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Minimal Generators for the Rees Algebra of Rational Space Curves of Type $(1, 1, d-2)$

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Abstract. We provide an algorithm to find a minimal set of generators for the Rees algebra associated to rational space curves of type $(1, 1, d-2)$ in projective 3-space based solely on a μ-basis of the curve. We also illustrate the geometry behind the generators via a case study of rational quartic space curves.

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1. Introduction

The Rees algebra of an ideal $I \subset R$ defined as the graded algebra (with the elements of R having degree 0 and the elements of I having degree 1)

$$\mathrm{Rees}(I) = R \oplus I \oplus I^2 \oplus \ldots$$

is a classical algebraic structure which has been studied for decades by the Commutative Algebra community, see [25]. One motivation for this study is that it is related to a classical problem in elimination theory: the implicitization problem. The implicitization problem is to find an algorithm to convert a parametrization given by a rational map

$$f : \mathbb{P}^m \to \mathbb{P}^n$$

into defining equations for the closure X of the image $f(\mathbb{P}^m)$. Rational curves and surfaces are widely used in Computer Aided Design, since it is easy to describe the points on these

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curves and surfaces by means of their parameter values. However, it is not convenient to use the parametric representations to describe the set of points that are common to two different parametrically defined curves or surfaces. Thus there is a need to go back and forth between a parametric and an implicit description of a curve or surface. This is, in essence, the implicitization problem: to develop efficient algorithms to generate implicit equations for a curve or surface for which one knows a parametric representation. Implicitization algorithms have been most highly developed in the case that X is a hypersurface, so X is defined by only one equation $F = 0$. It turns out that F is related to the structure of the Rees algebra of I.

For instance consider the implicitization problem for rational surfaces in $\mathbb{P}^3$. Algebraically the problem is this: Given an ideal $I = \langle f_0, f_1, f_2, f_3\rangle \subset R$ where the f_i are homogeneous polynomials of degree d in the standard $\mathbb{Z}$-graded ring $R := \mathbb{K}[s,t,u]$ over an infinite field $\mathbb{K}$, find a minimal set of generators for the kernel of the map h: $R[x_0, x_1, x_2, x_3] \to \text{Rees}(I)$, where $h(x_i) = f_i$ for $i = 0, 1, 2, 3$. Under certain general circumstances, the implicit equation F is the element of $\ker(h)$ of degree 0 in the variables s, t, u.

Elements of $\ker(h)$, under the name of moving lines and moving planes, were introduced into Computer Aided Geometric Design by Sederberg, Cox and their collaborators in order to develop robust, efficient algorithms for implicitizing rational curves and surfaces [10], [20], [21], [22]. In the past two years, [4], [7], [8], and [16] utilized the method of moving curves and surfaces to determine the defining equations for the Rees algebra of plane algebraic curves. They each develop different methods and algorithms for finding explicit moving curves that are a minimal set of generators for the associated Rees algebra. The approach in [7], [8] is based on iterations of Sylvester determinants, regular sequences, and local cohomology computations

This approach to the implicitization problem, which has been developed especially by Jouanoulou, Busé, Chardin, Cox, D'Andrea and others, utilizes the structure of a free resolution of I as an R-module (see [2], [3]). However, these studies have been largely limited to the case of hypersurface parametrizations, and they lead to expressions for F as determinants of certain complexes. See also [1], [4], [5], [15].

The corresponding problems for codimension two (and higher) are much more difficult. Given an ideal $I = \langle f_0, f_1, \ldots, f_n\rangle \subset R$ of height two where $f_0, f_1, \ldots, f_n$ are homogeneous polynomials of degree d in the standard $\mathbb{Z}$-graded ring $R := \mathbb{K}[s,t]$ over an infinite field $\mathbb{K}$, the goal is to find a minimal set of generators for the kernel of the map h: $R[x_0, x_1, \ldots, x_n] \to \text{Rees}(I)$, where $h(x_i) = f_i$ for $i = 0, 1, \ldots, n$.

The first nontrivial case is that of rational space curves, $n = 3$. Finding a minimal set of generators for the Rees algebra of the ideal I of a rational space curve solves the implicitization problem for space curves. Since the ideal I gives rise to a rational function $\mathbb{P}^1 \to \mathbb{P}^n$ mapping $(s,t) \to (x_0, \ldots, x_n)$, the image of this map is a curve $\mathscr{C} \subset \mathbb{P}^n$ with homogeneous coordinate ring $\mathbb{K}[x_0, \ldots, x_n]$. The ideal theoretic implicit equations of the curve $\mathscr{C}$ are among the minimal generators of the defining equations for the Rees algebra associated to the space curve. For example, [6, 11, 12, 19, 23], have all investigated minimal generators for the Rees algebra of the ideal of a rational space curve.

Recently, Kustin, Polini and Ulrich [18] studied minimal generators of the defining equations for the Rees algebra of a height two ideal I with a minimal free resolution of the following form:

$$0 \to R(-d-1)^{n-1} \bigoplus R(-2d+n+1) \longrightarrow R^n(-d) \xrightarrow{[f_0,\ldots,f_n]} I \to 0\cdot$$

Since the homogeneous coordinate ring $\mathbb{K}[x_0,\ldots,x_n]$ is isomorphic to the special fiber ring of I, their approach to determining the generators for the Rees algebra of the curve $\mathscr{C}$ is to find the defining ideal of this special fiber ring. They pay close attention to the depth and algebraic properties of this fiber ring of I, and give an explicit description of the minimal generators.

In this paper, we specialize the setting in Kustin, Polini and Ulrich [18] to rational space curves of type $(1,1,d-2)$ in projective 3-space. We do not prove any new theorems about the structure of Rees algebras; rather our primary goal is to provide an algorithmic approach and to give elementary constructions for the minimal generators for the Rees algebra associated to these curves based solely on their μ-basis. Our approach is to study separately the cases when the rational curve is either singular or non-singular. If the rational space curve is singular, then we study separately the cases when the degree of the curve is either even or odd. The generators of the Rees algebra are all expressed entirely in terms of the three elements of the μ-basis. We will prove our results by comparing the generators produced by our algorithm with those described in [18]. Our algorithm shows that the very complicated description of the generators of the Rees algebra given in [18] can be simplified considerably in the case of rational space curves of type $(1,1,d-2)$. The second goal of this paper is to illustrate the geometry behind the generators via a case study of rational quartic space curves. We will construct the implicit equations of the curve and the defining equations for the Rees algebra in a simple manner from the elements of the μ-basis.

We proceed in the following fashion. In Section 2 we review the basic notion of moving surfaces and μ-bases, and recall some results concerning how to detect the singularities of rational space curves using μ-bases. In Section 3 we provide an algorithm to find minimal generators for the Rees algebra of the ideal of a rational space curve based solely on the three elements of a μ-basis of the rational space curve. In Section 4 we illustrate the geometry behind the generators with a case study of rational quartic space curves.

2. Moving Planes and μ-bases

Throughout this paper, we shall consider rational space curves $\mathscr{C}$ in three-dimensional projective space over a field $\mathbb{K}$ of characteristic 0, given as the image of a generic 1-1 rational parametrization:

$$\mathbf{F}(s,t) = (f_0(s,t),\ f_1(s,t),\ f_2(s,t),\ f_3(s,t)),\ (s,t) \neq (0,0), \tag{1}$$

where f_0, f_1, f_2, f_3 are linearly independent homogeneous polynomials of the same degree $d \geq 3$ in the standard $\mathbb{Z}$-graded ring $R := \mathbb{K}[s,t]$, and $\gcd(f_0, f_1, f_2, f_3) = 1$.

We begin by briefly recalling some basic definitions.

Definition 1. *A **moving surface** of degree r is a polynomial*

$$\sum_{i+j+\ell+k=r} A_{ij\ell k}(s,t)x^i y^j z^\ell w^k, \quad A_{ij\ell k} \in R.$$

*This polynomial is said to **follow** the parametrization (1) if*

$$\sum_{i+j+\ell+k=r} A_{ij\ell k}(s,t) f_0(s,t)^i f_1(s,t)^j f_2(s,t)^\ell f_3(s,t)^k \equiv 0.$$

Hence a moving surface of degree r follows the parametrization (1) if and only if

$$(A_{ij\ell k})_{i+j+\ell+k=r} \in \mathrm{Syz}(I^r),$$

*where I is the ideal $\langle f_0, f_1, f_2, f_3 \rangle \subset R$. The set of all moving surfaces that follow the parametrization (1) is an ideal in $R[x,y,z,w]$, called the **moving surface ideal**.*

Remark 1. *If f is a moving surface, then f is in the bigraded ring $\mathbb{K}[s,t;x,y,z,w] = R[x,y,z,w]$, and $\deg(f) = (d_1, d_2)$ where d_1 is the degree in s, t, and d_2 is the degree in x, y, z, w.*

Definition 2. *A **moving plane** is a moving surface of degree $d_2 = 1$. An **axial moving plane** is a moving plane where all the planes of the family pass through either a common point A or a common line $\overleftrightarrow{AB}$. The point A is called an **axis point**, and the line $\overleftrightarrow{AB}$ is called the **axis line** or **axis** of the moving plane. Similarly, a **moving quadric** is a moving surface of degree $d_2 = 2$. When we refer to the degree of a moving plane (or moving quadric), we are referring to the degree in s,t, i.e., d_1.*

For example, consider the rational quintic space curve where $(f_0, f_1, f_2, f_3) = (s^5, s^3t^2, s^2t^3, t^5)$. The polynomial $t^3x - s^3z$ is an axial moving plane of degree 3 that follows the curve with axis $\overleftrightarrow{AB}$ where $A = (0,0,0,1)$ and $B = (0,1,0,1)$.

Remark 2. *A moving plane of degree one in s,t always has an axis line; a moving plane of degree two in s,t always has an axis point, and may have an axis line.*

Indeed if we write a moving plane of degree one in s,t as

$$f(x,y,z,w)s + g(x,y,z,w)t, \text{ where } \deg(f) = \deg(g) = 1, \ \gcd(f,g) = 1,$$

then the variety $\mathbb{V}(f,g)$ is the axis of the moving plane. It is easy to see that $\mathbb{V}(f,g)$ is a linear variety of dimension at least one; hence a moving plane of degree one in s,t always has an axis line.

In the language of Commutative Algebra, the collection of moving planes that follow a parametrization $\mathbf{F}(s,t) = (f_0, f_1, f_2, f_3)$ is exactly $\mathrm{Syz}(I)$, where $I = \langle f_0, f_1, f_2, f_3 \rangle$. The Hilbert-Burch theorem [9, Chapter 6] says that the minimal free resolution of the ideal I has the following form:

$$0 \to R(-d-\mu_1) \oplus R(-d-\mu_2) \oplus R(-d-\mu_3) \xrightarrow{\mathbf{p},\mathbf{q},\mathbf{r}} R^4(-d) \xrightarrow{f_0,f_1,f_2,f_3} I \to 0,$$

where $\mu_1 \le \mu_2 \le \mu_3$, $\mu_1 + \mu_2 + \mu_3 = d = \deg(f_i)$. Thus the syzygies $\mathrm{Syz}(f_0, f_1, f_2, f_3)$ are generated by $\mathbf{p}, \mathbf{q}, \mathbf{r}$. The generators $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are called a *μ-basis*. We sometimes write $\mathbf{p}, \mathbf{q}, \mathbf{r}$ as three independent moving planes p, q, r, where $p = \mathbf{p} \cdot \mathbf{X}$, $q = \mathbf{q} \cdot \mathbf{X}$, $r = \mathbf{r} \cdot \mathbf{X}$ and $\mathbf{X} = (x, y, z, w)$, of homogeneous degrees $\mu_1 \leq \mu_2 \leq \mu_3$ in s, t:

$$
\begin{aligned}
p &= p_x x + p_y y + p_z z + p_w w,\\
q &= q_x x + q_y y + q_z z + q_w w,\\
r &= r_x x + r_y y + r_z z + r_w w,
\end{aligned}
$$

and these three elements p, q, r are also called a *μ-basis*. There is a simple algorithm to compute a μ-basis from the polynomials f_0, f_1, f_2, f_3 using only Gaussian elimination [24]. Although the μ-basis elements are not unique, the degrees μ_1, μ_2, μ_3 of the μ-basis elements are unique. From now on, we will use (μ_1, μ_2, μ_3) to denote the type of a rational space curve. Throughout this paper, we focus on rational space curves of type $(1, 1, d-2)$.

Here we recall two results concerning the relation between μ-bases and points on rational space curves.

