EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 3, No. 4, 2010, 737-747
ISSN 1307-5543 – www.ejpam.com

Existence, Uniqueness of solutions for Set Differential Equations involving causal Operators with Memory

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Abstract. In this paper, we obtain existence and uniqueness results of IVP for set differential equations involving causal operators with memory. This paper is 2nd in sequel. In the first one we obtained inequality results and existence results for set causal operators involving memory.

2000 Mathematics Subject Classifications: 34A12

Key Words and Phrases: Set differential equations, Existence, Uniqueness and Comparison theorems and Extremal solutions.

1. Introduction

Owing to the generalization encompassed in both set differential equations and the causal operators, the study of set differential equations involving causal operators with memory has been initiated in [6]. The basic differential inequality and existence results have been developed for both set causal operators with memory and set differential equations involving causal operators with memory.

The Set differential equations [4] have certain advantages that dictate the continued interest in them. They are useful to study multivalued differential inclusions or multivalued differential equations. Moreover, they include the theory of ordinary differential equations and ordinary differential systems as special cases. This yields the theory of ordinary differential equations and that of systems in a semilinear metric space instead of a linear metric space which is an additional benefit.

A causal operator [1,3] or a non anticipative operator is a term adopted from engineering literature. The study of causal or Volterra operators envelopes the study of several dynamic systems such as ordinary differential equations [2], delay differential equations[2], integro

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differential equations[5] and integral equations to name a few.

In this paper we continue to combine these two areas and study set differential equations involving causal operators with memory. This will provide a unified treatment of the basic theory of set differential equations (SDE's), SDE's with delay and set integro differential equations which in turn include ordinary dynamic systems of the corresponding type.

2. Preliminaries

We begin with the definitions of $K_c(\mathbb{R}^n)$, the semilinear space in which we work. We next define the Hausdorff Metric, the Hukuhara difference, the Hukuhara derivative and the Hukuhara Integral. We also state all the important properties that are useful in this paper. We further define a partial order in $K_c(\mathbb{R}^n)$. We also state all the required results developed in [6] that will be used in this paper.
Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Define the Hausdorff metric by

$$D[A,B] = \max[\sup_{x\in B} d(x,A),\ \sup_{y\in A} d(y,B)], \tag{1}$$

where $d(x,A) = \inf[d(x,y) : y \in A]$, A,B are bounded sets in $\mathbb{R}^n$. We note that $K_c(\mathbb{R}^n)$ with this metric is a complete metric space.

It is known that if the space $K_c(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and non-negative scalar multiplication, then $K_c(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space.

The Hausdorff metric (1) satisfies the following properties:

$$D[A+C,B+C] = D[A,B] \text{ and } D[A,B] = D[B,A], \tag{2}$$

$$D[\lambda A,\lambda B] = \lambda D[A,B], \tag{3}$$

$$D[A,B] \le D[A,C] + D[C,B], \tag{4}$$

for all $A,B,C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$.

Let $A,B \in K_c(\mathbb{R}^n)$. The set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F : I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$\lim_{h\to 0+} \frac{F(t_0+h) - F(t_0)}{h} \text{ and } \lim_{h\to 0+} \frac{F(t_0) - F(t_0-h)}{h}$$

exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in $\mathbb{R}$.

With these preliminaries, we consider the set differential equation

$$D_H U = F(t,U),\ \ U(t_0) = U_0 \in K_c(\mathbb{R}^n),\ \ t_0 \ge 0, \tag{5}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$.

The mapping $U \in C^1[J, K_c(\mathbb{R}^n)], J = [t_0, t_0 + a]$ is said to be a solution of (5) on J if it satisfies (5) on J.
Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s) ds, \ \ t \in J. \tag{6}$$

Hence, we can associate with the IVP (5) the Hukuhara integral

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s)) ds, \ \ t \in J. \tag{7}$$

where the integral is the Hukuhara integral which is defined as,

$$intF(s)ds = \{\int f(s)ds : f \text{ is any continuous selector of } F\}$$

Observe that $U(t)$ is a solution of (5) on J iff it satisfies (7) on J.

We now define a partial order in the metric space $K_c(\mathbb{R}^n)$. To do so, we need the definition of a cone in $K_c(\mathbb{R}^n)$, which is given below.

