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$S-$Linear Almost Distributive Lattices

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Abstract. The concept of an $S-$Linear ADL is defined and characterized in terms of the $S-$prime ideals and $S-$prime filters. Equivalent condition for an ADL R to become a (dually)$B-$relatively normal ADL in terms of minimal prime ideals(filters) and $B-$maximal ideals(filters) is obtained, where B is the Birkhoff centre of R.

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1. Introduction

The concepts of $S-$completely normal lattice and dually $S-$completely normal lattice were given by Cignoli [2]. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [9] as a common abstraction of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. The concept of an ideal in an ADL was introduced in [9] analogous to that in a distributive lattice and it was observed that the set $PI(R)$ of all principal ideals of R forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. In our paper [5], we introduced the concept of an $S-$normal ADL R, where S is a uni subADL of R and obtained necessary and sufficient conditions for an ADL R to become an $S-$normal ADL in terms of $S-$prime filters, $S-$maximal filters. $B-$normal ADLs were also studied, where B is the Birkhoff centre of R. In this paper, we define the concept of an $S-$relative annihilator of any two elements of R and characterize an $S-$normal ADL in terms of $S-$relative annihilators.

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We introduce the concepts of $S-$relatively normal ADL and dually $S-$relatively normal ADL. We characterize the (dually)$S-$relatively normal ADL in terms of $S-$prime filters(ideals). If B is the Birkhoff centre of R, then we define the concept of (dually)$B-$relatively normal ADL and characterize it in terms of minimal prime ideals(filters) and $B-$maximal ideals(filters)of R.

2. Preliminaries

Definition 1 ([9]). *An Almost Distributive Lattice with zero or simply ADL is an algebra* $(R,\vee,\wedge,0)$ *of type* $(2,2,0)$ *satisfying:*

1. $(x\vee y)\wedge z=(x\wedge z)\vee(y\wedge z)$
2. $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$
3. $(x\vee y)\wedge y=y$
4. $(x\vee y)\wedge x=x$
5. $x\vee(x\wedge y)=x$
6. $0\wedge x=0$
7. $x\vee 0=x$.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0\in X$. Define the binary operations $\vee,\wedge$ on X by

$$x\vee y=\begin{cases}x & \text{if } x\neq x_0\\ y & \text{if } x=x_0\end{cases}\qquad x\wedge y=\begin{cases}y & \text{if } x\neq x_0\\ x_0 & \text{if } x=x_0.\end{cases}$$

Then $(X,\vee,\wedge,x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL. If $(R,\vee,\wedge,0)$ is an ADL, for any $a,b\in R$, define $a\leq b$ if and only if $a=a\wedge b$ (or equivalently, $a\vee b=b$), then $\leq$ is a partial ordering on R.

Theorem 1 ([9]). *If* $(R,\vee,\wedge,0)$ *is an ADL, for any* $a,b,c\in R$*, we have the following:*

1. $a\vee b=a\Leftrightarrow a\wedge b=b$
2. $a\vee b=b\Leftrightarrow a\wedge b=a$
3. $\wedge$ *is associative in* R
4. $a\wedge b\wedge c=b\wedge a\wedge c$
5. $(a\vee b)\wedge c=(b\vee a)\wedge c$
6. $a\wedge b=0\Leftrightarrow b\wedge a=0$

7. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

8. $a \wedge (a \vee b) = a,\ (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

9. $a \leq a \vee b$ *and* $a \wedge b \leq b$

10. $a \wedge a = a$ *and* $a \vee a = a$

11. $0 \vee a = a$ *and* $a \wedge 0 = 0$

12. *If* $a \leq c,\ b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

13. $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL R a distributive lattice. That is

Theorem 2 ([9])**.** *Let* $(R, \vee, \wedge, 0)$ *be an ADL with 0. Then the following are equivalent:*

1. $(R, \vee, \wedge, 0)$ *is a distributive lattice*

2. $a \vee b = b \vee a$*, for all* $a, b \in R$

3. $a \wedge b = b \wedge a$*, for all* $a, b \in R$

4. $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*, for all* $a, b, c \in R$.

As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$.

Theorem 3 ([9])**.** *Let* R *be an ADL and* $m \in R$*. Then the following are equivalent:*

1. m *is maximal with respect to* $\leq$

2. $m \vee a = m$*, for all* $a \in R$

3. $m \wedge a = a$*, for all* $a \in R$

4. $a \vee m$ *is maximal, for all* $a \in R$.