Proposition 1. *[[26]] Let $\mathbf{F}(s,t)$ be a rational space curve with a μ-basis $\mathbf{p}, \mathbf{q}, \mathbf{r}$. Then a point Q is on the space curve $\mathbf{F}(s,t)$ if and only if*

$$\deg(\gcd(\mathbf{p} \cdot Q,\ \mathbf{q} \cdot Q,\ \mathbf{r} \cdot Q)) \geq 1.$$

Moreover, the roots of this gcd *are the parameters with proper multiplicity corresponding to the point Q on the curve $\mathbf{F}(s,t)$.*

Proposition 2. *[[26]] Suppose $\mathbf{p}(s,t)$, $\mathbf{q}(s,t)$ and $\mathbf{r}(s,t)$ are a μ-basis of degrees $(1, 1, d-2)$ for the rational space curve $\mathbf{F}(s,t)$. Then*

1. *$\mathbf{F}(s,t)$ has no singularities if and only if the axes of $\mathbf{p}$ and $\mathbf{q}$ do not intersect.*
2. *$\mathbf{F}(s,t)$ has exactly one singular point A which is of order $d-2$ if and only if the axes of $\mathbf{p}$ and $\mathbf{q}$ intersect at the point A.*

Moreover, from two μ-basis elements, we can generate a quadric surface that contains a space curve of type $(1, 1, d-2)$.

Lemma 1. *Let $p = p_1 s + p_0 t$ and $q = q_1 s + q_0 t$ be two μ-basis elements of degree 1 of a rational quartic space curve $\mathscr{C}$. Then the curve $\mathscr{C}$ is contained in the irreducible quadric surface defined by* $\mathrm{Sylv}_{s,t}(p,q) = \det \begin{pmatrix} p_1 & p_0 \\ q_1 & q_0 \end{pmatrix} = 0$.

Proof. Since $\deg(p) = \deg(q) = (1,1)$, p and q are relatively prime over the ring C. Hence p and q form a regular sequence over C, and therefore $\mathrm{Sylv}_{s,t}(p,q) \not\equiv 0$. Moreover, since $\deg(p_i) = \deg(q_i) = 1$ for $i = 0, 1$, $\mathrm{Sylv}_{s,t}(p,q)$ is of homogeneous degree 2. In addition, $\mathrm{Sylv}_{s,t}(p,q)$ vanishes on the curve $\mathscr{C}$, since p, q are two moving planes that follow the curve. Finally, this quadric is irreducible, otherwise the curve $\mathscr{C}$ would be contained in one of the linear factors, contradicting the fact that the curve $\mathscr{C}$ is non-planar.

3. Minimal Generators of the Rees Algebra

In this section, we will investigate minimal generators for the Rees algebra associated to rational space curves of type $(1,1,d-2)$. We will provide a simple algorithm to find these generators based solely on the three μ-basis elements of these rational space curves. Our approach is to study separately the cases when the curve $\mathbf{F}(s,t)$ is either singular or non-singular. If $\mathbf{F}(s,t)$ is singular, then we also study separately the cases when the degree of the curve is either even or odd.

3.1. Singular Rational Space Curves of Type $(1,1,d-2)$

Here we will find a minimal set of generators for the Rees algebra associated to singular rational space curves of type $(1,1,d-2)$.

Remark 3. *Let $p = p_1 s + p_0 t$ and $q = q_1 s + q_0 t$ be two μ-basis elements of a singular rational space curve $\mathbf{F}(s,t)$ of type $(1,1,d-2)$, where p_1, p_0, q_1, q_0 are linear forms in x, y, z, w. Then the polynomials p_1, p_0, q_1, q_0 are linearly dependent of rank 3.*

Proof. By Proposition 2, there is a unique singular point on the curve, which is the intersection of the axes of p and q. Thus, $\mathbb{V}(p_1, p_0, q_1, q_0)$ consists of just one point. Therefore, the polynomials p_1, p_0, q_1, q_0 are linearly dependent of rank 3.

Lemma 2. *Let p, q, r be a μ-basis for a singular rational space curve $\mathbf{F}(s,t)$ of type $(1,1,d-2)$. By a linear transformation on the basis elements p, q, and by a projective change of coordinates in x, y, z, w, we can adjust the elements of the μ-basis so that $p = ys - xt$, $q = zs - yt$, and transform the singular point to $(0,0,0,1)$.*

Proof. Let $p = p_1 s + p_0 t$ and $q = q_1 s + q_0 t$ where p_1, p_0, q_1, q_0 are linear forms in x, y, z, w. Since p, q are μ-basis elements, $\gcd(p_1, p_0) = \gcd(q_1, q_0) = 1$; otherwise the curve would be contained in the plane $p_1 = p_0 = 0$ or $q_1 = q_0 = 0$. Also, note that $\gcd(p_1, q_1) = \gcd(p_0, q_0) = 1$. Otherwise, if $q_1 = ap_1$ for some non-zero constant a, then $q = q_1 s + q_0 t = ap_1 s + q_0 t$, and $q - ap = (ap_1 s + q_0 t) - a(p_1 s + p_0 t) = (q_0 - ap_0)t$. But this is impossible, since $q - ap$ is a moving plane that follows the space curve $\mathbf{F}(s,t)$, so the space curve $\mathbf{F}(s,t)$ would be contained in the plane $q_0 - ap_0$. Thus $\gcd(p_1, q_1) = 1$. Similarly, $\gcd(p_0, q_0) = 1$.

Since $\gcd(q_1, q_0) = 1$, by a change of coordinates we can let $y' = q_1$ and $x' = -q_0$. Hence $q = y's - x't$. Since $\gcd(p_1, q_1) = 1$, it follows that $p_1 \neq ay'$ for any non-zero constant. We will now prove the lemma by establishing the following two cases.

Case 1: $p_1 = ax' + by'$ for some $a, b \in \mathbb{K}$ and $a \neq 0$. Then $p = (ax' + by')s + p_0 t$. Therefore, the moving plane $\frac{1}{a}p - \frac{b}{a}q = x's + \frac{p_0 + bx'}{a}t$ is linearly independent from the moving plane q, and follows the space curve $\mathbf{F}(s,t)$. Thus, q, $\frac{1}{a}p - \frac{b}{a}q$ and r also form a μ-basis for the space curve $\mathbf{F}(s,t)$. Since the space curve $\mathbf{F}(s,t)$ is singular, the axes of q and $\frac{1}{a}p - \frac{b}{a}q$ intersect at one point. Thus, by Remark 3, $x', y', \frac{p_0 + bx'}{a}$ must have rank 3. Hence, by the projective change of coordinates, $x = -\frac{p_0 + bx'}{a}$, $y = x'$, $z = y'$, we obtain a μ-basis with elements $p = ys - xt$ and $q = zs - yt$.

Case 2: $p_1 \neq ax' + by'$ for all possible $a, b \in \mathbb{K}$. Then p_1 is linearly independent of x', y'. Without loss of generality, we can let $z' = p_1$. Then $p = z's + p_0 t$. Since the space curve $\mathbf{F}(s,t)$ is singular, by Remark 3 x', y', z', p_0 has rank 3, so $p_0 = ax' + by' + cz'$ for some non-zero constant a, b, c. Moreover, $\gcd(p_0, q_0) = 1$ implies that b, c cannot both be zero. Now consider the two moving planes

$$\begin{aligned} bq + cp &= b(y's - x't) + c[z's + (ax' + by' + cz')t] \\ &= (by' + cz')s + [(ac - b)x' + bcy' + c^2 z']t, \\ -aq - p &= -a(y's - x't) - [z's + (ax' + by' + cz')t] \\ &= (-ay' - z')s - (by' + cz')t. \end{aligned}$$

Notice that $ac - b \neq 0$, otherwise the moving plane $bq + cp = (by' + cz')s + [bcy' + c^2 z']t = (by' + cz')(s + ct)$, which is not possible because the curve $\mathbf{F}(s,t)$ is not planar. Thus $bq + cp$ and $-aq - p$ are two linearly independent moving planes that follow the space curve $\mathbf{F}(s,t)$. Therefore, by a projective change of coordinates, if we let $x = -[(ac - b)x' + bcy' + c^2 z']$, $y = by' + cz'$, $z = -ay' - z'$, then we obtain a μ-basis in the desired form: $p = ys - xt$ and $q = zs - yt$.

Finally, by Proposition 2, the singular point is $(0, 0, 0, 1)$.

By Lemma 2, we can choose the μ-basis elements so that $tp + sq = zs^2 - xt^2$. Therefore, since the μ-basis vanishes on the curve, we have the following relations for any point on the space curve $\mathbf{F}(s,t)$

$$\frac{t}{s} = \frac{y}{x}, \quad \frac{t^2}{s^2} = \frac{z}{x}, \quad \frac{s}{t} = \frac{y}{z}, \quad \frac{s^2}{t^2} = \frac{x}{z}. \tag{2}$$

We will find the implicit equations of these space curves, and also the defining equations for the Rees algebras of these space curve by studying separately the case when d is even and the case when d is odd.

3.1.1. Degree $d = 2k$, $k \geq 2$

First, observe that when $d = 2k$, the μ-basis element r can be written as

$$\begin{aligned} r &= r_{d-2}s^{d-2} + r_{d-3}s^{d-3}t + \cdots + r_0 t^{d-2} = \sum_{j=0}^{d-2} r_j s^j t^{d-2-j} = s^{2i}[\sum_{j=0}^{d-2} r_j \frac{s^j t^{d-2-j}}{s^{2i}}] \\ &= s^{2i}[\sum_{j=0}^{2i-1} r_j t^{d-2-2i}(\frac{t}{s})^{2i-j} + \sum_{j=2i}^{d-2} r_j s^{j-2i} t^{d-2-j}] \\ &= s^{2i}\{\sum_{j=0}^{i-1}[r_{2j} t^{d-2-2i}(\frac{t^2}{s^2})^{i-j} + r_{2j+1} t^{d-2-2i}(\frac{t^2}{s^2})^{i-j-1}(\frac{t}{s})] + \sum_{j=0}^{d-2-2i} r_{2i+j} s^j t^{d-2-2i-j}\}, \end{aligned}$$

For all $i = 1, \ldots, k-1$, let

$$r_i'(A,B) = \sum_{j=0}^{i-1}[r_{2j} t^{d-2-2i}(B)^{i-j} + r_{2j+1} t^{d-2-2i}(B)^{i-j-1}(A)] + \sum_{j=0}^{d-2-2i} r_{2i+j} s^j t^{d-2-2i-j}.$$

Then

$$r_i'(\frac{y}{x},\frac{z}{x}) = t^{d-2-2i}\sum_{j=0}^{i-1}[r_{2j}(\frac{z}{x})^{i-j} + r_{2j+1}(\frac{z}{x})^{i-j-1}(\frac{y}{x})] + \sum_{j=0}^{d-2-2i} r_{2i+j}s^j t^{d-2-2i-j}. \tag{3}$$

Theorem 1. *A minimal set of generators for the defining equation of the Rees algebra associated to a singular rational space curve of type $(1,1,d-2)$ where $d=2k$ are given by the following $k+3$ polynomials:*

1. *three μ-basis elements: p, q, r, where $\deg(p)=\deg(q)=(1,1)$, and $\deg(r)=(d-2,1)$;*
2. *two implicit equations: $\mathrm{Sylv}_{s,t}(p,q)$ of degree $(0,2)$, and $x^{k-1}r_{k-1}'(\frac{y}{x},\frac{z}{x})$ of degree $(0,k)$;*
3. *$k-2$ moving surfaces: $x^i r_i'(\frac{y}{x},\frac{z}{x})$ of degree $(d-2-2i, i+1)$ for $i=1,\dots,k-2$;*

where $r_i'(\frac{y}{x},\frac{z}{x})$ is defined as in Equation (3) for $i=1,\dots k-1$.

Proof. We will apply the results of Kustin, Polini and Ulrich [18], by listing the minimal set of generators in their paper, and comparing these generators with the generators listed in our theorem.