Let $K(K^0)$ be the subfamily of $K_c(\mathbb{R}^n)$ consisting of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a nonnegative (positive) vector of n-components satisfying $u_i \geq 0 (u_i > 0)$ for $i = 1, 2, 3, \ldots, n$. Then K is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K.

Definition 1. *For any U and $V \in K_c(\mathbb{R}^n)$, if there exists a $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and $U = V + Z$ then we say that $U \geq V$ $(U > V)$. Similarly we can define $U \leq V$ $(U < V)$.*

To define the causal operator we introduce the following notation.
Let $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ and $E_0 = C[[t_0 - h_1, T], K_c(\mathbb{R}^n)]$, where $U \in E_0$ implies $U(t) = \Phi_0(t), t_0 - h_1 \leq t \leq t_0$ and $U(t)$ is any arbitrarily continuous function on $[t_0, T]$.

We define a norm on E as follows: for $U, V \in E$

$$D_0[U, V] = Sup_{t_0 \leq t \leq T} D[U(t), V(t)]$$

where D denotes the Hausdorff Metric.

Definition 2. *By a causal operator or a Volterra operator or a nonanticipative operator we mean a mapping $Q : E \to E$ satisfying the property that if $U(s) = V(s), t_0 \leq s \leq t < T$ then $(QU)(s) = (QV)(s), t_0 \leq s \leq t < T$.*
By a causal operator with memory we mean a mapping $Q : E_0 \to E$ such that for $U(s) = V(s), t_0 \leq s \leq t < T$,

$$Q(U, \Phi_0)(s) = Q(V, \Phi_0)(s), t_0 \leq s \leq t < T \text{ and } \Phi_0 \in C_1 = C[[t_0 - h_1, t_0], \ K_c(\mathbb{R}^n)].$$

We now state the following theorems from [6] which are needed to prove results in the next sections. Before proceeding further, we set

$$D_0[U,V] = \sup_{t_0 \le s \le T} D[U(s),V(s)]$$

Theorem 1. *Assume that*

(i) Q *is nondecreasing in* U *for each* $t \in I = [t_0, T]$.

(ii) $D_H V(t) \le (QV)(t)$
$D_H W(t) \ge (QW)(t)$
where $V, W \in C^1[I, K_c(\mathbb{R}^n)]$ *and*

(iii) $V(t_0) < W(t_0)$.

Then $V(t) < W(t), t \in I$, *provided one of the above differential inequalities is strict.*

Theorem 2. *Assume that* $Q(U,\Phi_0) \in C[B,E]$ *is continuous and compact, where* $B \subseteq E_0$ *and*

$$B = \{U \in K_c(\mathbb{R}^n) : D_0[U, \Phi_0(t_0)] \le b \text{ and } D_0[U_{t_0}, \Phi_0] = 0, t \in I\}$$

Then there exists a solution of the IVP

$$D_H U = Q(U,\Phi_0)(t),$$
$$U_{t_0} = \Phi_0 \in C_1,$$

on some interval $[t_0, t_0+\delta]$, *where* $t_0 + \delta < T$, *and* $C_1 = C[t_0 - h_1, t_0], K_c(\mathbb{R}^n)]$.

3. Existence and Uniqueness Results

Consider the IVP for set differential equations involving causal operator with memory given by

$$D_H U(t) = Q(U,\Phi_0)(t) \tag{8}$$

$$U_{t_0} = \Phi_0 \in C_1 \tag{9}$$

with $Q(U,\Phi_0) : E_0 \to E$ is a causal or a Volterra Operator with memory. Let

$$B = \{U \in E_0 : D_0[U, \Phi_0(t_0)] < b \text{ and } D_0[U_{t_0}, \Phi_0] = 0, t \in I\}$$

Now the IVP (8) and (9) is equivalent to the set Hukuhara integral equation.

$$U(t) = \Phi_0(t_0) + \int_{t_0}^{t} Q(U,\Phi_0)(s)ds, \tag{10}$$

and

$$U_{t_0} = \Phi_0 \text{ on } [t_0 - h_1, t_0]. \tag{11}$$

We now prove below an existence and uniqueness result for the IVP (8) and (9), when Q satisfies a Lipschitz condition. We apply contraction mapping theorem to reach our goal.