As in distributive lattices [1, 3], a non-empty sub set I of an ADL R is called an ideal of R if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in R$. Also, a non-empty subset F of R is said to be a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in R$.

The set $I(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion in which, for any $I, J \in I(R)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of R is called a prime ideal if, for any $x, y \in R$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of R is said to be maximal if it is not properly contained in any proper ideal of R. It can be observed that every maximal ideal of R is a prime ideal. Every proper ideal of R is contained in a maximal ideal. For any subset S of R the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in R \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq R$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in R \text{ and } n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

Theorem 4 ([9])**.** *For any x, y in R the following are equivalent:*

1. $(x] \subseteq (y]$
2. $y \wedge x = x$
3. $y \vee x = y$
4. $[y) \subseteq [x)$.

For any $x, y \in R$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(R)$ of all principal ideals of R is a sublattice of the distributive lattice $I(R)$ of ideals of R.

3. $S-$relatively Normal ADLs

If R is an ADL and S is a subADL with 0, then the concept of $S-$normality in R introduced in [5] and its properties were discussed. R. Cignoli [2] gave the concept of $S-$completely normal lattice. In this section we define the concept of $S-$relative normality in an ADL R through its principal ideal lattice $PI(R)$. A subADL of an ADL with 0 carries the usual meaning where 0 is treated as a nullary operation. Through out this paper R represents an ADL and S stands for a subADL of R with 0. By a uni subADL of R we mean a subADL of R containing all maximal elements of R.

In [8], the concept of relative annihilator in an ADL was given.

If $x, y \in R$, then $\lfloor x, y \rfloor = \{a \in R \mid y \wedge a \wedge x = a \wedge x\}$ is called a relative annihilator in R and $\lfloor x, 0 \rfloor = (x)^*$ is the annihilator of x in R. Now we define the concept of an $S-$relative annihilator in R as follows.

Definition 2. *Let $x, y \in R$. Define $\lfloor x, y \rfloor_S = \{a \in S \mid y \wedge a \wedge x = a \wedge x\}$. We call $\lfloor x, y \rfloor_S$ an $S-$relative annihilator.*
It can be observed that $a \in \lfloor x, y \rfloor_S$ iff $y = y \vee (a \wedge x)$. Clearly $\lfloor x, y \rfloor_S$ is an ideal of S. The following result can be verified easily.

Lemma 1. *Let $x, y \in R$. Then for any $a \in S$, $a \in \lfloor x, y \rfloor_S$ iff $x \wedge a \leq y \wedge a$.*

The following definition is taken from [5].

Definition 3. *Let S be a subADL of R. An ideal I of R is called an S-ideal of R if I is generated by the set $I \cap S(I = (I \cap S])$. An $S-$ideal I is called an $S-$prime ideal of R if $I \cap S$ is a prime ideal of S and $S-$maximal ideal if $I \cap S$ is a maximal ideal of S. It can be observed that every $S-$maximal ideal of R is an $S-$prime ideal.*

The concepts of $S-$ filters, $S-$prime filters and $S-$maximal filters are defined analogously. Now, the following lemma can be verified easily.

Lemma 2. *Let R be an ADL, S a subADL of R and F_1 a filter of S. Then the filter F of R is generated by F_1 is $S-$filter of R and $F_1 = F \cap S$.*

We recall the following from [5].

Definition 4. *Let R be an ADL with maximal elements and S a uni subADL of R. R is called $S-$normal if for any $x, y \in R$ such that $x \wedge y = 0$ then there exist elements $a, b \in S$ such that $x \wedge a = 0 = y \wedge b$ and $a \vee b$ is a maximal element.*

In the following theorem, we characterize the $S-$normal ADL in terms of $S-$relative annihilators.

Theorem 5. *Let R be an ADL with maximal elements and S a uni subADL of R. Then the following conditions are equivalent:*

1. *R is $S-$normal*
2. *$\lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S = S$, for any $x, y \in R$ with $x \wedge y = 0$*
3. *For any prime filter F of S and for any $x, y \in R$ with $x \wedge y = 0$, there exists $a \in F$ such that $x \wedge a$ and $y \wedge a$ are comparable.*

Proof.

$(1) \Rightarrow (2)$: Assume that R is an $S-$normal ADL. Let $x, y \in R$ such that $x \wedge y = 0$. Then there exist $a, b \in S$ such that $a \wedge x = 0 = b \wedge y$ and $a \vee b$ is a maximal element. That implies $y \wedge a \wedge x = a \wedge x = 0 = x \wedge b \wedge y = b \wedge y$. Therefore $\lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S = S$.