First, we note that the notation x,y,z,w,t,s,r_i, $i=0,\dots,d-2$ in this paper is the same as T_1,T_2,T_3,T_4, x,y,c_i, $i=0,\dots,d-2$ in their notation. Moreover, in our setting, we identify the following items in their paper for singular curves of even degrees:

$$\rho=1,\ \ell=2,\ \sigma_1=2,\ \sigma_2=1,\ T_{2,1}=s,\ T_{2,2}=t,\ T_{1,1}=x,\ T_{1,2}=y,\ T_{1,3}=z.$$

By Theorem 3.2 in [18], we have

$$\mathbf{a}=(a_1)=0,1,\dots,k-2;\ f(\mathbf{a})=f(a_1)=d-3-2a_1;$$
$$r(\mathbf{a})=r(a_1)=1;\ f(\emptyset)=k-2;\ \ r(\emptyset)=1;\ T^{\emptyset}=1.$$

By definition 3.5 and the description in [18], the generators for the Rees algebra are $p,q,\mathrm{Sylv}_{s,t}(p,q)$, f_1 and $g_{a_1,1}$. We shall now write f_1 and $g_{a_1,1}$ explicitly and compare these expressions with the generators listed in the statement of our theorem. We have

$$\begin{aligned} f_1 &= z\sum_{i+j=k-3} x^i z^j(r_{2i}z + r_{2i+1}y) + x^{k-2}(r_{d-2}x + r_{d-3}y + r_{d-4}z) \\ &= \sum_{i+j=k-3}[r_{2i}x^i z^{j+2} + r_{2i+1}x^i y z^{j+1}] + (r_{d-2}x^{k-1} + r_{d-3}x^{k-2}y + r_{d-4}x^{k-2}z) \\ &= x^{k-1}\{r_{d-2} + r_{d-3}(\frac{y}{x}) + r_{d-4}(\frac{z}{x}) + \sum_{i=0}^{k-3}[r_{2i}(\frac{z}{x})^{k-1-i} + r_{2i+1}(\frac{z}{x})^{k-2-i}(\frac{y}{x})]\} \\ &= x^{k-1}\{r_{d-2} + \sum_{i=0}^{k-2}[r_{2i}(\frac{z}{x})^{k-1-i} + r_{2i+1}(\frac{z}{x})^{k-2-i}(\frac{y}{x})]\} \end{aligned}$$

$$= x^{k-1} r'_{k-1}(\frac{y}{x}, \frac{z}{x}).$$

Hence f_1 is one of our implicit equations and $\deg(f_1) = (0, k)$.

$$\begin{aligned}
g_{a_1,1} &= t^{d-2-2a_1} \sum_{i+j=a_1-1} x^i z^j (r_{2i} z + r_{2i+1} y) + x^{a_1} t \times \\
&\quad \sum_{i+j=d-4-2a_1} s^i t^j r_{2a_1+i} t + x^{a_1} s^{d-3-2a_1} (r_{d-2} s + r_{d-3} t) \\
&= x^{a_1} \{ t^{d-2-2a_1} \sum_{i=0}^{a_1-1} [r_{2i} (\frac{z}{x})^{a_1-i} + r_{2i+1} (\frac{z}{x})^{a_1-i-1} (\frac{y}{x})] + \sum_{i=0}^{d-2-2a_1} r_{2a_1+i} s^i t^{d-2-2a_1-i} \} \\
&= \begin{cases} x^{a_1} r'_{a_1}(\frac{y}{x}, \frac{z}{x}) & \text{for } a_1 = 1, \cdots, k-2, \\ r & \text{for } a_1 = 0. \end{cases}
\end{aligned}$$

Hence $g_{a_1,1}$ are our moving surfaces, and $\deg(g_{a_1,1}) = (d - 2 - 2a_1, a_1 + 1)$. Therefore, the generators provided by our theorem are the same as the generators described in [18]. Thus, we have proved our claim.

3.1.2. Degree $d = 2k + 1$, $k \geq 2$

First, observe that when $d = 2k + 1$, the μ-basis element r can be written as

$$\begin{aligned}
r &= r_{d-2} s^{d-2} + r_{d-3} s^{d-3} t + \cdots + r_0 t^{d-2} = \sum_{j=0}^{d-2} r_j s^j t^{d-2-j} = s^{d-2} \sum_{j=0}^{d-2} r_j \frac{t^{d-2-j}}{s^{d-2-j}} \\
&= s^{d-2} \sum_{j=0}^{k-1} [r_{2j} (\frac{t^2}{s^2})^{k-1-j} (\frac{t}{s}) + r_{2j+1} (\frac{t^2}{s^2})^{k-1-j}].
\end{aligned}$$

Let

$$r''(A, B) = \sum_{j=0}^{k-1} [r_{2j} (B)^{k-1-j} (A) + r_{2j+1} (B)^{k-1-j}].$$

Then

$$r''(\frac{y}{x}, \frac{z}{x}) = \sum_{j=0}^{k-1} [r_{2j} (\frac{z}{x})^{k-1-j} (\frac{y}{x}) + r_{2j+1} (\frac{z}{x})^{k-1-j}]. \tag{4}$$

On the other hand, the μ-basis element r also can be written as

$$\begin{aligned}
r &= r_{d-2} s^{d-2} + r_{d-3} s^{d-3} t + \cdots + r_0 t^{d-2} = \sum_{j=0}^{d-2} r_j s^j t^{d-2-j} = t^{d-2} \sum_{j=0}^{d-2} r_j \frac{s^j}{t^j} \\
&= t^{d-2} \{ \sum_{j=0}^{k-1} [r_{2j} (\frac{s^2}{t^2})^j + r_{2j+1} (\frac{s^2}{t^2})^j (\frac{s}{t})] \}.
\end{aligned}$$

Let

$$r'''(A,B) = \sum_{j=0}^{k-1}[r_{2j}(B)^j + r_{2j+1}(B)^j(A)];$$

then

$$r'''(\frac{y}{z},\frac{x}{z}) = \sum_{j=0}^{k-1}[r_{2j}(\frac{x}{z})^j + r_{2j+1}(\frac{x}{z})^j(\frac{y}{z})]. \tag{5}$$

Moreover, for all $i = 1,\dots,k-1$,

$$\begin{aligned}
r &= r_{d-2}s^{d-2} + r_{d-3}s^{d-3}t + \cdots + r_0t^{d-2} = \sum_{j=0}^{d-2} r_j s^j t^{d-2-j} = s^{2i}[\sum_{j=0}^{d-2} r_j \frac{s^j t^{d-2-j}}{s^{2i}}] \\
&= s^{2i}[\sum_{j=0}^{2i-1} r_j t^{d-2-2i}(\frac{t}{s})^{2i-j} + \sum_{j=2i}^{d-2} r_j s^{j-2i}t^{d-2-j}] \\
&= s^{2i}\{\sum_{j=0}^{i-1}[r_{2j}t^{d-2-2i}(\frac{t^2}{s^2})^{i-j} + r_{2j+1}t^{d-2-2i}(\frac{t^2}{s^2})^{i-j-1}(\frac{t}{s})] + \sum_{j=0}^{d-2-2i} r_{2i+j}s^j t^{d-2-2i-j}\}.
\end{aligned}$$

For all $i = 1,\dots,k-1$, let

$$r''_i(A,B) = t^{d-2-2i}\sum_{j=0}^{i-1}[r_{2j}(B)^{i-j} + r_{2j+1}(B)^{i-j-1}(A)] + \sum_{j=0}^{d-2-2i} r_{2i+j}s^j t^{d-2-2i-j};$$

then

$$r''_i(\frac{y}{x},\frac{z}{x}) = t^{d-2-2i}\sum_{j=0}^{i-1}[r_{2j}(\frac{z}{x})^{i-j} + r_{2j+1}(\frac{z}{x})^{i-j-1}(\frac{y}{x})] + \sum_{j=0}^{d-2-2i} r_{2i+j}s^j t^{d-2-2i-j}. \tag{6}$$

Theorem 2. *A minimal set of generators for the Rees algebra associated to a singular rational space curve of type* $(1,1,d-2)$ *where* $d = 2k+1$ *are given by the following* $k+5$ *polynomials:*

1. *three* μ*-basis elements:* p, q, r, *where* $\deg(p) = \deg(q) = (1,1)$ *and* $\deg(r) = (d-2,1)$*;*
2. *three implicit equations:* $\mathrm{Sylv}_{s,t}(p,q)$ *of degree* $(0,2)$*,* $x^k r''(\frac{y}{x},\frac{z}{x})$ *and* $z^k r'''(\frac{y}{z},\frac{x}{z})$ *of degree* $(0,k+1)$*;*
3. $k-1$ *moving surfaces:* $x^i r''_i(\frac{y}{x},\frac{z}{x})$ *of degree* $(d-2-2i, i+1)$ *for* $i = 1,\dots,k-1$*;*

where $r''(\frac{y}{x},\frac{z}{x})$*,* $r'''(\frac{y}{z},\frac{x}{z})$ *and* $r''_i(\frac{y}{x},\frac{z}{x})$ *are defined in Equations* (4)*,* (5) *and* (6)*.*

Proof. We will apply the results of Kustin, Polini and Ulrich [18], by listing the set of minimal generators in their paper, and comparing these generators with the generators listed in the statement of our theorem.

First, we note that the notation x, y, z, w, t, s, r_i, $i = 0, \dots, d-2$ in this paper is the same as T_1, T_2, T_3, T_4, x, y, c_i, $i = 0, \dots, d-2$ in their notation. Moreover, in our setting, we identify the following items in their paper for singular curves of odd degrees:

$$\rho = 1,\ \ell = 2,\ \sigma_1 = 2,\ \sigma_2 = 1,\ s = T_{2,1},\ t = T_{2,2},\ T_{1,1} = x,\ T_{1,2} = y,\ T_{1,3} = z.$$

By Theorem 3.2 in [18],

$$\mathbf{a} = (a_1) = 0, 1, \dots, k-1;\ f(\mathbf{a}) = f(a_1) = d - 3 - 2a_1;$$
$$r(\mathbf{a}) = r(a_1) = 1;\ f(\emptyset) = k-1;\ r(\emptyset) = 2;\ T^{\emptyset} = 1.$$

By definition 3.5 and the description in [18], the generators for the Rees algebra are $p, q, \mathrm{Sylv}_{s,t}(p, q)$, f_1 and $g_{a_1,1}$. We shall now write f_1 and $g_{a_1,1}$ explicitly and compare these expressions with the generators listed in the statement of our theorem.

$$\begin{aligned}
f_1 &= y \sum_{i+j=k-2} x^i z^j (r_{2i} z + r_{2i+1} y) + x^{k-1}(r_{d-2} x + r_{d-3} y) \\
&= \sum_{i=0}^{k-2} [r_{2i} x^i z^{k-1} y + r_{2i+1} x^i z^{k-2-i} y^2] + r_{d-2} x^k + r_{d-3} x^k y \\
&= x^k \{ \sum_{i=0}^{k-2} [r_{2i} \frac{z^{k-1} y}{x^{k-i}} + r_{2i+1} \frac{z^{k-2-i} y^2}{x^{k-i}}] + r_{d-2} + r_{d-3} \frac{y}{x} \} \\
&= x^k \sum_{i=0}^{k-1} [r_{2i} (\frac{z}{x})^{k-1-i} (\frac{y}{x}) + r_{2i+1} (\frac{z}{x})^{k-1-i}], \quad \text{since } y^2 = xz \text{ on the curve} \\
&= x^k r''(\frac{y}{x}, \frac{z}{x}).
\end{aligned}$$

Hence f_1 is one of our implicit equations and $\deg(f_1) = (0, k+1)$.

$$\begin{aligned}
f_2 &= z \sum_{i+j=k-2} x^i z^j (r_{2i} z + r_{2i+1} y) + x^{k-1}(r_{d-2} y + r_{d-3} z) \\
&= \sum_{i=0}^{k-2} [r_{2i} x^i z^{k-i} + r_{2i+1} x^i z^{k-1-i} y] + r_{d-2} x^{k-1} y + r_{d-3} x^{k-1} z \\
&= z^k \{ \sum_{i=0}^{k-2} [r_{2i} (\frac{x}{z})^i + r_{2i+1} (\frac{x}{z})^i (\frac{y}{z})] + r_{d-2} (\frac{x}{z})^{k-1} (\frac{y}{z}) + r_{d-3} (\frac{x}{z})^{k-1} \} \\
&= z^k \sum_{i=0}^{k-1} [r_{2i} (\frac{x}{z})^i + r_{2i+1} (\frac{x}{z})^i (\frac{y}{z})] \\
&= z^k r'''(\frac{y}{z}, \frac{x}{z}).
\end{aligned}$$

Hence f_2 is another one of our implicit equations and $\deg(f_2) = (0, k+1)$.

$$g_{a_1,1} \quad = \quad t^{d-2-2a_1} \sum_{i+j=a_1-1} x^i z^j (r_{2i} z + r_{2i+1} y) + x^{a_1} t \times$$

$$
\begin{aligned}
&\sum_{i+j=d-4-2a_1} s^i t^j r_{2a_1+i} t + x^{a_1} s^{d-3-2a_1}(r_{d-2}s + r_{d-3}t) \\
&= x^{a_1}\{t^{d-2-2a_1}\sum_{i=0}^{a_1-1}[r_{2i}(\frac{z}{x})^{a_1-i} + r_{2i+1}(\frac{z}{x})^{a_1-1-i}(\frac{y}{x})] + \sum_{i=0}^{d-2-2a_1} r_{2a_1+i}s^i t^{d-2-2a_1-i}\}
\end{aligned}
$$

$$
g_{a_1,1} = \begin{cases} x^{a_1} r''_{a_1}(\frac{y}{x}, \frac{z}{x}) & \text{for } a_1 = 1, \cdots, k-1, \\ r & \text{for } a_1 = 0. \end{cases}
$$

Hence $g_{a_1,1}$ are our moving surfaces and $\deg(g_{a_1,1}) = (d-2-2a_1, a_1+1)$. Therefore, the generators provided by our theorem are the same as the generators described in [18]. Thus, we have proved our claim.

