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On the Partial Differential Equations with Non-Constant Coefficients and Convolution Method

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Abstract. In this study we consider the linear second order partial differential equations with non-homogenous forcing term and having singular variable data. In the special case we solve the one dimensional wave equation by using the double integral transform.

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1. Introduction

A number of problems in engineering give rise to the following well-known partial differential equations: wave equation, one dimensional heat flow, two dimensional heat flow, which in steady state becomes the two dimensional Laplace's equation, Poisson's equation, these equation arises in electrostatics and elasticity theory and transmission line equation.

In a long electrical cable or a telephone wire both the current and voltage depend upon position along the wire as well as the time. By using basic laws of electrical circuit theory, that the electrical current $i(x,t)$ satisfies the PDE

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + GL)\frac{\partial i}{\partial t} + RGi. \tag{1}$$

where the constant R, L, C and G are the resistance, inductance, capacitance and leakage conductance and the distance measured along the length of the cable represented by x. The voltage also satisfies equation (1). Several special cases of (1) arise in particular situations,

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for example, for a submarine cable G is negligible and frequencies are low so inductive effects can also be neglected, so that one may place

$$G = L = 0.$$

In this case equation (1) becomes

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}, \tag{2}$$

which is called the submarine equation or telegraph equation, then we see that equation (2) satisfies the one dimensional heat equation. For high frequency alternating current, again with negligible leakage, then equation (1) can be approximated by

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}, \tag{3}$$

which is called the high frequency line equation, also this equation satisfy the one dimensional wave equation. Another application of wave equation, wave propagation under moving load is considered in the present study. In [1], James discussed the basic equation of moving load in wave equation with a non-homogenous forcing term,

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t)\,, \quad t > 0, \qquad 0 \le x \le l \\ u(x,0) &= 0, \qquad u_t(x,0) = 0 \\ u(0,t) &= 0, \qquad u(l,t) = 0, \end{aligned} \tag{4}$$

by using single Laplace transform. In the next we will generalize the James's setting and will solve by using double Laplace transform(DLT) with moving data as follows

$$\begin{aligned} u_{tt} - u_{xx} &= f(x,t)\,, \quad t > 0, \quad x > 0 \\ u(0,t) &= g_1(t),\ u(x,0) = h_1(x) \\ u_x(0,t) &= g_1'(t), \quad u_t(x,0) = h_1'(x). \end{aligned} \tag{5}$$

where $c = 1$.

Now let us slightly modify and consider high frequency line equation as follow

$$\frac{\partial^2 i}{\partial t^2} - \frac{\partial^2 i}{\partial x^2} = H(x) \otimes H(t), \quad t,\ x > 0, \tag{6}$$

where the symbol $\otimes$ represents tensor product and H heaviside function, under the initial conditions

$$\begin{aligned} i(0,t) &= \delta(t), \quad i(x,0) = \delta(x) \\ i_x(0,t) &= \delta'(t), \ \ i_t(x,0) = \delta'(x). \end{aligned} \tag{7}$$

Here we consider i inductance and capacitance to be each equal to unity and δ the Dirac delta function. We solve this equation, by using DLT technique as follow: Taking the DLT of the equation (6), we get

$$s^2 I(s,p) - sI(p,0) - \frac{\partial}{\partial t} I(p,0) - (p^2 I(s,p) - pI(0,s) - \frac{\partial}{\partial x} I(0,s)) = \frac{1}{sp}, \tag{8}$$

and on taking single Laplace transform for initial data i.e. (7), we get

$$I(0,s) = 1,\ I(p,0) = 1,\ \ I_t(p,0) = p \text{ and } I_x(0,s) = s. \tag{9}$$

Substituting (5) into (8), we obtain

$$I(p,s) = \frac{1}{sp(s^2 - p^2)}. \tag{10}$$

Now, on using double inverse Laplace transform for both sides of equation (10) we obtain the solution which is known as current in the form of

$$i(x,t) = \frac{1}{2} t^2.$$

Now, let us reconsider the equation (5), and multiply the left hand side of this equation by the polynomial $\Psi(x,t)$ in order to have non-constant coefficients on applying double convolutions, we get

$$\Psi(x,t) ** \left[u_{tt} - u_{xx}\right] = f(x,t) \tag{11}$$

under the same initial conditions. Now by taking double Laplace transform for the equation (11) and single Laplace transform for initial conditions, the solution of equation (11) given by

$$u(x,t) = L_s^{-1} L_p^{-1} \left[\begin{array}{c} \dfrac{sH_1(p)}{(s^2 - p^2)} - \dfrac{pG_1(s)}{(s^2 - p^2)} + \dfrac{pH_1(p) - H_1(0)}{(s^2 - p^2)} \\ - \dfrac{sG_1(s) - G_1(0)}{(s^2 - p^2)} + \dfrac{F(p,s)}{(s^2 - p^2)\, \Psi(p,s)} \end{array} \right] \tag{12}$$

provided the inverse double Laplace transform exist. In [2] and [3] the authors consider the convolution terms are polynomials.

In the next we compare two solutions of the equations (5) and (11). Now consider the equations (5) and (11) have solutions $F(x,t)$ and $K(x,t)$ respectively, then we can easily check whether $F(x,t) ** K(x,t)$ is a solution for a similar type of wave equation.

