 2. *Let* $\alpha, \gamma \in \mathbb{C}^*$*, and let* q *be univalent in* U*, with* $q(0) = 1$ *and* $q(z) \neq 0$ *for all* $z \in$ U*, such that* q *satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0,\ z \in \mathrm{U}. \tag{21}$$

Let $g \in \mathscr{A}$*, and for all functions* $f \in \mathscr{A}$ *with* $(f * g)(z) \neq 0$*,* $z \in \dot{\mathrm{U}}$*, suppose that*

$$1 + \gamma\alpha\left(\frac{z(f * g)'(z)}{(f * g)(z)} - 1\right) \prec 1 + \gamma\frac{zq'(z)}{q(z)}. \tag{22}$$

Then,

$$\left(\frac{(f * g)(z)}{z}\right)^{\alpha} \prec q(z),$$

and q *is the best dominant of (22). (The power is the principal one)*

Proof. If we define the function ϕ by

$$\phi(z) = \left(\frac{(f * g)(z)}{z}\right)^{\alpha}, \tag{23}$$

then ϕ is analytic in U and $\phi(0) = 1$. Differentiating (23) logarithmically with respect to z, we get

$$\frac{z\phi'(z)}{\phi(z)} = \alpha\left(\frac{z(f * g)'(z)}{(f * g)(z)} - 1\right).$$

Using the above relation in (22), we have

$$1 + \gamma\frac{z\phi'(z)}{\phi(z)} \prec 1 + \gamma\frac{zq'(z)}{q(z)}.$$

Setting $\theta(w) = 1$ and $\varphi(w) = \gamma/w$, then φ and θ are analytic in $\mathbb{C}^*$. A simple computation shows that

$$Q(z) = zq'(z)\varphi(q(z)) = \gamma\frac{zq'(z)}{q(z)},$$
$$h(z) = \theta(q(z)) + Q(z) = 1 + \gamma\frac{zq'(z)}{q(z)},$$

and it is easily to see that the conditions of Lemma 1 are satisfied whenever (21) holds. Then, by applying Lemma 1, our conclusion follows.

Putting $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$ in Theorem 2, it is easy to check that the condition (21) holds whenever $-1 \le B < A \le 1$, hence we obtain:

Corollary 4. *Let $-1 \leq B < A \leq 1$. Let $g \in \mathscr{A}$, and for all functions $f \in \mathscr{A}$ with $(f * g)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, suppose that*

$$1 + \alpha \left(\frac{z(f * g)'(z)}{(f * g)(z)} - 1 \right) \prec 1 + \frac{(A - B)z}{(1 + Az)(1 + Bz)}. \tag{24}$$

Then,

$$\left(\frac{(f * g)(z)}{z} \right)^{\alpha} \prec \frac{1 + Az}{1 + Bz},$$

and $(1 + Az)/(1 + Bz)$ is the best dominant of (24). (The power is the principal one)

Putting $q(z) = (1 + Bz)^{\alpha(A-B)/B}$ $(-1 \leq B < A \leq 1,\ B \neq 0)$ and $\gamma = 1$ in Theorem 2, and according to Lemma 4, we have the following result:

Corollary 5. *Let $-1 \leq B < A \leq 1$, with $B \neq 0$, such that*

$$\left| \frac{\alpha(A - B)}{B} - 1 \right| \leq 1 \quad \textit{or} \quad \left| \frac{\alpha(A - B)}{B} + 1 \right| \leq 1.$$

Let $g \in \mathscr{A}$, and for all functions $f \in \mathscr{A}$ with $(f * g)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, suppose that

$$1 + \alpha \left(\frac{z(f * g)'(z)}{(f * g)(z)} - 1 \right) \prec \frac{1 + [B + \alpha(A - B)]\, z}{1 + Bz}. \tag{25}$$