3.2. Non-Singular Rational Space Curve of Type $(1, 1, d-2)$

Now, we will find a minimal set of generators for the Rees algebra associated to non-singular rational space curves of type $(1, 1, d-2)$.

Lemma 3. *Let p, q, r be a μ-basis for a non-singular rational space curve of type $(1, 1, d-2)$. By a linear transformation on the basis elements p, q, and by a projective change of coordinates in x, y, z, w, we can adjust the elements of the μ-basis so that $p = ys - xt,\ q = ws - zt$.*

Proof. First write $p = p_1 s + p_0 t$ and $q = q_1 s + q_0 t$ where p_1, p_0, q_1, q_0 are linear forms in $\mathbb{K}[x, y, z, w]$. Since the curve is non-singular, by Proposition 2, $\mathbb{V}(p_1, p_0, q_1, q_0) = \emptyset$. Hence the polynomials p_1, p_0, q_1, q_0 are linearly independent of rank 4. Thus, without loss of generality, we can set $p = ys - xt$ and $q = ws - zt$.

Now since the μ-basis vanishes on the curve, we have the following relations for any point on the space curve $\mathbf{F}(s, t)$

$$
\frac{t}{s} = \frac{y}{x},\ \frac{t}{s} = \frac{w}{z}. \tag{7}
$$

For $i = 0, 1, \cdots, d-2$, the μ-basis element r can be written as

$$
\begin{aligned}
r &= r_{d-2}s^{d-2} + r_{d-3}s^{d-3}t + \cdots + r_0 t^{d-2} = \sum_{j=0}^{d-2} r_j s^j t^{d-2-j} = s^{d-2}\sum_{j=0}^{d-2} r_j(\frac{t}{s})^{d-2-j} \\
&= s^{d-2}[\sum_{j=0}^{i-1} r_j(\frac{t}{s})^{d-2-i}(\frac{t}{s})^{i-j} + \sum_{j=i}^{d-2} r_j(\frac{t}{s})^{d-2-j}], \quad \text{where } \sum_{j=0}^{i-1} r_j(\frac{t}{s})^{d-2-i}(\frac{t}{s})^{i-j} := 0 \text{ if } i = 0 \\
&= s^{d-2}[\sum_{j=0}^{i-1} r_j(\frac{t}{s})^{d-2-i}(\frac{t}{s})^{i-j} + \sum_{j=0}^{d-2-i} r_{i+j}(\frac{t}{s})^{d-2-i-j}].
\end{aligned}
$$

For $i = 0, 1, \cdots, d-2$, let

$$
r'_i(A, B) = \sum_{j=0}^{i-1} r_j(B)^{d-2-i}(A)^{i-j} + \sum_{j=0}^{d-2-i} r_{i+j}(B)^{d-2-i-j}, \text{ where } \sum_{j=0}^{i-1} r_j(B)^{d-2-i}(A)^{i-j} := 0 \text{ if } i = 0.
$$

Then

$$r_i'(\frac{y}{x},\frac{w}{z}) = \sum_{j=0}^{i-1} r_j(\frac{w}{z})^{d-2-i}(\frac{y}{x})^{i-j} + \sum_{j=0}^{d-2-i} r_{i+j}(\frac{w}{z})^{d-2-i-j}, \tag{8}$$

where $\sum_{j=0}^{i-1} r_j(\frac{w}{z})^{d-2-i}(\frac{y}{x})^{i-j} := 0$ if $i=0$.

Moreover, for all $i=1,\dots,d-3$, and for all $0 \le j \le i$, we can also write

$$\begin{aligned}
r &= r_{d-2}s^{d-2} + r_{d-3}s^{d-3}t + \cdots + r_0 t^{d-2} = \sum_{\ell=0}^{d-2} r_\ell s^\ell t^{d-2-\ell} \\
&= s^i[\sum_{\ell=0}^{i-1} r_\ell t^{d-2-i}(\frac{t}{s})^{i-\ell} + \sum_{\ell=i}^{d-2} r_\ell t^{d-2-\ell}s^{\ell-i}] \\
&= s^i\{t^{d-2-i}[\sum_{\ell=0}^{j-1} r_\ell(\frac{t}{s})^{i-j}(\frac{t}{s})^{j-\ell} + \sum_{\ell=j}^{i-1} r_\ell(\frac{t}{s})^{i-\ell}] + \sum_{\ell=i}^{d-2} r_\ell t^{d-2-\ell}s^{\ell-i}\}, \\
&\quad \text{where } \sum_{\ell=j}^{i-1} r_\ell(\frac{t}{s})^{i-\ell} := 0, \text{ if } j=i \\
&= s^i\{t^{d-2-i}[\sum_{\ell=0}^{j-1} r_\ell(\frac{t}{s})^{i-j}(\frac{t}{s})^{j-\ell} + \sum_{\ell=0}^{i-1-j} r_{j+\ell}(\frac{t}{s})^{i-j-\ell}] + \sum_{\ell=0}^{d-2-i} r_{i+\ell}t^{d-2-i-\ell}s^\ell\}.
\end{aligned}$$

For all $i=1,\dots,d-3$, and for all $0 \le j \le i$, let

$$r'_{j,i-j}(A,B) = t^{d-2-i}[\sum_{\ell=0}^{j-1} r_\ell(B)^{i-j}(A)^{j-\ell} + \sum_{\ell=0}^{i-1-j} r_{j+\ell}(B)^{i-j-\ell}] + \sum_{\ell=0}^{d-2-i} r_{i+\ell}t^{d-2-i-\ell}s^\ell,$$

where $\sum_{\ell=0}^{i-1-j} r_{j+\ell}(B)^{i-j-\ell} := 0$, if $j=i$. Then

$$\begin{aligned}
&r'_{j,i-j}(\frac{y}{x},\frac{w}{z}) \\
&= t^{d-2-i}[\sum_{\ell=0}^{j-1} r_\ell(\frac{w}{z})^{i-j}(\frac{y}{x})^{j-\ell} + \sum_{\ell=0}^{i-1-j} r_{j+\ell}(\frac{w}{z})^{i-j-\ell}] + \sum_{\ell=0}^{d-2-i} r_{i+\ell}t^{d-2-i-\ell}s^\ell,
\end{aligned} \tag{9}$$

where $\sum_{\ell=0}^{i-1-j} r_{j+\ell}(\frac{w}{z})^{i-j-\ell} := 0$, if $j=i$.

Theorem 3. *A minimal set of generators for the Rees algebra associated to a non-singular rational space curve type* $(1,1,d-2)$ *are given by the following* $3+d+\dfrac{d(d-3)}{2}$ *polynomials:*

1. *three* μ*-basis elements:* p, q, r *where* $\deg(p)=\deg(q)=(1,1)$*, and* $\deg(r)=(d-2,1)$*;*
2. d *implicit equations:* $\mathrm{Sylv}_{s,t}(p,q)$ *of degree* 2*, and* $d-1$ *implicit equations* $x^i z^{d-2-i} r_i'(\frac{y}{x},\frac{w}{z})$ *of degree* $d-1$ *for* $i=0,1,\dots,d-2$*;*

3. *$\frac{d(d-3)}{2}$ moving planes: $x^j z^{i-j} r'_{j,i-j}(\frac{y}{x}, \frac{w}{z})$ of degree $(d-2-i, i+1)$ for all $i = 1, , \ldots, d-3$ and $0 \le j \le i$;*

where $r'_i(\frac{y}{x}, \frac{w}{z})$ and $r'_{j,i-j}(\frac{y}{x}, \frac{w}{z})$ are defined in Equations (8) *and* (9).

Proof. We will apply the results of Kustin, Polini and Ulrich [18], by listing the set of minimal generators in their paper, and comparing these generators with the generators listed in our theorem.

First, we note that the notation x, y, z, w, t, s, r_i, $i = 0, \ldots, d-2$ in this paper is the same as $T_1, T_2, T_3, T_4, x, y, c_i$, $i = 0, \ldots, d-2$ in their notation. Moreover, in our setting, we identify the following items in their paper for non-singular curves:

$$\rho = 2, \ell = 3, \sigma_1 = \sigma_2 = \sigma_3 = 1, [T_{3,1}, T_{3,2}; T_{1,1}, T_{1,2}, T_{1,3}; T_{2,1}, T_{2,2}] = [s, t; x, y, z; z, w].$$

By Theorem 3.2 in [18], we have

$$\mathbf{a} = (a_1, a_2),\ 0 \le a_1 + a_2 \le d-3;\ f(a_1) = d-3-a_1;\ f(a_1, a_2) = d-3-a_1-a_2;$$

$$r(a_1) = 1;\ r(a_1, a_2) = 1;\ f(\emptyset) = d-3;\ r(\emptyset) = 1;\ T^{\emptyset} = 1.$$

By definition 3.5 and the description in [18], the generators for the Rees algebra are $p, q, \mathrm{Sylv}_{s,t}(p,q)$, f_1, $g_{a_1,1}$, and $h(a_1, a_2)$. We shall now write f_1, $g_{a_1,1}$, and $h(a_1, a_2)$ explicitly and compare these expressions with the generators listed in the statement of our theorem. We have

$$\begin{aligned} f_1 &= y \sum_{i+j=d-4} x^i y^j (r_i y) + x^{d-3}(r_{d-2}x + r_{d-3}y) = \sum_{i=0}^{d-4} r_i x^i y^{d-2-i} + r_{d-2}x^{d-2} + r_{d-3}x^{d-3}y \\ &= x^{d-2} \sum_{i=0}^{d-2} r_i (\frac{y}{x})^{d-2-i} = x^{d-2} r'_{d-2}(\frac{y}{x}, \frac{w}{z}). \end{aligned}$$

Hence f_1 is one of our implicit equations and $\deg(f_1) = (0, d-1)$.

$$\begin{aligned} g_{a_1,1} &= w^{d-2-a_1} \sum_{i+j=a_1-1} x^i y^j (r_i y) + x^{a_1} w \times \\ &\quad \sum_{i+j=d-4-a_1} z^i w^j r_{a_1+i} w + x^{a_1} z^{d-3-a_1}(r_{d-2}z + r_{d-3}w) \\ &= \sum_{i=0}^{a_1-1} r_i x^i y^{a_1-i} w^{d-2-a_1} + \sum_{i=0}^{d-4-a_1} r_{a_1+i} x^{a_1} z^i w^{d-2-a_1-i} + r_{d-2} x^{a_1} z^{d-2-a_1} + r_{d-3} x^{a_1} z^{d-3-a_1} w \\ &= x^{a_1} z^{d-2-a_1} [\sum_{i=0}^{a_1-1} r_i (\frac{w}{z})^{d-2-a_1} (\frac{y}{x})^{a_1-i} + \sum_{i=0}^{d-2-a_1} r_{a_1+i} (\frac{w}{z})^{d-2-a_1-i}] \\ &= x^{a_1} z^{d-2-a_1} r'_{a_1}(\frac{y}{x}, \frac{w}{z}). \end{aligned}$$

Hence each $g_{a_1,1}$ is one of our implicit equations and $\deg(g_{a_1,1}) = (0, d-1),\ a_1 = 0, 1, \ldots, d-3$.

$$\begin{aligned}
h_{a_1,a_2} &= t^{d-2-a_1-a_2}[w^{a_2} \sum_{i+j=a_1-1} x^i y^j (r_i y) + x^{a_1} \sum_{i+j=a_2-1} z^i w^j r_{a_1+i} w] \\
&\quad + x^{a_1} z^{a_2} \sum_{i=0}^{d-2-a_1-a_2} r_{a_1+a_2+i} t^{d-2-a_1-a_2-i} s^i \\
&= t^{d-2-a_1-a_2}[\sum_{i=0}^{a_1-1} r_i x^i y^{a_1-i} w^{a_2} + \sum_{i=0}^{a_2-1} r_{a_2+i} x^{a_1} z^i w^{a_2-i}] \\
&\quad + x^{a_1} z^{a_2} \sum_{i=0}^{d-2-a_1-a_2} r_{a_1+a_2+i} t^{d-2-a_1-a_2-i} s^i \\
&= x^{a_1} z^{a_2} \{ t^{d-2-a_1-a_2} [\sum_{i=0}^{a_1-1} r_i (\frac{w}{z})^{a_2} (\frac{y}{x})^{a_1-i} + \sum_{i=0}^{a_2-1} r_{a_2+i} (\frac{w}{z})^{a_2-i}] \\
&\quad + \sum_{i=0}^{d-2-a_1-a_2} r_{a_1+a_2+i} t^{d-2-a_1-a_2-i} s^i \} \\
&= \begin{cases} x^{a_1} z^{a_2} r'_{a_1,a_2}(\frac{y}{x}, \frac{w}{z}), & a_1 + a_2 \neq 0,\ 0 < a_1 + a_2 \leq d-3, \\ r & a_1 + a_2 = 0. \end{cases}
\end{aligned}$$

Hence h_{a_1,a_2} are our moving surfaces and $\deg(h_{a_1,a_2}) = (d-2-a_1-a_2, a_1+a_2+1),\ 0 \leq a_1 + a_2 \leq d-3$.