Theorem 3. *Suppose that Q is such that*

$$D[Q(U,\Phi_0)(t),Q(V,\Phi_0)(t)] \leq LD[U(t),V(t)],$$

for $U,V \in \Omega$, $L > 0, t \in I$ *where*

$$\Omega = \{U,V \in E_0 : \max_{s\in[t_0-h_1,t]} D[U(s),V(s)] = D[U(t),V(t)]t \in I\}$$

Then there exists a unique solution $U(t)$ *of the IVP(8) and (9) provided* $T - t_0 < \frac{1}{L}$.

Proof. *Define*

$$D_0[U,V](t) = \max_{s\in[t_0-h_1,t]} D[U(s),V(s)]$$

For any $U \in E_0$, *define the Hukuhara integral operator* T *on* I *by*

$$(TU)(t) = \Phi_0(t_0) + \int_{t_0}^{t} Q(U,\Phi_0)(s)ds \tag{12}$$

$$U_{t_0} = \Phi_0 \in C_1 \text{ on } [t_0 - h_1, t_0] \tag{13}$$

Now for $U,V \in \Omega$, *using the properties of Hausdorff metric and hypothesis of the theorem,*

$$\begin{aligned}
D[TU(t),TV(t)] &= D[\Phi_0(t_0) + \int_{t_0}^{t} Q(U,\Phi_0)(s)ds, \Phi_0(t_0) + \int_{t_0}^{t} Q(V,\Phi_0)(s)ds] \\
&= D[\int_{t_0}^{t} Q(U,\Phi_0)(s)ds, \int_{t_0}^{t} Q(V,\Phi_0)(s)ds] \\
&\leq \int_{t_0}^{t} D[Q(U,\Phi_0)(s),Q(V,\Phi_0)(s)]ds \\
&\leq \int_{t_0}^{t} Lmax_{t_0\leq s\leq t} D[U(s),V(s)]ds \\
&= L\int_{t_0}^{t} D[U(t),V(t)]ds \\
&\leq LD_0[U,V](t-t_0) \\
&\leq L(T-t_0)D_0[U,V]
\end{aligned}$$

which is a contraction, when $L(T - t_0) < 1$ *or* $(T - t_0) < \frac{1}{L}$.
Thus, since $(T-t_0) < \frac{1}{L}$, T *is contraction from* E *to* E *and hence by contraction mapping theorem there exists a* $U \in E$ *such that* $TU = U$, $U_{t_0} = \Phi_0$ *we get that* U *is unique solution for the IVP (8) and (9) whenever* $(T - t_0) < \frac{1}{L}$. *Hence the proof is complete.*

Remark 1. *The restriction $(T - t_0) < \frac{1}{L}$ can be avoided by using a weighted norm. We define*

$$D_0[U,V] = max_{s\in[t_0,T]} D[U(s),V(s)]e^{-\lambda t}.$$

where λ is to be chosen suitably.
Now using the relation (12), and using the properties of Hausdorff metric we arrive at,

$$\begin{aligned}
D[(TU)(t),(TV)(t)] &= D[\Phi_0(t_0) + \int_{t_0}^{t} Q(U,\Phi_0)(s)ds, \Phi_0(t_0) + \int_{t_0}^{t} Q(V,\Phi_0)(s)ds] \\
&= D[\int_{t_0}^{t} Q(U,\Phi_0)(s)ds, \int_{t_0}^{t} Q(V,\Phi_0)(s)ds] \\
&\leq \int_{t_0}^{t} D[Q(U,\Phi_0)(s), Q(V,\Phi_0)(s)]ds \\
&\leq \int_{t_0}^{t} L\ max_{s\in[t_0,T]} D[U(s),V(s)]e^{-\lambda s}e^{\lambda s}ds \\
&= D_0[U,V]L\int_{t_0}^{t} e^{\lambda s}ds \\
&= LD_0[U,V]\int_{t_0}^{t} e^{\lambda s}ds \\
&= \frac{L}{\lambda}D_0[U,V][e^{\lambda t} - e^{\lambda t_0}] \\
&\leq \frac{L}{\lambda}D_0[U,V]e^{\lambda t}
\end{aligned}$$

This gives

$$e^{-\lambda t}D[TU(t),TV(t)] \leq \frac{L}{\lambda}D_0[U,V]$$

Thus,

$$D_0[TU,TV] \leq \frac{L}{\lambda}D_0[U,V]$$

Now to choose λ, we observe that $\frac{L}{\lambda} < \frac{1}{2}$, yields that T is a contraction, so we can choose $\lambda < \frac{2}{L}$. Hence we get that T is a contraction and that there exists a unique $U \in E_0$ such that U is a solution of the IVP (8) and (9).