Then,

$$\left(\frac{(f * g)(z)}{z} \right)^{\alpha} \prec (1 + Bz)^{\alpha(A-B)/B},$$

and $(1 + Bz)^{\alpha(A-B)/B}$ is the best dominant of (25). (The power is the principal one)

Taking $\gamma = 1/ab$, $(a, b \in \mathbb{C}^*)$, $\alpha = a$ and $q(z) = (1 - z)^{-2ab}$ in Theorem 2 and combining this together with Lemma 4, we obtain the next corollary:

Corollary 6. *Let $a, b \in \mathbb{C}^*$ such that*

$$|2ab - 1| \leq 1 \quad \textit{or} \quad |2ab + 1| \leq 1.$$

Let $g \in \mathscr{A}$, and for all functions $f \in \mathscr{A}$ with $(f * g)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, suppose that

$$1 + \frac{1}{b} \left(\frac{z(f * g)'(z)}{(f * g)(z)} - 1 \right) \prec \frac{1 + z}{1 - z}. \tag{26}$$

Then,

$$\left(\frac{(f * g)(z)}{z} \right)^{a} \prec (1 - z)^{-2ab},$$

and $(1 - z)^{-2ab}$ is the best dominant of (26). (The power is the principal one)

Remark 3. *(i) Taking $g(z) = z/(1-z)$ in Corollary 6, we obtain the result of Obradović et al. [16, Theorem 1].*

(ii) For $g(z) = z/(1-z)$ and $a = 1$, Corollary 6 reduces to the recent result of Srivastava and Lashin [22, Theorem 3].

(iii) The special case of Corollary 6, when $g(z) = z/(1-z)$, $\gamma = e^{i\lambda}/(ab\cos\lambda)$ $(a, b \in \mathbb{C}^*,\ |\lambda| < \pi/2)$, and $q(z) = (1-z)^{-2ab\cos\lambda e^{-i\lambda}}$, is due to Aouf et al. [3, Theorem 1].

Theorem 3. *Let q be convex in U, and let $\alpha, \eta \in \mathbb{C}^*$ with*

$$\operatorname{Re}\frac{\alpha}{\eta} > 0. \tag{27}$$

Let $g \in \mathscr{A}$, and for all functions $f \in \mathscr{A}$ with $(f * g)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, suppose that $\left(\frac{(f*g)(z)}{z}\right)^{\alpha} \in H[q(0),1] \cap \mathscr{Q}$, and that $\chi_g(\alpha,\eta;f)$ is univalent in U, where $\chi_g(\alpha,\eta;f)$ is given by (9).

Then,

$$q(z) + \frac{\eta}{\alpha} z q'(z) \prec \chi_g(\alpha,\eta;f)(z), \tag{28}$$

implies

$$q(z) \prec \left(\frac{(f*g)(z)}{z}\right)^{\alpha},$$

and q is the best subordinant of (28). (All the powers are the principal ones)

Proof. If we let the function ψ be given by (11), a simple computation shows that

$$\psi(z) + \frac{\eta}{\alpha} z\psi'(z) = \chi_g(\alpha,\eta;f)(z).$$

Setting $\theta(w) = w$ and $\varphi(w) = \eta/\alpha$, then θ and φ are analytic in $\mathbb{C}$, and from (27) we have

$$\operatorname{Re}\frac{\theta'(q(z))}{\varphi(q(z))} = \operatorname{Re}\frac{\alpha}{\eta} > 0,\ z \in \mathrm{U}.$$

Since q is a convex function, it follows that $h(z) = zq'(z)\varphi(q(z)) = \left(\eta z q'(z)\right)/\alpha$ is starlike in U, and using Lemma 3 we obtain our result.