$(2) \Rightarrow (3)$: Let F be any prime filter of S and $x, y \in R$ such that $x \wedge y = 0$. Then $\lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S = S$. Let m be any maximal element in S. Then $m = a \vee b$, for some $a \in \lfloor x, y \rfloor_S$ and $b \in \lfloor y, x \rfloor_S$. That implies $a \wedge x = y \wedge a \wedge x = 0$ and $b \wedge y = x \wedge b \wedge y = 0$. Since $a \vee b \in F$, we get either $a \in F$ or $b \in F$. Suppose $a \in F$. Since $a \in \lfloor x, y \rfloor_S$, we get $x \wedge a \leq y \wedge a$. Thus there is an element $a \in F$ such that $x \wedge a$ and $y \wedge a$ are comparable. Similarly, we get $x \wedge b$ and $y \wedge b$ are comparable, if $b \in F$.

$(3) \Rightarrow (1)$: Let $x, y \in R$ such that $x \wedge y = 0$. Suppose that $((x)^* \cap S) \vee ((y)^* \cap S) \neq S$. Then there exists a maximal ideal M of S such that $((x)^* \cap S) \vee ((y)^* \cap S) \subseteq M$. That implies $S \setminus M$ is a prime filter of S. By (3), there exists $x \in S \setminus M$ such that $x \wedge a$ and $y \wedge a$ are comparable. Suppose $x \wedge a \leq y \wedge a$. Then $x \wedge a = x \wedge a \wedge y \wedge a = 0$. Then $a \in \lfloor x, y \rfloor_S \cap (S \setminus M)$, which is a contradiction. Therefore $((x)^* \cap S) \vee ((y)^* \cap S) = S$. Hence R is $S-$normal.

In [7], the concept of relatively normal ADL was given as follows.

Definition 5. *Let R be an ADL with maximal elements. Then R is called relatively normal if for any $x, y \in R$, there exist $a, b \in R$ such that $y \wedge a \wedge x = a \wedge x,\ x \wedge b \wedge y = b \wedge y$ and $a \vee b$ is a maximal element.*

The following definition is taken from Cignoli [2].

Definition 6. *Let $(L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice and S a sublattice of L containing 0 and 1. Then L is called $S-$completely normal, if for any $x, y \in L$, there exist $a, b \in S$ such that $x \wedge a \leq y,\ y \wedge b \leq x$ and $a \vee b = 1$.*

Now we define the concept of an $S-$relatively normal ADL in the following.

Definition 7. *Let R be an ADL with maximal elements and S a uni subADL of R. R is called $S-$relatively normal if $PI(R)$ is $PI(S)-$completely normal lattice.*

The following lemma can be verified directly.

Lemma 3. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is $S-$relatively normal if and only if for any $x, y \in R$, there exist $a, b \in S$ such that $y \wedge a \wedge x = a \wedge x,\ x \wedge b \wedge y = b \wedge y$ and $a \vee b$ is a maximal element.*

Example 1. *Let A be a discrete ADL and B a Boolean algebra. Then $R = A \times B$ is an ADL. Let D be a subADL of A containing at least two elements. Then $S = D \times B$ is a subADL of R. Let $x, y \in R$. Then $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Let t be any non-zero element of D. Suppose $x_1, y_1 \neq 0$. Write $a = (t, y_2 \vee x_2')$ and $b = (t, x_2 \vee y_2')$. Now, $y \wedge a \wedge x = (y_1, y_2) \wedge (t, y_2 \vee x_2') \wedge (x_1, x_2) = (y_1 \wedge t \wedge x_1, y_2 \wedge (y_2 \vee x_2') \wedge x_2) = (x_1, y_2 \wedge x_2) = a \wedge x$ and $x \wedge b \wedge y = (x_1 \wedge t \wedge y_1, x_2 \wedge (x_2 \vee y_2') \wedge y_2) = (y_1, x_2 \wedge y_2) = b \wedge y$. Also $a \vee b = (t, 1)$. Now, suppose $x_1 = 0$ and $y_1 \neq 0$. Take $a = (t, y_2 \vee x_2')$ and $b = (0, x_2 \vee y_2')$. Now, $y \wedge a \wedge x = (y_1 \wedge t \wedge 0, y_2 \wedge (y_2 \vee x_2') \wedge x_2) = (0, y_2 \wedge x_2) = a \wedge x$ and $x \wedge b \wedge y = (0 \wedge 0 \wedge y_1, x_2 \wedge (x_2 \vee y_2') \wedge y_2) = (0, x_2 \wedge y_2) = b \wedge y$. Clearly $a \vee b = (t, 1)$. Thus R is an $S-$relatively normal ADL.*