Since $F(x,t)$ and $K(x,t)$ are solutions, by substitution we obtain

$$(F(x,t) ** K(x,t))_{tt} - (F(x,t) ** K(x,t))_{xx} \stackrel{?}{=} f(x,t). \tag{13}$$

on using the definition of partial derivative with convolution, we have

$$F_{tt}(x,t) ** K(x,t) - F_{xx}(x,t) ** K(x,t) = F(x,t) ** K_{tt}(x,t) - F(x,t) ** K_{xx}(x,t)$$

then the equation (13) can be written in the form

$$F(x,t) ** \left[K_{tt}(x,t) - K_{xx}(x,t)\right] \stackrel{?}{=} f(x,t) \tag{14}$$

or

$$\left[F_{tt}(x,t) - F_{xx}(x,t)\right] ** K(x,t) \stackrel{?}{=} f(x,t) \tag{15}$$

by substituting we obtain

$$F(x,t) ** \frac{1}{i!j!} f(x,t) \neq f(x,t) \tag{16}$$

and

$$f(x,t) ** K(x,t) \neq f(x,t) \tag{17}$$

thus the convolution $F(x,t) ** K(x,t)$ is not a solution for equations (5) and (11). But it is a solution for another type of equation as in the following theorem.

Theorem 1. *Let $F(x,t)$ be a solution of*

$$u_{tt} - u_{xx} = f(x,t)$$

and similarly $K(x,t)$ be a solution for

$$p(x,t) ** \left(u_{tt} - u_{xx}\right) = f(x,t) \qquad (x,t) \in \mathbb{R}^2_+$$

under the initial conditions

$$\begin{aligned} u(0,t) &= g_1(t),\ u(x,0) = h_1(x) \\ u_x(0,t) &= g_1'(t),\quad u_t(x,0) = h_1'(x) \end{aligned}$$

then $F(x,t) ** K(x,t)$ is a solution for the following type of equation

$$u_{tt}(x,t) - u_{xx}(x,t) - \theta(x,t) = f(x,t) \quad (x,t) \in \mathbb{R}^2_+ \tag{18}$$

where the initial conditions as above and $p(x,t)$ is a polynomial.

Proof. Since $F(x,t)$ is a solution of equation (5) then

$$F_{tt}(x,t) - F_{xx}(x,t) = f(x,t) \tag{19}$$

true and $K(x,t)$ is a solution of equation (11) rendering

$$K_{tt}(x,t) - K_{xx}(x,t) = \frac{1}{i!j!} f(x,t). \tag{20}$$

to be true. Now it follows

$$(F(x,t) ** K(x,t))_{tt} - (F(x,t) ** K(x,t))_{xx} - \theta(x,t) = f(x,t). \tag{21}$$

On using partial derivative of convolution, we have

$$F_{tt}(x,t)**K(x,t)-F_{xx}(x,t)**K(x,t)=F(x,t)**K_{tt}(x,t)-F(x,t)**K_{xx}(x,t) \quad (22)$$

then the equation (21) can be written in the form

$$\left[F_{tt}(x,t)-F_{xx}(x,t)\right]**K(x,t)-\theta(x,t)=f(x,t). \quad (23)$$

By substituting equation (19) in (23) we have

$$f(x,t)**K(x,t)-\theta(x,t)=f(x,t). \quad (24)$$

Thus we see that the convolution $F(x,t)**K(x,t)$ is a solution of equation (18).

We can also apply the same method to solve non-homogenous one dimensional heat equation with non-constant coefficient as well as for Laplace's equation in two dimensions.

In the next we consider the one dimensional wave equation in the form

$$\begin{aligned} u_{tt}-u_{xx} &= e^{x+t} \qquad (t,x)\in\mathbb{R}_+^2 && (25)\\ u(x,0) &= xe^x, \;\; u_t(x,0)=xe^x+e^x && (26)\\ u(0,t) &= te^t, \;\; u_x(0,t)=te^t+e^t && (27)\end{aligned}$$

by taking double Laplace transform for equation (25) and single Laplace transform for equations (26) and (27), we obtain

$$\begin{aligned} U(p,s) &= \frac{s}{(p-1)^2\left(s^2-p^2\right)}+\frac{p}{(p-1)^2\left(s^2-p^2\right)}-\frac{p}{(s-1)^2\left(s^2-p^2\right)}\\ &\quad -\frac{s}{(s-1)^2\left(s^2-p^2\right)}+\frac{1}{(p-1)(s-1)\left(s^2-p^2\right)}. \end{aligned} \quad (28)$$

On using the double inverse Laplace transform for equation (28), we obtain the solution of equation (25) as follows

$$u(x,t)=\frac{3}{2}e^{t+x}t+e^{t+x}x-\frac{1}{4}e^{t+x}+\frac{1}{4}e^{-t+x}. \quad (29)$$

We note that in the literature there is no systematic way to generate a partial differential equation with variable coefficients from the PDE with constant coefficients, however the most of the partial differential equations with variable coefficients depend on nature of particular problems, see [2] and [3]. In the next we use the convolution technique to generate a PDE with variable coefficients by using the equation (25) and compare the solution with the solution of the equation (25).

Now, if we consider to multiply the left hand side of equation (25) by non-constant coefficient term x^3t^2 and using the double convolution with respect to x and t respectively, then we have the following equation

$$x^3t^2**\left(u_{tt}-u_{xx}\right) = e^{x+t} \qquad (t,x)\in\mathbb{R}_+^2 \quad (30)$$

$$u(x,0) = xe^x, \ \ u_t(x,0) = xe^x + e^x \tag{31}$$
$$u(0,t) = te^t, \ \ u_x(0,t) = te^t + e^t. \tag{32}$$

By using the same technique we obtain the solution of equation (30) as

$$v(t,x) = \frac{5}{48}e^{t+x} + e^{t+x}x - \frac{1}{48}e^{-t+x} + \frac{25}{24}e^{t+x}t. \tag{33}$$

Now when we compare the equations (29) and (33) we see the relationship between these two solutions as in the following form

$$(x^3t^2) ** (v_{tt} - v_{xx}) = (u_{tt} - u_{xx}) + \theta(x,t).$$

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