Therefore, the generators provided by our theorem are the same as the generators described in [18]. Thus, we have proved our claim.

3.3. Algorithm

Based on Theorems 1, 2, and 3, we now provide a simple algorithm to find a minimal set of generators for the Rees algebra associated to rational space curves of type $(1, 1, d-2)$ in projective 3-space based solely on a μ-basis of the curve.

Algorithm 4. *Given a rational space curve $\mathscr{C}$ as the image of a generic 1-1 rational parametrization $\mathbf{F}(s,t)$ as in Equation (1), we compute a minimal set of generators for the associated Rees algebra.*

1. *Compute a μ-basis p, q, r, and write in the following form:*

$$p = p_1 s + p_0 t,\ \ q = q_1 s + q_0 t,\ \ r = r_{d-2} s^{d-2} + r_{d-3} s^{d-3} t + \cdots + r_0 t^{d-2}.$$

2. *Compute $\mathbb{V}(p_1, p_0, q_1, q_0)$.*

3. If $\mathbb{V}(p_1, p_0, q_1, q_0) \neq \emptyset$, and if $p_1 = bq_1 - aq_0$ for some $a, b \in \mathbb{K}$ and $a \neq 0$, then let

$$X = -\frac{p_0 - bq_0}{a}, \;\; Y = -q_0, \;\; Z = q_1, \;\; W = w$$

Otherwise, find non-zero constant $a, b, c \in \mathbb{K}$, such that $p_0 = -aq_0 + bq_1 + cp_1$, and let

$$X = (ac - b)q_0 - bcq_1 - c^2 p_1, \;\; Y = bq_1 + cp_1, \;\; Z = -aq_1 - p_1, \;\; W = w,$$

so that

$$p = Ys - Xt, \;\; q = Zs - Yt, \;\; r = R_{d-2}s^{d-2} + R_{d-3}s^{d-3}t + \cdots + R_0 t^{d-2},$$

where R_i are linear in X, Y, Z, W.

(1) If $d = 2k$, then a minimal set of generators for the associated Rees algebra is

$$p, \; q, \; r, \; \mathrm{Sylv}_{s,t}(p, q), \; X^i R_i'\left(\frac{Y}{X}, \frac{Z}{X}\right), \; i = 0, \ldots, k-1,$$

where

$$\begin{aligned} X^i R_i'\left(\frac{Y}{X}, \frac{Z}{X}\right) &= t^{d-2-2i} \sum_{j=0}^{i-1} \left[R_{2j} \left(\frac{Z}{X}\right)^{i-j} + r_{2j+1} \left(\frac{Z}{X}\right)^{i-j-1} \left(\frac{Y}{X}\right) \right] \\ &+ \sum_{j=0}^{d-2-2i} R_{2i+j} s^j t^{d-2-2i-j}. \end{aligned}$$

(2) If $d = 2k+1$, then a minimal set of generators for the associated Rees algebra is

$$p, \; q, \; r, \; \mathrm{Sylv}_{s,t}(p, q), \; R'', \; R''', \; X^i R_i'\left(\frac{Y}{X}, \frac{Z}{X}\right), \; i = 1, \ldots, k-1,$$

where

$$\begin{aligned} R''\left(\frac{Y}{X}, \frac{Z}{X}\right) &= \sum_{j=0}^{k-1} \left[R_{2j} \left(\frac{Z}{X}\right)^{k-1-j} \left(\frac{Y}{X}\right) + R_{2j+1} \left(\frac{Z}{X}\right)^{k-1-j} \right] \\ R'''\left(\frac{Y}{Z}, \frac{X}{Z}\right) &= \sum_{j=0}^{k-1} \left[R_{2j} \left(\frac{X}{Z}\right)^{j} + R_{2j+1} \left(\frac{X}{Z}\right)^{j} \left(\frac{Y}{Z}\right) \right] \\ R_i''\left(\frac{Y}{X}, \frac{Z}{X}\right) &= t^{d-2-2i} \sum_{j=0}^{i-1} \left[R_{2j} \left(\frac{Z}{X}\right)^{i-j} + R_{2j+1} \left(\frac{Z}{X}\right)^{i-j-1} \left(\frac{Y}{X}\right) \right] \\ &+ \sum_{j=0}^{d-2-2i} R_{2i+j} s^j t^{d-2-2i-j}. \end{aligned}$$

4. *If $\mathbb{V}(p_1, p_0, q_1, q_0) = \emptyset$, then let $X = -p_0$, $Y = p_1$, $Z = -q_0$, $W = q_1$, so that*

$$p = Ys - Xt,\ \ q = Ws - Zt,\ \ r = R_{d-2}s^{d-2} + R_{d-3}s^{d-3}t + \cdots + R_0 t^{d-2},$$

where R_i are linear in X, Y, Z, W. A minimal set of generators for the associated Rees algebra is

$$p,\ q,\ r,\ \mathrm{Sylv}_{s,t}(p,q),\ X^\ell Z^{d-2-\ell} R'_\ell\left(\frac{Y}{X}, \frac{W}{Z}\right), X^j Z^{i-j} R'_{j,i-j}\left(\frac{Y}{X}, \frac{W}{Z}\right),$$
$$0 \le \ell \le d-2,\ 0 \le j \le i = 1, \ldots, d-3,$$

where

$$\begin{aligned} R'_\ell\left(\frac{Y}{X}, \frac{W}{Z}\right) &= \sum_{j=0}^{\ell-1} R_j \left(\frac{W}{Z}\right)^{d-2-\ell} \left(\frac{Y}{X}\right)^{\ell-j} + \sum_{j=0}^{d-2-\ell} R_{\ell+j} \left(\frac{W}{Z}\right)^{d-2-\ell-j}, \\ R'_{j,i-j}\left(\frac{Y}{X}, \frac{W}{Z}\right) &= t^{d-2-i} \left[\sum_{\ell=0}^{j-1} R_\ell \left(\frac{W}{Z}\right)^{i-j} \left(\frac{Y}{Z}\right)^{j-\ell} + \sum_{\ell=0}^{i-1-j} R_{j+\ell} \left(\frac{W}{Z}\right)^{i-j-\ell} \right] \\ &\quad + \sum_{\ell=0}^{d-2-i} R_{i+\ell} t^{d-2-i-\ell} s^\ell. \end{aligned}$$

Below we present two examples to show how to use Algorithm 4 to find a minimal set of generators for the associated Rees algebra both for singular and for non-singular rational space curves.

Example 1. *Consider the rational quintic space curve given by*

$$\mathbf{F}(s,t) = (s^4t + s^3t^2 - 2s^2t^3,\ s^5 + 5s^4t + 6s^3t^2 - 4s^2t^3 - 8st^4,\ s^4t - 3s^2t^3 + 2st^4,\ t^5).$$

Compute a μ-basis for $\mathbf{F}(s,t)$:

$$p = xs - (4z + y - 8x)t,\ \ q = (x - z)s - xt,\ \ r = (s^3 + s^2t - 2st^2)w + t^3(z - x).$$

The point $(0,0,0,1)$ is a singular point of order 3.

By a projective changes of coordinates, let

$$X = -8x + y + 4z,\ Y = x,\ Z = x - z,\ W = w.$$

Then the new μ-basis can be written as

$$p = Ys - Xt,\ \ q = Zs - Yt,\ \ r = Ws^3 + Ws^2t - 2Wst^2 - Zt^3.$$

The minimal generators for the Rees algebra associated to this curve are p, q, r and the following polynomials:

$$\mathrm{Sylv}_{s,t}(p,q) = XZ - Y^2 = -9x^2 + xy + 12xz - yz - 4z^2;$$

$$X^2r''(\frac{Y}{X},\frac{Z}{X}) = X^2\{[1+(\frac{Y}{X})-2(\frac{Z}{X})]W-(\frac{Y}{X})(\frac{Z}{X})Z\}$$
$$= -x^3+2x^2z-xz^2+72x^2w-17xyw+y^2w-84xzw+10yzw+24z^2w;$$
$$Z^2r'''(\frac{Y}{Z},\frac{X}{Z}) = Z^2\{[(\frac{Y}{Z})(\frac{X}{Z})+(\frac{X}{Z})-2(\frac{Y}{Z})]W-Z\}$$
$$= -x^3+3x^2z-3xz^2+z^3-18x^2w+2xyw+18xzw-yzw-4z^2w,$$
$$Xr_1''(\frac{Y}{X},\frac{Z}{X}) = X\{[(s+t-2t(\frac{Y}{X})]W-t(\frac{Z}{X})Z\}$$
$$= -8xws+yws+4zws-x^2t+2xzt-z^2t-10xwt+ywt+4zwt.$$

Example 2. *Consider the non-singular rational degree 7 space curve given by*

$$\mathbf{F}(s,t) = (s^7,\, s^6t,\, st^6,\, t^7).$$

Compute a μ-basis for $\mathbf{F}(s,t)$: $p = ys - xt$, $q = ws - zt$, $r = zs^5 - yt^5$.

The minimal generators for the Rees algebra associated to this curve consist of p, q, r, the following quadric and sextic implicit equations of the space curve:

$$\mathrm{Sylv}_{s,t}(p,q) = xw - yz;$$
$$z^5r_i'(\frac{w}{z},\frac{y}{x}) = z^5[z-y(\frac{w}{z})^5] = z^6 - yw^5;$$
$$xz^4r_i'(\frac{w}{z},\frac{y}{x}) = xz^4[z-y(\frac{w}{z})^4(\frac{y}{x})] = xz^5 - y^2w^4;$$
$$x^2z^3r_i'(\frac{w}{z},\frac{y}{x}) = x^2z^3[z-y(\frac{w}{z})^3(\frac{y}{x})^2] = x^2z^4 - y^3w^3;$$
$$x^3z^2r_i'(\frac{w}{z},\frac{y}{x}) = x^3z^2[z-y(\frac{w}{z})^2(\frac{y}{x})^3] = x^3z^3 - y^4w^2;$$
$$x^4zr_i'(\frac{w}{z},\frac{y}{x}) = x^4z[z-y(\frac{w}{z})(\frac{y}{x})^4] = x^4z^2 - y^5w;$$
$$x^5r_i'(\frac{w}{z},\frac{y}{x}) = x^5[z-y(\frac{y}{x})^5] = x^5z - y^6,$$

together with $\dfrac{d(d-3)}{2} = \dfrac{7(4)}{2} = 14$ *moving planes of degree $(d-2-i, i+1)$,*

$$x^jz^{i-j}r_{j,i-j}'(\frac{w}{z},\frac{y}{x}), \quad \forall i = 1,\dots,d-3=4,\ 0 \le j \le i, \text{ i.e.,}$$

$$z[zs^4-yt^4(\frac{w}{z})] = z^2s^4 - ywt^4, \qquad x[zs^4-yt^4(\frac{y}{x})] = xzs^4 - y^2t^4,$$
$$z^2[zs^3-yt^3(\frac{w}{z})^2] = z^3s^3 - yw^2t^3, \qquad xz[zs^3-yt^3(\frac{y}{x})(\frac{w}{z})] = xz^2s^3 - y^2wt^3,$$
$$x^2[zs^3-yt^3(\frac{y}{x})^2] = x^2zs^3 - y^3t^3, \qquad z^3[zs^2-yt^2(\frac{w}{z})^3] = z^4s^2 - yw^3t^2,$$
$$xz^2[zs^2-yt^2(\frac{y}{x})(\frac{w}{z})^2] = xz^3s^2 - y^2w^2t^2, \qquad x^2z[zs^2-yt^2(\frac{y}{x})^2(\frac{w}{z})] = x^2z^2s^2 - y^3wt^2,$$

$$x^3[zs^2 - yt^2(\frac{y}{x})^3] = x^3zs^2 - y^4t^2, \qquad z^4[zs - yt(\frac{w}{z})^4] = z^5s - yw^4t,$$
$$xz^3[zs - yt(\frac{y}{x})(\frac{w}{z})^3] = xz^4s - y^2w^3t, \qquad x^2z^2[zs - yt(\frac{y}{x})^2(\frac{w}{z})^2] = x^2z^3s - y^3w^2t,$$
$$x^3z[zs - yt(\frac{y}{x})^3(\frac{w}{z})] = x^3z^2s - y^4wt, \qquad x^4[zs - yt(\frac{y}{x})^4] = x^4zs - y^5t.$$

4. Rational Quartic Space Curves

In this section, we are going to discuss some of the geometry behind the generators for the Rees algebra by studying rational quartic space curves. First, we would like to find implicit equations for rational quartic space curves. We need to consider both singular and non-singular curves. We begin with some generic results.