Lemma 4. *Let R be an ADL with maximal elements and S a uni subADL of R. If R is $S-$relatively normal, then, for each pair $a, b \in S$ such that $a < b$, the segment $[a, b]$ is an $S \cap [a, b]-$normal lattice.*

Proof. Let $x, y \in [a, b]$ such that $x \wedge y = a$. Since R is $S-$relatively normal, there exist $c, d \in S$ such that $y \wedge c \wedge x = c \wedge x,\ x \wedge d \wedge y = d \wedge y$ and $c \vee d$ is a maximal element. Now, take $c_1 = a \vee (c \wedge b)$ and $d_1 = a \vee (d \wedge b)$. Clearly $c_1, d_1 \in [a, b] \cap S$. Now, $c_1 \wedge x = (a \vee (c \wedge b)) \wedge x = (a \wedge x) \vee (c \wedge b \wedge x) = a \vee (c \wedge x) = a \vee (c \wedge y \wedge x) = a \vee (c \wedge a) = a$ and $d_1 \wedge y = (a \vee (d \wedge b)) \wedge y = (a \wedge y) \vee (d \wedge b \wedge y) = a \vee (d \wedge y) = a \vee (d \wedge x \wedge y) = a \vee (d \wedge a) = a$. Clearly $c_1 \vee d_1 = b$. Therefore $[a, b]$ is $S \cap [a, b]-$normal lattice.

The following two results can be verified easily.

Lemma 5. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is $S-$relatively normal if and only if for any $x, y \in R$, $\lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S = S$.*

Lemma 6. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is $S-$relatively normal if and only if for any prime filter F of S and for any $x, y \in R$, there exists $a \in F$ such that $x \wedge a$ and $y \wedge a$ are comparable.*

Theorem 6. *Let R be an ADL with maximal elements, S a uni subADL of R, F an $S-$filter of R and K a non-empty subset of R, which is closed under the operation join such that $F \cap K = \emptyset$. Then there exists an $S-$prime filter P of R such that $F \subseteq P$ and $P \cap K = \emptyset$.*

Theorem 7. *Let R be an ADL with maximal elements and S a uni subADL of R. Then the following conditions are equivalent:*

1. *R is $S-$relatively normal*
2. *For each pair $x, y \in R$, there is no proper ideal of S contain both $\lfloor x, y \rfloor_S$ and $\lfloor y, x \rfloor_S$*
3. *The set of all filters of R that contain a given $S-$prime filter of R form a chain*
4. *The set of all prime filters of R that contain a given $S-$prime filter of R form a chain*
5. *Any proper filter of R that contain a given $S-$prime filter of R is prime.*

Proof.

$(1) \Rightarrow (2)$: It follows from lemma 5.

$(2) \Rightarrow (3)$: Assume (2). Suppose P is an $S-$prime filter of R and F_1, F_2 are two filters of R such that $P \subseteq F_1$ and $P \subseteq F_2$. Suppose $F_1 \nsubseteq F_2$ and $F_2 \nsubseteq F_1$. Choose $x \in F_1 \setminus F_2$ and $y \in F_2 \setminus F_1$. Let $a \in \lfloor x, y \rfloor_S$. Then $y \wedge a \wedge x = a \wedge x$. Suppose $a \notin S \setminus (P \cap S)$. Then $a \in P \cap S$. That implies $a \in F_1$ and $x \in F_1$. Hence $a \wedge x \in F_1$. Thus $y \vee (a \wedge x) = y \in F_1$, which is a contradiction. Therefore $a \in S \setminus (P \cap S)$. Hence $\lfloor x, y \rfloor_S \subseteq S \setminus (P \cap S)$ and similarly, we have $\lfloor y, x \rfloor_S \subseteq S \setminus (P \cap S)$. Since $S \setminus (P \cap S)$ is a prime ideal of S, this is a contradiction.

$(3) \Rightarrow (4)$: Clear.

$(4) \Rightarrow (5)$: Assume (4). Let P be an $S-$prime filter of R and F a proper filter of R such that $P \subseteq F$. Suppose F is not prime filter of R. Then there exist $a, b \in R$ such that $a \notin F$, $b \notin F$ and $a \vee b \in F$. Then there exist prime filters P_a, P_b of R such that $a \notin P_a, b \notin P_b$ and $F \subseteq P_a \cap P_b$. Since $a \vee b \in P_a \cap P_b$, we get $b \in P_a$ and $a \in P_b$. Therefore $P_a \nsubseteq P_b$ and $P_b \nsubseteq P_a$, which is a contradiction. Hence F is a prime filter of R.