In order to establish existence and uniqueness result using generalized Lipschitz condition. We need the following comparison theorem on $\mathbb{R}_+$ from [4].

Theorem 4. *Assume that $m \in C[I,\mathbb{R}_+]$, $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$ and for $t \in I$,*

$$D_-m(t) \leq g[t| \, m \, |_0(t)], \tag{14}$$

where $|\, m \,|_0(t) = sup_{t_0 \leq s \leq t} |\, m(s) \,|$.

Suppose that $r(t) = r(t, t_0, w_0)$ is the maximal solution of the scalar differential equation

$$w' = g(t, w), w(t_0) = w_0 \geq 0 \tag{15}$$

existing on I. Then $m(t_0) \leq w_0$ implies $m(t) \leq r(t), t \in I$.

We now state a Comparison Theorem that connects an estimate on the solution of the IVP (8) and (9) with the maximal solution of the initial value problem (15).

Theorem 5. *Let $Q \in C[E_0, E]$ be a causal map such that for $t \in I$,*

$$D[(QU)(t), (QV)(t)] \leq g(t, D_0[U, V](t)],$$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$. Suppose further that the maximal solution $r(t, t_0, w_0)$ of the scalar differential equation (15) exists on I. Then, if $U(t)$, $V(t)$ are any two solutions of (8) and (9) with initial function $U_{t_0} = V_{t_0} = \Phi_0 \in C_1$, then we have

$$D[U(t), V(t)] \leq r(t, t_0, w_0), t \in I$$

Proof. We first observe that $D_0[U_{t_0},\ V_{t_0}] \leq w_0$ is satisfied automatically. The proof of the theorem is exactly same as that of Theorem 5.7.2 in [4]. Hence we avoid the proof.

We are now in a position to state the existence and uniqueness result using successive approximations and generalized Lipschitz condition. Once again the proof is very much similar to the corresponding theorem, Theorem 5.7.3 in [4]. Hence we omit it. Observe that the only difference between the two results is that the following theorem has memory included in its set up.

Theorem 6. *Suppose that*

(i) *$Q \in C[B, E]$ be a causal map, where $B \subseteq E_0$*
with $B = \{U \in E_0 : D_0[U, \Phi_0(t_0)] \leq b, D_0[U_{t_0}, \Phi_0] = 0, t \in I\}$
and $D_0[Q(U, \Phi_0), \theta] \leq M_1$ on $\mathbb{B}$.

(ii) *$g \in C[I \times \mathbb{R}_+, \mathbb{R}_+] g(t, u) \leq M_2$ on $I \times [0, 2b], g(t, 0) = 0, g(t, u)$ be nondecreasing in u for each $t \in I$ and $w(t) \equiv 0$ is the only solution of*

$$w' = g(t, w), w(t_0) = 0 \text{ on } I \tag{16}$$

(iii) *$D[Q(U, \Phi_0)(t), Q(V, \Phi_0)(t)] \leq g(t, D_0[U, V](t)]$, on $\mathbb{B}$*
Then the successive approximations defined by

$$U_{n+1}(t) = \Phi_0(t_0) + \int_{t_0}^{t} Q(U_n, \Phi_0)(s) ds.$$

$$U_{n+1t_0} = \Phi_0 \in C_1,\ n = 0, 1, 2, 3 \ldots$$

exist on $I_0 = [t_0, t_0 + \eta]$ where $\eta = min[T - t_0, \frac{b}{2M}]$ $M = max[M_1, M_2]$ and converge uniformly to a unique solution $U(t)$ of (8) and (9).

4. Global Existence

Let $\widehat{E} = C[[t_0, \infty], K_c(\mathbb{R}^n)]$ and $\widehat{E_0} = C[[t_0 - h_1, \infty], K_c(\mathbb{R}^n]$.
We now state and prove a global existence result.