Letting g be of the form (3) in Theorem 3 and using the identity (15), we get the following result obtained by Murugusundaramoorthy and Magesh [15, Theorem 3.9]:

Corollary 7. *Let q be convex in U, and suppose that $\alpha, \eta \in \mathbb{C}^*$ satisfies the condition (27). For all functions $f \in \mathscr{A}$ with $H_{l,s}(\alpha_1)f(z)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, suppose that $\left(\frac{H_{l,s}(\alpha_1)f(z)(z)}{z}\right)^{\alpha} \in H[q(0),1] \cap \mathscr{Q}$, and that $\chi_1(\alpha_1;\alpha,\eta;f)$ is univalent in U, where $\chi_1(\alpha_1;\alpha,\eta;f)$ is given by (16).*

Then,

$$q(z) + \frac{\eta}{\alpha} z q'(z) \prec \chi_1(\alpha_1;\alpha,\eta;f)(z), \tag{29}$$

implies

$$q(z) \prec \left(\frac{H_{l,s}(\alpha_1)f(z)}{z}\right)^{\alpha},$$

and q is the best subordinant of (29). (All the powers are the principal ones)

Letting g be of the form (4) in Theorem 3 and using the identity (19), we have:

Corollary 8. *Let q be convex in* U*, and suppose that $\alpha, \eta \in \mathbb{C}^*$ satisfies the condition (27). For all functions $f \in \mathscr{A}$ with $\mathrm{I}(m,\lambda,l)f(z) \neq 0$, $z \in \dot{\mathrm{U}}$ $\left(\lambda > 0,\ l \geq 0,\ m \in \mathbb{N}_0\right)$, suppose that $\left(\frac{\mathrm{I}(m,\lambda,l)f(z)}{z}\right)^{\alpha} \in H[q(0),1] \cap \mathscr{Q}$, and that $\chi_2(m,\lambda,l;\alpha,\eta;f)$ is univalent in* U*, where $\chi_2(m,\lambda,l;\alpha,\eta;f)$ is given by (19).*

Then,

$$q(z) + \frac{\eta}{\alpha} z q'(z) \prec \chi_2(m,\lambda,l;\alpha,\eta;f)(z), \tag{30}$$

implies

$$q(z) \prec \left(\frac{\mathrm{I}(m,\lambda,l)f(z)}{z}\right)^{\alpha},$$

and q is the best subordinant of (30). (All the powers are the principal ones)

Combining Theorem 1 and Theorem 3, we deduce the following *sandwich theorem*:

Theorem 4. *Let q_1 and q_2 be convex functions in* U*. Suppose that $\alpha, \eta \in \mathbb{C}^*$ satisfies (27) and q_2 satisfies (8).*

Let $g \in \mathscr{A}$, and for all functions $f \in \mathscr{A}$ with $(f * g)(z) \neq 0$, $z \in \dot{\mathrm{U}}$, suppose that $\left(\frac{(f*g)(z)}{z}\right)^{\alpha} \in H[q(0),1] \cap \mathscr{Q}$, and that $\chi_g(\alpha,\eta;f)$ is univalent in U*, where $\chi_g(\alpha,\eta;f)$ is given by (9).*

Then,

$$q_1(z) + \frac{\eta}{\alpha} z q_1'(z) \prec \chi_g(\alpha,\eta;f)(z) \prec q_2(z) + \frac{\eta}{\alpha} z q_2'(z), \tag{31}$$

implies

$$q_1(z) \prec \left(\frac{(f*g)(z)}{z}\right)^{\alpha} \prec q_2(z),$$

and, moreover, q_1 and q_2 are respectively, the best subordinant and the best dominant of (31). (All the powers are the principal ones)

Remark 4. *Combining Corollary 2 and Corollary 7, we get the sandwich result obtained by Murugusundaramoorthy and Magesh [15, Theorem 3.10].*

From Corollary 3 and Corollary 8, we get the next *sandwich theorem*:

Theorem 5. *Let q_1 and q_2 be convex functions in* U. *Suppose that $\alpha, \eta \in \mathbb{C}^*$ satisfies (27) and q_2 satisfies (8). For all functions $f \in \mathscr{A}$ with $\mathrm{I}(m,\lambda,l)f(z) \neq 0$, $z \in \dot{\mathrm{U}}$ $\left(\lambda > 0,\ l \geq 0,\ m \in \mathbb{N}_0\right)$, suppose that $\left(\frac{\mathrm{I}(m,\lambda,l)f(z)}{z}\right)^{\alpha} \in H[q(0),1] \cap \mathscr{Q}$, and that $\chi_2(m,\lambda,l;\alpha,\eta;f)$ is univalent in* U*, where $\chi_2(m,\lambda,l;\alpha,\eta;f)$ is given by (19).*

Then,

$$q_1(z) + \frac{\eta}{\alpha} z q_1'(z) \prec \chi_2(m,\lambda,l;\alpha,\eta;f)(z) \prec q_2(z) + \frac{\eta}{\alpha} z q_2'(z), \tag{32}$$

implies

$$q_1(z) \prec \left(\frac{\mathrm{I}(m,\lambda,l)f(z)}{z}\right)^{\alpha} \prec q_2(z),$$

and, moreover, q_1 and q_2 are respectively, the best subordinant and the best dominant of (32). (All the powers are the principal ones)

References

[1] R. M. Ali, V. Ravichandran and K. G. Subramanian, *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci., 15(2004), no. 1, 87–94.

[2] F. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Internat. J. Math. Math. Sci., 27(2004), 1429–1436.

[3] M. K. Aouf, F. M. Al-Oboudi and M. M. Haidan, *On some results for λ-spirallike and λ-Robertson functions of complex order*, Publ. Institute Math. Belgrade, 77(2005), no. 91, 93–98.

[4] T. Bulboacă, *A class of superordination-preserving integral operators*, Indag. Math. (N. S.), 13(2002), no. 3, 301–311.

[5] T. Bulboacă, *Classes of first order differential superordinations*, Demonstratio Math. 35(2002), no. 2, 287–292.

[6] A. Cătaş, G. I. Oros and G. Oros, *Differential subordinations associated with multiplier transformations*, Abstract Appl. Anal., 2008 (2008), ID 845724, 1–11.

[7] N. E. Cho and T. G. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., 40(2003), no. 3, 399–410.

[8] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., 103(1999), 1–13.

[9] J. Dziok and H. M. Srivastava, *Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function*, Adv. Stud. Contemp. Math., 5(2002), 115–125.

[10] J. Dziok and H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct., 14(2003), 7–18.

[11] Yu. E. Hohlov, *Operators and operations in the univalent functions*, Izv. Vysh. Ucebn. Zaved. Mat., 10(1978), 83–89 (in Russian).

[12] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., 16(1965), 755–658.

[13] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Michig. Math. J., 32(1985), 185–195

[14] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, 48(2003), no. 10, 815–826.

[15] G. Murugusundaramoorthy and N. Magesh, *Differential subordinations and superordinations for analytic functions defined by the Dziok-Srivastava linear operator*, J. Inequal. Pure Appl. Math., 7(4)(2006), Art. 152, 1–9.

[16] M. Obradović, M. K. Aouf and S. Owa, *On some results for starlike functions of complex order*, Publ. Institute Math. Belgrade, 46(60)(1989), 79–85.

[17] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., 39(1987), 1057–1077.

[18] W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J., 12(1965), 385–387.

[19] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49(1975), 109–115.

[20] G. S. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math. (Springer-Verlag) 1013, (1983), 362–372.

[21] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, *Differantial sandwich theorems for some subclasses of analytic functions*, J. Austr. Math. Anal. Appl., 3(1)(2006), Art. 8, 1–11.

[22] H. M. Srivastava and A. Y. Lashin, *Some applications of the Briot-Bouquet differential subordination*, J. Inequal. Pure Appl. Math., 6(2)(2005), Art. 41, 1–7.