Lemma 4. *If a rational quartic space curve $\mathscr{C}$ is singular, then the curve $\mathscr{C}$ is a complete intersection of two quadric surfaces.*

If a rational quartic space curve $\mathscr{C}$ is non-singular, then the curve $\mathscr{C}$ is the projection of a rational normal quartic curve $S(4) \in \mathbb{P}^4$ from a point $Q \notin S(4)$.

Proof. It is known [14, Page 353] that the geometric genus of every rational quartic space curve $\mathscr{C}$ is zero, although the arithmetic genus of $\mathscr{C}$ may be either zero or one.

If $\mathscr{C}$ is singular, then the arithmetic genus of $\mathscr{C}$ is one, and $\mathscr{C}$ is a complete intersection of two quadric surfaces [13, Chapter 1, page 44].

If $\mathscr{C}$ is non-singular, then the arithmetic genus of $\mathscr{C}$ is zero, and $\mathscr{C}$ is contained in a unique non-singular quadric surface. Hence, by Theorem 6 [27], a rational non-singular quartic space curve is the projection of a rational normal curve $S(4) \in \mathbb{P}^4$ from a point $Q \notin S(4)$.

By Lemma 1, we know that a rational quartic space curve is contained in the quadric surface given $\mathrm{Sylv}_{s,t}(p,q) = 0$. Next we provide methods for finding implicit equations for both singular and non-singular rational quartic space curves using moving planes and μ-bases.

4.1. Implicit Equations of Singular Rational Quartic Space Curves

By Proposition 2, for a singular rational quartic space curve $\mathscr{C}$ with a μ-basis p, q, r, the only singular point P is the double point at the intersection of the axes of p and q.

Now suppose that P is a singular point on a rational quartic space curve $\mathscr{C}$. If we take two distinct points $\mathbf{F}(s_1, t_1)$, $\mathbf{F}(s_2, t_2)$ different from the singular point P on the space curve $\mathscr{C}$, then by Theorem 3.3 [26] there are two moving planes $\mathbf{L}_1, \mathbf{L}_2$ of degree one that follow the space curve $\mathscr{C}$ with axes $P\mathbf{F}(s_1, t_1)$ and $P\mathbf{F}(s_2, t_2)$. Moreover we can easily choose $\mathbf{F}(s_1, t_1), \mathbf{F}(s_2, t_2)$ so that these axes are distinct. Since the axes of $\mathbf{L}_1, \mathbf{L}_2$ are distinct, the moving planes $\mathbf{L}_1, \mathbf{L}_2$ are linearly independent. Therefore, $\mathbf{L}_1, \mathbf{L}_2, \mathbf{r}$ form another μ-basis for the curve $\mathscr{C}$.

Hence, without loss of generality, we can assume that the axes of $\mathbf{p}$ and $\mathbf{q}$ intersect the space curve $\mathscr{C}$ at two distinct points $\mathbf{F}(s_1, t_1)$ and $\mathbf{F}(s_2, t_2)$ other than the singular point P. In this case, the plane that contains the axes of $\mathbf{p}$ and $\mathbf{q}$ has the implicit equation

$$a(\mathbf{X}) = [\mathbf{F}(s_1, t_1), \mathbf{F}(s_2, t_2), P] \cdot \mathbf{X} = 0, \tag{10}$$

where $\mathbf{X} = (x, y, z, w)$ and $[\cdot]$ denotes the outer product.

Next we are going to find two quadric surfaces that contain the singular rational quartic space curve $\mathscr{C}$.

To compute the implicit equations of a singular rational quartic space curve, we write the μ-basis in the form:

$$p = p_1 s + p_0 t,\;\; q = q_1 s + q_0 t,\;\; r = r_2 s^2 + r_1 st + r_0 t^2,$$

where $p_1, p_0, q_1, q_0, r_2, r_1, r_0$ are homogeneous polynomials of degree 1 in x, y, z, w. Let $M(p, q, r)$ be the 3×3 coefficient matrix of the moving planes $(t_1 s - s_1 t)p, (t_2 s - s_2 t)q, r$. Since

$$\begin{aligned}
(t_1 s - s_1 t)p &= t_1 p_1 s^2 + (t_1 p_0 - s_1 p_1)st - s_1 p_0 t^2,\\
(t_2 s - s_2 t)q &= t_2 q_1 s^2 + (t_2 q_0 - s_2 q_1)st - s_2 q_0 t^2,
\end{aligned}$$

it follows that

$$M(p,q,r) = \begin{bmatrix} t_1 p_1 & t_1 p_0 - s_1 p_1 & -s_1 p_0 \\ t_2 q_1 & t_2 q_0 - s_2 q_1 & -s_2 q_0 \\ r_2 & r_1 & r_0 \end{bmatrix}. \tag{11}$$

Now we quote some results concerning the geometric construction of the implicit equations of the singular quartic space curve. Detailed proofs can be found in [17, Theorem 4.9 and Theorem 4.11].

Theorem 5. *Let $\mathscr{C}$ be a singular rational quartic space curve, and let $M = M(p, q, r)$ be the matrix in Equation (11) constructed from a μ-basis for the curve. Then*

$$\det(M) = ah,$$

where $a(x, y, z, w) = 0$ is the implicit equation of the plane that contains the axes of p and q, and $h(x, y, z, w) = 0$ is a quadric surface that contains the singular rational quartic space curve $\mathscr{C}$.

Theorem 6. *Let $\mathscr{C}$ be a singular rational quartic space curve, and let $M = M(p, q, r)$ be the matrix in Equation (11) constructed from a μ-basis p, q, r for the curve $\mathscr{C}$. Suppose nthe plane that contains the axes of p, q has implicit equation $a = 0$. Then the two quadric surfaces $f = \mathrm{Sylv}_{s,t}(p, q) = 0$ and $g = \frac{\det(M)}{a} = 0$ form set-theoretic complete intersection generators for the curve $\mathscr{C}$.*

Next we will use a very simple example to illustrate our method for finding both the singular point and the implicit equations for a singular rational quartic space curve.

Example 3. *Let the singular rational quartic space curve $\mathscr{C}$ be given as the image of the parameterization:*

$$(x, y, z, w) = (s^4,\, s^3 t,\, s^2 t^2,\, t^4).$$

Compute a μ-basis using the algorithm in [24]:

$$p = (y-z)s + (y-x)t,\ \ q = zs - yt,\ \ r = ws^2 - zt^2.$$

The axis of p is the line through the points $(0,0,0,1)$ and $(1,1,1,1)$, and the axis of q is the line through the points $(1,0,0,0)$ and $(0,0,0,1)$. These axes intersect at the point $(0,0,0,1)$—the only singular point, a point of order 2—which corresponds to the parameters $(s_0, t_0) = (0,1)$. The curve $\mathscr{C}$ intersects the axis of p at the point $(1,1,1,1)$ with parameters $(s_1, t_1) = (1,1)$, and intersects the axis of q at the point $(1,0,0,0)$ with parameters $(s_2, t_2) = (1,0)$. Now

$$\begin{aligned}
f &= \mathrm{Sylv}_{s,t}(p,q) = \det\begin{pmatrix} y-z & y-x \\ z & -y \end{pmatrix} = xz - y^2, \\
g &= \det(M(p,q,r)) = \det\begin{pmatrix} t_1p_1 & t_1p_0 - s_1p_1 & -s_1p_0 \\ t_2q_1 & t_2q_0 - s_2q_1 & -s_2q_0 \\ r_2 & r_1 & r_0 \end{pmatrix} = \det\begin{pmatrix} y-z & z-x & x-y \\ 0 & -z & y \\ w & 0 & -z \end{pmatrix} \\
&= z^2y - xyw - z^3 + xzw.
\end{aligned}$$

The implicit equation of the plane containing the axes of $p(s,t)$ and $q(s,t)$ is:

$$a = \begin{vmatrix} x & y & z & w \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = y - z = 0.$$

Therefore the other quadric surface that contains the quartic curve is

$$h = \frac{g}{a} = z^2 - xw.$$

Hence the implicit equations of the singular rational quartic curve $\mathscr{C}$ are (see Figure 1)

$$xz - y^2 = 0,\ \ z^2 - xw = 0.$$

4.2. Implicit Equations of Non-Singular Rational Quartic Space Curves

Next we consider the case where the rational quartic space curve $\mathscr{C}$ given by the parametrization f_0, f_1, f_2, f_3 is smooth. By Lemma 4, we know that the curve $\mathscr{C}$ is the image of the projection of the rational normal scroll $S(4)$ from a point $P \notin S(4) \subset \mathbb{P}^4$ to a hyperplane in $\mathbb{P}^4$.

Lemma 5. *The implicit equations of a non-singular rational quartic space curve $\mathscr{C}$ are given by 4 equations $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, $F_4 = 0$, one quadric surface and three cubic surfaces. Moreover, the quadric surface is given by $F_1 = \mathrm{Sylv}_{s,t}(p,q) = 0$, where p, q are the degree 1 elements of a μ-basis for $\mathscr{C}$.*

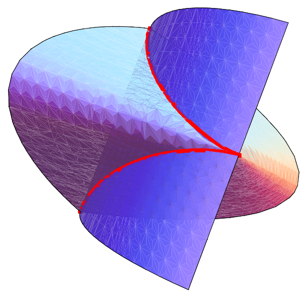

Figure 1: Set-theoretic complete intersection of a singular rational quartic space curve.