$(5) \Rightarrow (1)$: Assume (5). Let $x, y \in R$. Suppose $\lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S \neq S$. Let m be any maximal element in R. Then $m \notin \lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S$ and hence there exists an prime filter P' of S such that $(\lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S) \cap P' = \emptyset$. So that $\lfloor x, y \rfloor_S \cap P' = \emptyset$ and $\lfloor y, x \rfloor_S \cap P' = \emptyset$. Let P be the filter of R generated by P'. By the lemma 2, we get that P is an $S-$prime filter of R and $P' = P \cap S$. If $0 \in P \vee [x \vee y)$, then $0 = p \wedge (x \vee y)$ and hence $p \wedge x = 0$ and $p \wedge y = 0$. Since $p \in P$, there exists $s \in P \cap S = P'$ such that $p \vee s = p$. Now, we prove that the filter $P \vee [x \vee y)$ is a proper filter of R. Now, $s \wedge x = p \wedge s \wedge x = 0$. So that $s \in \lfloor x, y \rfloor_S \cap P'$, which is a contradiction. Therefore $P \vee [x \vee y)$ is a proper filter of R containing P. By our assumption, $P \vee [x \vee y)$ is a prime filter of R. Without loss of generality, suppose $x \in P \vee [x \vee y)$. Then $x = t \wedge (x \vee y)$, for some $t \in P$. Since $t \in P$, there exists $s_1 \in P \cap S$ such that $t \vee s_1 = t$. Now, $s_1 \wedge x = s_1 \wedge t \wedge (x \vee y) = (s_1 \wedge x) \vee (s_1 \wedge y)$ and hence $s_1 \wedge y = s_1 \wedge x \wedge s_1 \wedge y = x \wedge s_1 \wedge y$. That implies $s_1 \in \lfloor y, x \rfloor_S \cap P$, which is a contradiction. Therefore $\lfloor x, y \rfloor_S \vee \lfloor y, x \rfloor_S = S$.

Corollary 1. *Let R be an ADL with maximal elements and S_1, S_2 uni subADLs of R such that $S_1 \subseteq S_2$. Then the following conditions are equivalent:*

1. *R is S_1-relatively normal*
2. *R is S_2-relatively normal and the filters generated in S_2 by prime filters of S_1 are prime.*

Proof.

$(1) \Rightarrow (2)$: Assume that R is S_1-relatively normal. Clearly R is S_2-relatively normal and S_2 is S_1-relatively normal. Let P be a prime filter of S_1. We have to prove that $[P)$ is an S_1-prime filter of S_2, where $[P) = \{s \vee a \mid s \in S_2 \text{ and } a \in P\}$. Let $x, y \in [P)$. Then $x = s_1 \vee a_1$ and $y = s_2 \vee a_2$, for some $s_1, s_2 \in S_2$ and $a_1, a_2 \in P$. Now, $x \wedge y = (s_1 \vee a_1) \wedge (s_2 \vee a_2) = (s_1 \wedge (s_2 \vee a_2)) \vee (a_1 \wedge (s_2 \vee a_2)) = (s_1 \wedge (s_2 \vee a_2)) \vee ((a_1 \wedge s_2) \vee (a_1 \wedge a_2))$ and hence $(x \wedge y) \wedge (a_1 \wedge a_2) = ((s_1 \wedge (s_2 \vee a_2)) \vee ((a_1 \wedge s_2) \vee (a_1 \wedge a_2))) \wedge (a_1 \wedge a_2) = (a_1 \wedge a_2)$. Thus $x \wedge y = (x \wedge y) \vee (a_1 \wedge a_2)$ and hence $x \wedge y \in [P)$. Let $x \in [P)$ and $r \in S_2$. Then $x = s \vee a$, for some $s \in S_2$ and $a \in P$. Now, $(r \vee x) \wedge a = (r \vee (s \vee a)) \wedge a = a$ and hence $r \vee x = (r \vee x) \vee a$. Therefore $r \vee x \in [P)$. Hence $[P)$ is a filter of S_2. Let $x \in [P)$. Then $x = s \vee a$, for some $s \in S_2$ and $a \in P$. Now, $x \vee a = (s \vee a) \vee a = s \vee a = x$. Hence $[P)$ is an S_1-filter of R. Let $a, b \in S_1$ such that $a \vee b \in [P) \cap S_1$. Then $a \vee b = s \vee x$, for some $s \in S_2$ and $x \in P$. Now, $x = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) \in P$ (since $x \in P$). That implies either $a \wedge x \in P$ or $b \wedge x \in P$. Suppose $a \wedge x \in P$. Then $a \wedge x \in [P)$. That implies $a \vee (a \wedge x) \in [P) \cap S_1$. Hence $a \in [P) \cap S_1$. Thus $[P)$ is an S_1- prime filter of S_2. Since S_2 is S_1-relatively normal, $[P)$ is a prime filter of S_2.