Theorem 7. *Assume that* $Q \in C[\widehat{E_0}, \widehat{E}]$ *and is smooth enough to guarantee local existence of solutions of IVP (8) and (9) for any* $(t_0, \Phi_0(t_0)) \in R_+ \times K_c(\mathbb{R}^n)$*. Further* $Q(U, \Phi_0)$ *be such that*

$$D[Q(U, \Phi_0)(t), \theta] \leq g(t, D_0[U, \theta](t)) \tag{17}$$

where $g \in C[\mathbb{R}_+{}^2, \mathbb{R}]$, $g(t, w)$ *is nondecreasing in* w *for each* $t \in \mathbb{R}_+$*, and the maximal solution* $r(t) = r(t, t_0, w_0)$ *of scalar IVP (15) exists on* $[t_0, \infty)$*. Then the largest interval of existence for any solution* $U(t)$ *of (8) and (9) is* $[t_0, \infty)$*, whenever* $D_0[\Phi_0, \theta] \leq w_0$

Proof. Suppose that $U(t) = U(t, t_0, \Phi_0(t_0))$ with $U_{t_0} = \Phi_0$ be any solution of (8) and (9) existing on $[t_0, \beta), t_0 < \beta < \infty$, with $D_0[\Phi_0, \theta] \leq w_0$ and the value of β cannot be increased. Set $m(t) = D[U(t), \theta]$ then

$$m(t_0) = D[U(t_0), \theta] = D[\Phi_0(t_0), \theta] \leq D_0[\Phi_0, \theta] \leq w_0$$

Consider $D^+m(t) \leq D[D_H U(t), \theta] \leq D[Q(U, \Phi_0)(t), \theta] \leq g(t, D_0[U, \theta](t))$. Now using the comparison theorem, Theorem 4, we obtain that

$$m(t) \leq r(t), t_0 \leq t \leq \beta.$$

For any t_1, t_2 such that $t_0 < t_1 < t_2 < \beta$, we obtain, using the properties of the Hausdorff metric, and relation (17),

$$\begin{aligned} D[U(t_1), U(t_2)] &= D[\int_{t_0}^{t} Q(U, \Phi_0)(s)ds, \int_{t_0}^{t_2} Q(U, \Phi_0)(s)ds] \\ &\leq \int_{t_1}^{t_2} D[Q(U, \Phi_0)(s), \theta]ds \\ &\leq \int_{t_1}^{t_2} g(s, D_0[U, \theta])(s)ds \\ &= \int_{t_1}^{t_2} g(s, | m |_0 (s)])ds. \end{aligned}$$

Using the fact that $m(t) = D[U(t), \theta] \leq r(t)$ and $g(t, u)$ is nondecreasing in u for each t, we get

$$D[U(t_1), U(t_2)] \leq r(t_2) - r(t_1)$$

Since

$$\lim_{t \to \beta} r(t, t_0, w_0)$$

exists, taking the limit as $t_1, t_2 \to \beta^-$, we conclude that $\{U(t_k)\}$ is a Cauchy sequence and therefore the

$$lim_{t\to\beta^-} U(t, t_0, \Phi_0) = U_\beta$$

exists. Now define

$$\Phi_\beta(t) = \begin{cases} \Phi_0(t), & t_0 - h_1 \le t \le t_0 \\ U(t, t_0, \Phi_0), & t_0 \le t < \beta \\ U_\beta, & t = \beta \end{cases}$$

and consider the IVP

$$D_H U(t) = Q(U, \Phi_0)(t), t \ge \beta, U_\beta = \Phi_\beta \text{ on } [t_0 - h_1, \beta], t_0 \ge 0 \tag{18}$$

set $\widehat{E_0} = C[[t_0 - h_1, \beta + a], K_c(\mathbb{R}^n)]$ and $\widehat{E} = C[[t_0, \beta + a], K_c(\mathbb{R}^n)]$,
$a > 0, \widehat{B} \subset \widehat{E_0}$ where $\widehat{B} = \{U \in \widehat{E_0} : D_0[U, \Phi_0(t_0)] \le b, D_0[U(t_0), \Phi_0(t_0)] = 0, t \in J\}$
Then $Q : \widehat{B} \to \widehat{E}$ is a causal map such that it guarantees the local existence of a solution, hence there exists $U(t, \beta, U_\beta)$ satisfying (17) on some interval $[\beta, \beta + \alpha]$, $0 < \alpha < a$.