Proof. Let $P = (a, b, c, d, 1) \in \mathbb{P}^4$ be a point not contained in the rational normal curve $S(4)$ given by the parametric representation

$$x = s^4,\ \ y = s^3 t,\ \ z = s^2 t^2,\ \ w = st^3,\ \ u = t^4.$$

Without loss of generality, let the non-singular rational quartic space curve $\mathscr{C}$ be the image of the projection of the rational normal curve $S(4)$ from the point P to the hyperplane $u = 0$. Observe that the parametrization of the image can be described as $\mathbf{F}(s,t) = (s^4 - at^4, s^3 t - bt^4, s^2 t^2 - ct^4, st^3 - dt^4)$. Therefore the implicit equations of $\mathscr{C}$ are the generators of the following ideal:

$$\langle x - (s^4 - at^4),\ y - (s^3 t - bt^4),\ z - (s^2 t^2 - ct^4),\ w - (st^3 - dt^4)\rangle \bigcap \mathbb{K}(a,b,c,d)[x,y,z,w].$$

With the aid of a computer algebra system, we compute four polynomial generators: F_1, F_2, F_3, F_4; with one quadric F_1 and three cubics F_2, F_3, F_4–see below. Moreover, $F_1 = \mathrm{Sylv}_{s,t}(p,q)$, since by Lemma 4 any non-singular quartic space curve is contained in exactly one quadric.

$$\begin{aligned}
F_1 =&(-c+d^2)xz+(b-cd)xw+(c-d^2)y^2+(-b+cd)yz+(-a+2bd-c^2)yw+\\
&(a-bd)z^2+(-ad+bc)zw+(ac-b^2)w^2;\\
F_2 =&(c-d^2)y^2w+(-c+d^2)yz^2+(-b+cd)yzw+(2bd-2c^2)yw^2+(b-d^3)z^3+\\
&(-3bd+3cd^2)z^2w+(3bc-3c^2d)zw^2+(-b^2+c^3)w^3;\\
F_3 =&(-c^2+2cd^2-d^4)xyw+(b^2-3bcd+bd^3+c^3)xw^2+(c^2-2cd^2+d^4)y^2z+\\
&(bc-bd^2-c^2d+cd^3)y^2w+(-bc+bd^2+cd^3-d^5)yz^2+(-b^2+4bcd-2bd^3-\\
&3c^2d^2+2cd^4)yzw+(-ab+ad^3+2b^2d-bc^2-2bcd^2+2c^3d-c^2d^3)yw^2+\\
&(ab-ad^3-b^2d+bd^4)z^2w+(-abd-ac^2+2acd^2+b^2d^2+bc^2d-2bcd^3)zw^2+\\
&(2abc-abd^2-ac^2d-b^3+b^2cd-bc^3+bc^2d^2)w^3;\\
F_4 =&(-c^2+2cd^2-d^4)x^2w+(c^2-2cd^2+d^4)xyz+(bc-bd^2-c^2d+cd^3)xyw+\\
&(-bc+bd^2+cd^3-d^5)xz^2+(bcd-bd^3+c^3-3c^2d^2+2cd^4)xzw+(ab-
\end{aligned}$$

$3acd+2ad^3-bcd^2+2c^3d-c^2d^3)xw2+(-ab+3acd-2ad^3-bc^2+2bcd^2-$
$c^3d)yzw+(-a^2+2abd-2bc^2d+c^4)yw^2+(a^2-abd-ac^2-acd^2+ad^4+$
$b^2c-b^2d^2+bc^2d)z^2w+(-a^2d-abc+2abd^2+3ac^2d-2acd^3-$
$bc^3)zw^2+(2a^2c-a^2d^2-ab^2-2ac^3+ac^2d^2+b^2c^2)w^3.$

Lemma 5 is a theoretical result. Our next goal is to find simple explicit expressions for these four implicit equations. To proceed, we first dehomogenize the μ-basis elements by setting $t=1$, so that

$$p=p_1s+p_0,\ \ q=q_1s+q_0,\ \ r=r_2s^2+r_1s+r_0,$$

where $\deg(p_1)=\deg(p_0)=\deg(q_1)=\deg(q_0)=\deg(r_2)=\deg(r_1)=\deg(r_1)=1$.

If we take the resultants with respect to the variable s, then we generate the following three expressions:

$$\operatorname{Res}(p,q)=\det\begin{pmatrix}p_1 & p_0\\ q_1 & q_0\end{pmatrix}=\operatorname{Sylv}_{s,t}(p,q)=p_1q_0-p_0q_1,$$
$$\operatorname{Res}(p,r)=\det\begin{pmatrix}p_1 & 0 & r_2\\ p_0 & p_1 & r_1\\ 0 & p_0 & r_0\end{pmatrix}=r_2p_0^2-r_1p_1p_0+r_0p_1^2,$$
$$\operatorname{Res}(q,r)=\det\begin{pmatrix}q_1 & 0 & r_2\\ q_0 & q_1 & r_1\\ 0 & q_0 & r_0\end{pmatrix}=r_2q_0^2-r_1q_1q_0+r_0q_1^2,$$

where $\deg(\operatorname{Res}(p,q))=2$, $\deg(\operatorname{Res}(p,r))=3$, and $\deg(\operatorname{Res}(q,r))=3$ in x,y,z,w.

Remark 4. *If p,q,r are a μ-basis for the space curve $\mathscr{C}$, then $\mathscr{C}$ is contained in the three surfaces* $\operatorname{Res}(p,q)=0$, $\operatorname{Res}(p,r)=0$ *and* $\operatorname{Res}(q,r)=0$ *because each element of the μ-basis follows the curve.*

Lemma 6.

$$\gcd(\operatorname{Res}(p,q),\operatorname{Res}(p,r))=\gcd(\operatorname{Res}(p,q),\operatorname{Res}(q,r))=\gcd(\operatorname{Res}(p,r),\operatorname{Res}(q,r))=1.$$

Proof. Suppose that

$$\gcd(\operatorname{Res}(p,q),\operatorname{Res}(p,r))=g\neq 1.$$

Then for any point A satisfying $g(A)=0$, we have

$$\deg(\gcd(p(s,t,A),q(s,t,A)))\geq 1,\quad \text{and}\quad \deg(\gcd(p(s,t,A),r(s,t,A)))\geq 1.$$

Since $\deg(p)=\deg(q)=1$ and $\deg(r)=2$ in the variables s,t, we have

$$\deg(\gcd(p(s,t,A),q(s,t,A),r(s,t,A)))\geq 1.$$

Therefore by Proposition 1 the point A is on the rational space curve $\mathscr{C}$. Hence the surface $g = 0$ is contained in the space curve $\mathscr{C}$. This is impossible. Therefore, $\gcd(\mathrm{Res}(p,q), \mathrm{Res}(p,r)) = 1$. A similar argument applies to show that $\gcd(\mathrm{Res}(p,q), \mathrm{Res}(q,r)) = 1$.

Now suppose that $\gcd(\mathrm{Res}(p,r), \mathrm{Res}(q,r)) = g \neq 1$.

If $\deg(g) = 3$, then $\mathrm{Res}(p,r) = k\mathrm{Res}(q,r)$ for some nonzero constant k. Hence since the axis of p is contained in the cubic surface $\mathrm{Res}(p,r) = 0$, the axis of p is also contained in the surface $\mathrm{Res}(q,r) = 0$. Thus for a point A on the axis of p, we have $p(s,t,A) = 0$ and $\deg(\gcd(q(s,t,A), r(s,t,A))) \geq 1$. Hence

$$\deg(\gcd(p(s,t,A), q(s,t,A), r(s,t,A))) \geq 1,$$

so the point A is on the curve $\mathscr{C}$. Therefore, the axis line of p is contained in the rational space curve $\mathscr{C}$. This is impossible.

If $\deg(g) = 2$, then since a rational non-singular quartic curve can be contained in only one quadric surface, we would have $g = \mathrm{Res}(p,q)$, again contradicting the fact that $\gcd(\mathrm{Res}(p,q), \mathrm{Res}(p,r)) = 1$.

If $\deg(g) = 1$, then since $\mathscr{C}$ is a space curve contained in the cubic surface $\mathrm{Res}(p,r) = 0$, we would have $\mathscr{C}$ is contained in the quadric surface $\frac{\mathrm{Res}(p,r)}{g} = 0$. Again since the non-singular quartic space curve $\mathscr{C}$ can be contained in only one quadric surface, we would have $\frac{\mathrm{Res}(p,r)}{g} = \mathrm{Res}(p,q)$, again contradicting the fact that $\gcd(\mathrm{Res}(p,q), \mathrm{Res}(p,r)) = 1$. Therefore, $\gcd(\mathrm{Res}(p,r), \mathrm{Res}(q,r)) = 1$.

Now let $N(p,q,r)$ be the 3×3 coefficient matrix of the moving planes sp, tq, r. Since

$$sp = p_1 s^2 + p_0 st, \;\; tq = q_1 st + q_0 t^2, \;\; r = r_2 s^2 + r_1 st + r_0 t^2$$

it follows that

$$N(p,q,r) = \begin{bmatrix} p_1 & 0 & r_2 \\ p_0 & q_1 & r_1 \\ 0 & q_0 & r_0 \end{bmatrix}, \quad \text{and} \quad \det(N) = r_2 p_0 q_0 - r_1 p_1 q_0 + r_0 p_1 q_1. \tag{12}$$

Lemma 7. $\det(N(p,q,r))$ *is not identically zero.*

Proof. If $\det(N(p,q,r)) \equiv 0$, then the rows of the matrix N are linearly dependent over the ring $\mathbb{K}[x,y,z,w]$. Without loss of generality, by Lemma 3 we can assume that $p_1 = y, p_0 = -x, q_1 = w, q_0 = -z$. A dependence relation among the three rows of $N(p,q,r)$ would generate three linear equations

$$\begin{aligned} ay - bx &= 0 \\ bw - cz &= 0 \\ ar_2 + br_1 + cr_0 &= 0 \end{aligned}$$

with homogeneous polynomials $a(x,y,z,w), b(x,y,z,w), c(x,y,z,w)$ of the same degree in x,y,z,w at least one of which is not zero. From the first equation, we find that $a = \alpha x, b =$

αy. The second equation then shows that $\alpha = \beta z, c = \beta y w$. Substituting these results into the third equation gives $xzr_2 + yzr_1 + ywr_0 = 0$. In other words, (r_2, r_1, r_0) is a syzygy of (xz, yz, yw). But the syzygy module for (xz, yz, yw) is generated by $(y, -x, 0)$ and $(0, w, -z)$. Therefore, $r_2 = my, r_1 = -mx + nw, r_0 = -nz$ where m, n are necessarily constants because the r_i are linear in x, y, z, w. Hence, column three of the matrix N is a $\mathbb{K}$-linear combination of the first two columns. Thus, the moving plane r is a $\mathbb{K}$-linear combination of the moving planes sp and tq. But this is impossible because p, q, r are linearly independent over the ring $\mathbb{K}[s, t]$.

The surface $\det(N(p, q, r)) = 0$ is a cubic surface containing the non-singular rational quartic space curve $\mathscr{C}$ because the moving planes p, q, r follow the curve $\mathscr{C}$. Moreover, we have the following result.

Theorem 7. $\mathrm{Res}(p, q) = 0$, $\mathrm{Res}(p, r) = 0$, $\mathrm{Res}(q, r) = 0$, $\det(N(p, q, r)) = 0$ *are the implicit equations of the non-singular rational quartic space curve with μ-basis p, q, r.*

Proof. Since p, q, r are a μ-basis, the space curve $\mathscr{C}$ is contained in the surface $\det(N(p, q, r)) = 0$. By Remark 4, the space curve $\mathscr{C}$ is also contained in the surfaces $\mathrm{Res}(p, q) = 0$, $\mathrm{Res}(p, r) = 0$, and $\mathrm{Res}(q, r) = 0$. Hence the polynomials $\mathrm{Res}(p, q)$, $\mathrm{Res}(p, r)$, $\mathrm{Res}(q, r)$, and $\det(N(p, q, r))$ are contained in the ideal of the quartic space curve $\mathscr{C}$, which is generated by the four polynomials F_1, F_2, F_3, F_4 in Lemma 5.

To simplify our notation, let $N = N(p, q, r)$. Now we claim that the cubic surface $\det(N) = 0$ is irreducible. Otherwise, the cubic $\det(N)$ would have $\mathrm{Res}(p, q)$ as a factor, since a non-singular rational quartic space curve $\mathscr{C}$ is contained in exactly one quadric surface. But this is impossible, since there exists at least one point which lies on the surface $\mathrm{Res}(p, q) = 0$, but not on the surface $\det(N) = 0$. To see that such a point exists, recall that by Lemma 3 there is a linear transformation on the μ-basis elements p, q and a projective change of coordinates so that $p = ys - xt$, $q = ws - zt$, $r = r_2 s^2 + r_1 st + r_0 t^2$, and $\det(N) = r_2 xz + r_1 yz + r_0 yw$. Now observe that the surface $\mathrm{Res}(p, q) = xw - yz = 0$ contains the line $x = z = 0$. But there must be a point $P = (0, a, 0, b)$ on the line $x = z = 0$ such that $\det(N)(P) = r_0(P)ab \neq 0$. If not, then $r_0(P) = 0$, so $\deg(\gcd(\mathbf{p} \cdot P, \mathbf{q} \cdot P, \mathbf{r} \cdot P)) = \deg(\gcd(as, bs, r_2 s^2 + r_1 st)) = 1$. Hence by Proposition 1, $\mathscr{C}$ must contain a line, contradicting the assumption that $\mathscr{C}$ is a space curve.