$(2) \Rightarrow (1)$: Assume that R is S_2-relatively normal and the filters generated in S_2 by prime filters of S_1 are prime. Let P be an S_1-prime filter of R. Let F be a proper filter of R such that $P \subseteq F$. Clearly P is an S_2-filter of R. We have to prove that $[P \cap S_1) = P \cap S_2$. Let $a \in [P \cap S_1)$. Then $a = s \vee x$, for some $s \in S_2$ and $x \in P \cap S_1$. That implies $a \in P \cap S_2$. Therefore $[P \cap S_1) \subseteq P \cap S_2$. Let $a \in P \cap S_2$. Then there exists $s \in P \cap S_1$ such that $a \vee s = a$. That implies $a \in [P \cap S_1)$. Hence $P \cap S_2$ is a prime filter of S_2. That implies P is an S_2-prime filter of R. Therefore F is prime filter of R. Thus R is an S_1-relatively normal ADL.

Corollary 2. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is $S-$relatively normal if and only if R is relatively normal and the $S-$prime filters of R are prime.*

Proof. Take $S_1 = S$ and $S_2 = R$ in the above corollary.

Let R be an ADL and F a filter in R. Then the relation $\psi(F) = \{(x, y) \in R \times R \mid x \wedge t = y \wedge t$, for some $t \in F\}$ is a congruence relation on R and the set $R/\psi(F) = \{x/\psi(F) \mid x \in R\}$ is an ADL. Let $\prod$ be the natural homomorphism from R onto $R/\psi(F)$ defined by $\prod(x) = x/\psi(F)$ for all $x \in R$.

Theorem 8. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is $S-$relatively normal if and only if $R/\psi(F)$ is a chain, for each prime filter F of S.*

Proof. Assume that R is $S-$relatively normal. Let $x/\psi(F), y/\psi(F) \in R/\psi(F)$. Since $x, y \in R$, by theorem 6, there exists $a \in F$ such that $x \wedge a$ and $y \wedge a$ are comparable. With out loss of generality, suppose $x \wedge a \leq y \wedge a$. Then $x \wedge a = x \wedge a \wedge y \wedge a = x \wedge y \wedge a$. That implies $(x, x \wedge y) \in \psi(F)$ and hence $x/\psi(F) = (x \wedge y)/\psi(F) = x/\psi(F) \wedge y/\psi(F)$. Therefore $x/\psi(F) \leq y/\psi(F)$. Hence $R/\psi(F)$ is a chain. Conversely, assume that $R/\psi(F)$ is a chain. Let $x, y \in R$. Then $x/\psi(F), y/\psi(F) \in R/\psi(F)$. Since $R/\psi(F)$ is a chain, $x/\psi(F), y/\psi(F)$ are comparable. With out loss of generality, suppose $x/\psi(F) \leq y/\psi(F)$. Then $x/\psi(F) = x/\psi(F) \wedge y/\psi(F)$. That implies $(x, x \wedge y) \in \psi(F)$. Then $x \wedge a = x \wedge y \wedge a$, for some $a \in F$. Therefore $x \wedge a \leq y \wedge a$. Thus R is an $S-$relatively normal.

The following result follows directly from the above theorem.

Theorem 9. *Each $S-$relatively normal ADL is a subdirect product of the bounded chains $R \setminus P$, where P runs through the set of all prime ideals of S.*

4. Dually $S-$relatively Normal ADLs

The concept of a dually $S-$completely normal lattices was given by Cignoli [2]. In this section we define the concept of dually $S-$relative normality in an ADL R through its principal filter lattice $PF(R)$. We begin with the following.