Thus $U(t, t_0, \Phi_0)$ can be extended beyond β, contradicting our assumption that β cannot be increased. Thus every solution $U(t, t_0, \Phi_0)$ of (8), (9) such that $D_0[\Phi_0, \theta] \le w_0$ exists globally on $[t_0 - h_1, \infty)$. Hence the proof is complete.

Theorem 8. *Let $Q \in C^1[\widehat{E_0}, \widehat{E}]$ and satisfy the estimate.*

$$D[Q(U, \Phi_0)(t), \theta] \le g(t, D[U(t), \theta]], U \in \Omega \tag{19}$$

where

$$\Omega = \{U \in E_0 : \max_{t_0 - h_1 \le s \le t} D[U(s), \theta] = D[U(t), \theta], t \in I\}$$

and $g \in C[[t_0, \infty) \times \mathbb{R}_+, \mathbb{R}_+]$, $g(t, u)$ is monotone nondecreasing in u for each $t \in [t_0, \infty)$.

Assume that for every $t_0 > 0$, the scalar differential equation

$$u' = g(t, u), u(t_0) = u_0 \ge o, \tag{20}$$

has a solution $u(t)$ existing on $[t_0, \infty)$. Then for $\Phi_0 \in C_1$ such that $D_0[\Phi_0, \theta] \le u_0$ there exists a solution $U(t)$ of (8), (9) on $[t_0, \infty)$ satisfying

$$D[U(t), \theta] \le u(t), t \in [t_0, \infty) \tag{21}$$

Proof. Consider the space $\widehat{E}$, of all continuous functions from $[t_0, \infty)$ to $K_c(\mathbb{R}^n)$, and a family of pseudonorms $\{p_n(U)\}_{n=1}^{\infty}$ be defined for $U \in \widehat{E}$,

$$p_n(U) = \sup_{t_0 \le t \le n} D[U(t), \theta].$$

Let the topology on $\widehat{E}$ be generated by this family.
A fundamental system of neighborhoods is then given by $\{V_n(U\}_{n=1}^{\infty}$, where

$$V_n(U) = \{U \in \widehat{E} : p_n(U) \leq 1\}$$

Under this topology, $\widehat{E}$ becomes a complete, locally convex linear space. Now define a subset $\overline{E} \subset \widehat{E}$ as follows.

$$\overline{E} = \{U \in \Omega : D[U(t), \theta] \leq u(t), t \geq t_0\}$$

where $u(t)$ is a solution of (20) existing on $[t_0, \infty)$. Then under the topology of $\widehat{E}, \overline{E}$ is closed convex and bounded. Consider the integral operator defined by

$$(TU)(t) = \Phi_0(t_0) + \int_{t_0}^{t} Q(U, \Phi_0)(s)ds \text{ and}$$
$$U_{t_0} = \Phi_0 \in C_1$$

It is obvious that a fixed point of T will be a solution of the IVP (8),(9). The operator T is compact in the topology of $\widehat{E}$ and therefore closure of $T\overline{E}$ is compact since $\widehat{E}$ is bounded.
The proof of the theorem is complete, if we show that $T\overline{E} \subseteq \overline{E}$. Hence consider $U \in \overline{E}$. Then

$$\begin{aligned} D[(TU)(t), \theta] &= D[\Phi_0(t_0) + \int_{t_0}^{t} Q(U, \Phi_0)(s)ds, \theta] \\ &\leq D_0[\Phi_0, \theta] + \int_{t_0}^{t} D[Q(U, \Phi)(t), \theta] \\ &\leq D_0[\Phi_0, \theta] + \int_{t_0}^{t} g(t, D[U(t), \theta])ds \\ &\leq D_0[\Phi_0, \theta] + \int_{t_0}^{t} g(t, u(s))ds \end{aligned}$$

because of the relation (18),(20) and (21) and the monotonic nature of g, the definition of the set $\overline{E}$ and the fact that $u(t)$ is a solution of (20), with $D_0[\Phi_0, \theta] \leq u_0$. This yields

$$\begin{aligned} D[(TU)(t), \theta] &\leq u_0 + \int_{t_0}^{t} g(s, u(s))ds \\ &= u(t) \end{aligned}$$

Hence $TU \in \overline{E}$ or $T\overline{E} \subseteq \overline{E}$. Thus the proof is complete.

ACKNOWLEDGEMENTS This work has been done under the project no. SR/S4/MS: 491/07 sanctioned by Department of Science and Technology, Government of India. The author acknowledges their support

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