Moreover, the three cubic surfaces $\det(N) = 0$, $\mathrm{Res}(p, r) = 0$ and $\mathrm{Res}(q, r) = 0$ are distinct. Indeed, since both p and q are axial moving planes, the axis of p is contained in the surfaces $\mathrm{Res}(p, r) = 0$ and $\det(N) = 0$, but is not contained in the surface $\mathrm{Res}(q, r) = 0$; and similarly, the axis of q is contained in the surfaces $\mathrm{Res}(q, r) = 0$ and $\det(N) = 0$, but is not contained in the surface $\mathrm{Res}(p, r) = 0$. Indeed a point X is on $\mathrm{Res}(q, r) = 0$ if and only if there is an (s, t) such that X is on the planes $q(s, t) = 0, r(s, t) = 0$. If X is also on the axis of p, then the point X would also be on $p(s, t) = 0$, and thus by Proposition 1 on the curve $\mathscr{C}$. Therefore the entire axis of p would be on $\mathscr{C}$, which is absurd. Hence the three surfaces $\det(N) = 0$, $\mathrm{Res}(p, r) = 0$ and $\mathrm{Res}(q, r) = 0$ are different from each other.

Furthermore, the three cubic surfaces $\det(N) = 0$, $\mathrm{Res}(p, r) = 0$ and $\mathrm{Res}(q, r) = 0$ are linearly independent. For suppose that there exist some nonzero constants a, b, c such that

$$a\mathrm{Res}(p, r) + b\mathrm{Res}(q, r) + c \det(N) \equiv 0.$$

Then

$$\langle \text{Res}(p,r), \det(N)\rangle = \langle \text{Res}(p,r), \text{Res}(q,r)\rangle,$$

so

$$\mathbb{V}(\langle \text{Res}(p,r), \det(N)\rangle) = \mathbb{V}(\text{Res}(p,r), \text{Res}(q,r)\rangle).$$

Hence the axis of p is contained in the surface $\text{Res}(q,r) = 0$. Contradiction. Therefore, the three cubic surfaces $\det(N) = 0$, $\text{Res}(p,r) = 0$ and $\text{Res}(q,r) = 0$ are linearly independent.

Thus, by Lemma 5 $\{\text{Res}(p,q)$, $\text{Res}(p,r)$, $\text{Res}(q,r)$, $\det(N)\}$ must be a set of generators for the ideal of the non-singular rational quartic space curve $\mathscr{C}$. Therefore, $\text{Res}(p,q) = 0$, $\text{Res}(p,r) = 0$, $\text{Res}(q,r) = 0$, $\det(N) = 0$ are the implicit equations of the non-singular rational quartic space curve.

We illustrate our method for finding the implicit equations of a non-singular rational quartic space curve with the following simple example.

Example 4. *Let the non-singular rational quartic space curve $\mathscr{C}$ be given as the image of the parameterization:*

$$(x, y, z, w) = (s^4,\ s^2t(s+t),\ st^2(s-t),\ t^4).$$

Compute a μ-basis using the algorithm in [24]

$$p = 2tx + (-2s+t)y + sz,\ q = tx - sy + (s+t)z + sw,\ r = (-t^2 - st)x + s^2y.$$

Then

$$\begin{aligned}
\text{Res}(p,q) &= y^2 - xz - 3yz + z^2 - 2xw - yw, \\
\text{Res}(p,r) &= z^3 - xzw + 2yzw - z^2w - 2xw^2 + yw^2, \\
\text{Res}(q,r) &= yz^2 - xyw + 2xzw + 3yzw - z^2w + yw^2, \\
\det(N(p,q,r)) &= xz^2 - x^2w + 2xyw - 3xzw - 3yzw + z^2w - xw^2 - yw^2
\end{aligned}$$

are the implicit equations of the curve $\mathscr{C}$ (see Figure 2).

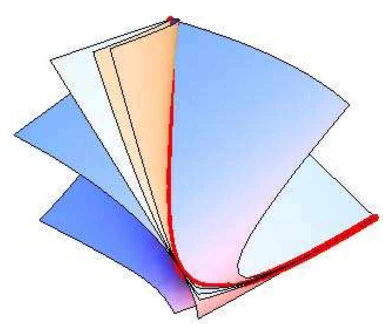

Figure 2: Set-theoretic generators of a non-singular rational quartic space curve.

4.3. Generators for the Rees Algebra Associated to Rational Quartic Space Curves

Now we can give a minimal set of generators for the Rees algebra associated to a rational quartic space curve using only the μ-basis of the curve.

Theorem 8. *Let p, q, r be a μ-basis for a rational quartic space curve. Then:*

1. *A minimal set of generators for the kernel K of the Rees algebra for a singular rational quartic space curve is given by*

$$p,\ q,\ r,\ \mathrm{Sylv}_{s,t}(p,q),\ \frac{\det(M(p,q,r))}{a(x,y,z,w)},$$

where a and M are defined in Equations (10) and (11).

2. *A minimal set of generators for the kernel K of the Rees algebra for a non-singular rational quartic space curve is given by*

$$p,\ q,\ r, \mathrm{Sylv}_{s,t}(p,r),\ \mathrm{Sylv}_{s,t}(q,r),\ \mathrm{Res}(p,q), \mathrm{Res}(p,r), \mathrm{Res}(q,r), \det(N(p,q,r)),$$

where $N(p,q,r)$ is defined in Equation (12).

Proof. We will prove the claim by comparing the generators listed above against the generators described in Theorems 1 and 3. First, we note that the μ-basis elements p, q, r and $\mathrm{Sylv}_{s,t}(p,q) = \mathrm{Res}(p,q)$ are among the generators in both theorems. We will focus therefore on the other generators.

For singular quartic space curves, recall that by Lemma 2 there is a linear transformation on the μ-basis elements p, q, and a projective change of coordinates so that $p = ys - xt$, $q = zs - yt$, and the singular point is located at $(0,0,0,1)$. Now the axial plane determined by the axes of p and q is defined by $a(x,y,z,w) = y = 0$.

In addition, notice that $p(1,0) = y$ and $q(0,1) = -y$. Therefore, since the μ-basis elements $p(s,t)$ and $q(s,t)$ follow the curve $\mathbf{F}(s,t)$, the quartic space curve $\mathbf{F}(s,t)$ intersects the axial plane $y = 0$ at the two points $\mathbf{F}(0,1)$ and $\mathbf{F}(1,0)$. Hence st must be a factor of y, so $y = st\beta$, where β is a homogenous form of degree 2 in s, t. Moreover, since the two μ-basis elements $p = ys - xt$, $q = zs - yt$ follow the curve $\mathbf{F}(s,t)$, we conclude that on the curve $\mathbf{F}(s,t)$:

$$x = s^2\beta,\ \ y = st\beta,\ \ z = t^2\beta, \quad \text{where } \beta \text{ is a homogeneous form in } s,t \text{ of degree 2.} \tag{13}$$

In fact, the roots of $\beta(s,t) = 0$ are the two parameters corresponding to the singular point $(0,0,0,1)$ on the curve $\mathbf{F}(s,t)$. We will study the generator $\det(M(p,q,r))/a(x,y,z,w)$ by investigating the roots of the polynomial $\beta(s,t) = 0$ in the following three cases.

Case 1: Neither $(0,1)$ nor $(1,0)$ is a root of $\beta(s,t) = 0$. Then $\mathbf{F}(0,1)$ and $\mathbf{F}(1,0)$ are both non-singular points on the curve $\mathbf{F}(s,t)$. Setting $(s_1,t_1) = (0,1)$ and $(s_2,t_2) = (1,0)$ in Equation (11) yields:

$$\det(M(p,q,r)) = \det\begin{pmatrix} y & -x & 0 \\ 0 & -z & y \\ r_2 & r_1 & r_0 \end{pmatrix} = -r_2xy - r_1y^2 - r_0yz;$$

$$\frac{\det(M(p,q,r))}{a(x,y,z,w)} = -[r_2x + r_1y + r_0z] = -x[r_2 + r_1(\frac{y}{x}) + r_0(\frac{z}{x})] = -xr_1'(\frac{y}{x}, \frac{z}{x}).$$

Thus, the above generators are the same as the generators described in Theorem 1.

Case 2: Either $(0,1)$ or $(1,0)$ is a root of $\beta(s,t) = 0$, but $\beta(1,1) \neq 0$ and $\beta(1,-1) \neq 0$. Then by Equation (13) $\mathbf{F}(s_1,t_1) = \mathbf{F}(1,-1) = (1,-1,1,*)$ and $\mathbf{F}(s_2,t_2) = \mathbf{F}(1,1) = (1,1,1,*)$ are two distinct non-singular points on the curve $\mathbf{F}(s,t)$. Therefore, we may choose a new μ-basis

$$p+q = (y+z)s - (x+y)t, \;\; p-q = (y-z)s - (x-y)t, \;\; r.$$

Now the axial moving plane is $a(x,y,z,w) = \det \begin{pmatrix} x & y & z & w \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & * \\ 1 & -1 & 1 & * \end{pmatrix} = 2(x-z) = 0$. In this case, Equation (11) again yields:

$$\det(M(p,q,r)) = \det \begin{pmatrix} -(y+z) & x-z & x+y \\ y-z & -x+z & x-y \\ r_2 & r_1 & r_0 \end{pmatrix} = 2(x-z)[r_2x + r_1y + r_0z];$$

$$\frac{\det(M(p,q,r)}{a(x,y,z,w)} = -[r_2x + r_1y + r_0z] = -x[r_2 + r_1(\frac{y}{x}) + r_0(\frac{z}{x})] = -xr_1'(\frac{y}{x}, \frac{z}{x}).$$

Thus, the above generators are the same as the generators described in Theorem 1.

Case 3: Either $(0,1)$ or $(1,0)$ and either $(1,1)$ or $(1,-1)$ is a root of $\beta(s,t) = 0$. Without loss of generality, assume that $\beta(0,1) \neq 0$ and $\beta(1,-1) \neq 0$. Then $\mathbf{F}(1,0) = \mathbf{F}(1,1) = (0,0,0,1)$ is the singular point, and by Equation (13) $\mathbf{F}(s_1,t_1) = \mathbf{F}(0,1) = (0,0,1,*)$ and $\mathbf{F}(s_2,t_2) = \mathbf{F}(1,-1) = (1,-1,1,*)$ are two distinct non-singular point on the curve $\mathbf{F}(s,t)$. Therefore, we may choose a new μ-basis

$$p = ys - xt, \;\; p+q = (y+z)s - (x+y)t, \;\; r.$$

Now the axial moving plane is $a(x,y,z,w) = \det \begin{pmatrix} x & y & z & w \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & * \\ 1 & -1 & 1 & * \end{pmatrix} = x + y = 0$. In this case, Equation (11) again gives:

$$\det(M(p,q,r)) = \det \begin{pmatrix} y & -x & 0 \\ -(y+z) & x-z & x+y \\ r_2 & r_1 & r_0 \end{pmatrix} = -(x+y)(r_2x + r_1y + r_0z);$$

$$\frac{\det(M(p,q,r))}{a(x,y,z,w)} = -[r_2x + r_1y + r_0z] = -x[r_2 + r_1(\frac{y}{x}) + r_0(\frac{z}{x})] = -xr_1'(\frac{y}{x}, \frac{z}{x}).$$

Thus, the above generators are the same as the generators described in Theorem 1.

For non-singular quartic space curves, recall that by Lemma 3 there is a linear transformation on the μ-basis elements p, q, and a projective change of coordinates so that $p = ys - xt$, $q = ws - zt$. Then

$$\begin{aligned}
\mathrm{Res}(p,r) &= r_2x^2 + r_1xy + r_0y^2 = x^2[r_2 + r_1(\frac{y}{x}) + r_0(\frac{y}{x})^2] = x^2r_2'(\frac{y}{x}, \frac{w}{z}),\\
\mathrm{Res}(q,r) &= r_2z^2 + r_1zw + r_0w^2 = z^2[r_2 + r_1(\frac{w}{z}) + r_0(\frac{w}{z})^2] = z^2r_0'(\frac{y}{x}, \frac{w}{z}),\\
\det(N) &= r_2xz + r_1xw + r_0yw = xz[r_2 + r_1(\frac{w}{z}) + r_0(\frac{w}{z})(\frac{y}{x})] = xzr_1'(\frac{y}{x}, \frac{w}{z}),\\
\mathrm{Sylv}_{s,t}(p,r) &= r_2xs + r_1xt + r_0yt = x[r_2s + r_1t + r_0t(\frac{y}{x})] = xr_{1,0}'(\frac{y}{x}, \frac{w}{z}),\\
\mathrm{Sylv}_{s,t}(q,r) &= r_2zs + r_1zt + r_0wt = z[r_2s + r_1t + r_0t(\frac{w}{z})] = zr_{0,1}'(\frac{y}{x}, \frac{w}{z}).
\end{aligned}$$

Thus, the generators listed in the statement of this theorem are exactly the same as the generators listed in the statement of Theorem 3.

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