Definition 8. *Let R be an ADL, S a uni subADL of R and $x, y \in R$. We define $\lceil x, y \rceil_S = \{a \in S \mid (x \vee a) \vee y = x \vee a\}$. We call $\lceil x, y \rceil_S$ an $S-$relative dual annihilator. It can be observed that $a \in \lceil x, y \rceil_S$ iff $y = (x \vee a) \wedge y$. Clearly $\lceil x, y \rceil_S$ is a filter of S.*

The usual lattice theoretic duality principle doesn't hold in ADLs. For example, in an ADL R, $\wedge$ is right distributive over $\vee$ but $\vee$ is not right distributive over $\wedge$. However, we get that the dual of many results of section 3, hold good in dually $S-$ relatively normal ADLs. For this reason we give only statements of these results.

Lemma 7. *Let R be an ADL with maximal elements and S a uni subADL of R. If m_1, m_2 are two maximal elements in R, then for any $x \in R$, $\lceil x, m_1 \rceil_S = \lceil x, m_2 \rceil_S$.*

Lemma 8. *Let P be any prime ideal of S. For any $x, y \in R$, if $y \in P \vee (x]$, then $P \cap \lceil x, y \rceil_S$ is non-empty.*

Definition 9. *Let R be an ADL with maximal elements and S a uni subADL of R. R is called dually S−normal if for any* $x, y \in R$ *with* $x \vee y$ *is a maximal element in R, then there exist* $a, b \in R$ *such that* $x \vee a,\ y \vee b$ *are maximal elements and* $a \wedge b = 0$.

Theorem 10. *Let R be an ADL with maximal elements and S a uni subADL of R. Then the following are equivalent:*

1. *R is dually S−normal*
2. $\lceil x, y\rceil_S \vee \lceil y, x\rceil_S = S$, *for any* $x, y \in R$ *with* $x \vee y$ *is a maximal element.*

The following definition is taken from [8].

Definition 10. *Let R be an ADL with maximal elements. Then R is called dually relatively normal if for any* $x, y \in R$ *there exist* $a, b \in R$ *such that* $(x \vee a) \vee y = x \vee a,\ (y \vee b) \vee x = y \vee b$ *and* $a \wedge b = 0$.

The following definition is taken from Cignoli [2].

Definition 11. *Let* $(L, \vee, \wedge, 0, 1)$ *be a bounded distributive lattice and S a sublattice of L containing 0 and 1. Then L is called dually S−completely normal, if for any* $x, y \in L$, *there exist* $a, b \in S$ *such that* $x \vee a \geq y,\ y \vee b \geq x$ *and* $a \wedge b = 0$.

Now we define the concept of dually S−relatively normal ADL in the following.

Definition 12. *Let R be an ADL with maximal elements and S a uni subADL of R. R is called dually S−relatively normal if* $PF(R)$ *is dually* $PF(S)$*−completely normal lattice.*

Lemma 9. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is dually S−relatively normal if and only if for any* $x, y \in R$, *there exist* $a, b \in R$ *such that* $(x \vee a) \vee y = x \vee a,\ (y \vee b) \vee x = y \vee b$ *and* $a \wedge b = 0$.

Lemma 10. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is dually S−relatively normal if and only if for any* $x, y \in R$, $\lceil x, y\rceil_S \vee \lceil y, x\rceil_S = S$.

Theorem 11. *Let R be an ADL with maximal elements and S a uni subADL of R. Then the following conditions are equivalent:*

1. *R is dually S−relatively normal*
2. *For each pair* $x, y \in R$, *there is no proper filter of S containing both* $\lceil x, y\rceil_S$ *and* $\lceil y, x\rceil_S$
3. *The set of all ideals of R that contain a given S−prime ideal of R form a chain*
4. *The set of all prime ideals of R that contain a given S−prime ideal of R form a chain*
5. *Any proper ideal of R that contain a given S−prime ideal of R is a prime.*

Corollary 3. *Let R be an ADL with maximal elements and* S_1, S_2 *uni subADLs of R such that* $S_1 \subseteq S_2$. *Then the following conditions are equivalent:*

1. *R is dually S_1-relatively normal*
2. *R is dually S_2-relatively normal and the ideals generated in S_2 by prime ideals of S_1 are prime.*

Corollary 4. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is dually $S-$relatively normal if and only if R is dually relatively normal and the $S-$prime ideals of R are prime.*

Proof. Take $S_1 = S$ and $S_2 = R$ in the above corollary.

Definition 13. *Let R be an ADL with maximal elements. R is called relatively normal if for any $x, y \in R$, there exist $a, b \in R$ such that $y \wedge a \wedge x = a \wedge x$, $x \wedge b \wedge y = b \wedge y$ and $a \vee b$ is a maximal element. R is called dually relatively normal if for any $x, y \in R$, there exist $a, b \in R$ such that $(x \vee a) \vee y = x \vee a$, $(y \vee b) \vee x = y \vee b$ and $a \wedge b = 0$.*

Definition 14. *Let R be an ADL with maximal elements. Then R is called a linear ADL if R is both relatively normal and dually relatively normal. If S is a uni subADL of R, then R is called an $S-$linear ADL if R is both $S-$relatively normal and dually $S-$relatively normal.*

The following theorem can be verified easily.

Theorem 12. *Let R be an ADL with maximal elements and S a uni subADL of R. Then R is $S-$linear if and only if*

1. *R is a linear ADL*
2. *The $S-$prime filters of R are prime in R*
3. *The $S-$prime ideals of R are prime in R.*

Definition 15. *Let R be an ADL with maximal elements. Then*

$$B = \{a \in R \mid \text{ there exists } b \in R \text{ such that } a \wedge b = 0 \text{ and } a \vee b \text{ is maximal}\}$$

is called the Birkhoff centre of R and $(B, \vee, \wedge)$ is a uni sub ADL of R which is also a relatively complemented ADL [10].

If $a \in B$, then an element $b \in R$ with the property $a \wedge b = 0$ and $a \vee b$ is maximal is called a complement of a in B. It was observed in [7] that every relatively complemented ADL is a normal ADL and hence B is normal.

We conclude this paper with the following characterization theorem.

Theorem 13. *Let R be an ADL with maximal elements and B the Birkhoff centre of R. Then the following conditions are equivalent:*

1. *R is $B-$relatively normal*

1.′ R is dually B−relatively normal

2. Given $x, y \in R$, there is $a \in B$ and a complement a' of a such that $y \wedge a \wedge x = a \wedge x$ and $x \wedge a' \wedge y = a' \wedge y$

2.′ Given $x, y \in R$, there is $a \in B$ a complement a' of a such that $x \vee a \vee y = x \vee a$ and $y \vee a' \vee x = y \vee a'$

3. R is a linear ADL and the minimal prime ideals of R are B−maximal ideals of R

3.′ R is linear ADL and the minimal prime filters of R are B−maximal filters of R.

Proof.

$(1) \Rightarrow (2)$: Assume (1). Let $x, y \in R$. Then there exist $a, b \in B$ such that $y \wedge a \wedge x = a \wedge x,\ x \wedge b \wedge y = b \wedge y$ and $a \vee b$ is a maximal element. Since $a \in B$, there exists $c \in R$ such that $a \wedge c = 0$ and $a \vee c$ is a maximal element. Now, $a \vee (c \wedge b) = (a \vee c) \wedge (a \vee b) = a \vee b$ and $a \wedge c \wedge b = 0$. So that $c \wedge b (= a'$ say) is a complement of a in B and $x \wedge a' \wedge y = x \wedge c \wedge b \wedge y = c \wedge x \wedge b \wedge y = c \wedge b \wedge y = a' \wedge y$.

$(2) \Rightarrow (2')$: Assume (2). Let $x, y \in R$. Then by our assumption there exists $a \in B$ and a complement a' of a in B such that $y \wedge a \wedge x = a \wedge x$ and $x \wedge a' \wedge y = a' \wedge y$. Now, $(x \vee a) \wedge y = ((x \vee (a' \wedge y) \vee a) \wedge y = (x \vee a \vee a') \wedge (x \vee a \vee y) \wedge y = (a \vee a') \wedge y = y$. Therefore $(x \vee a) \vee y = x \vee a$. Similarly, $(y \vee a') \vee x = y \vee a'$.

$(2') \Rightarrow (2)$: Assume $(2')$. Let $x, y \in R$. Then there exists $a \in B$ and a complement a' of a in B such that $(x \vee a) \vee y = x \vee a$ and $(y \vee a') \vee x = y \vee a'$. Now, $y \vee (a \wedge x) = y \vee (a \wedge (y \vee a') \wedge x) = y \vee ((a \wedge y \wedge x) \vee (a \wedge a' \wedge x)) = y \vee (a \wedge y \wedge x) = y$. Therefore $y \wedge (a \wedge x) = a \wedge x$. Similarly, $x \wedge (a' \wedge y) = a' \wedge y$.

$(2) \Rightarrow (3)$: Assume (2). Since (2) and $(2')$ are equivalent, R is a linear ADL. Let P be a minimal prime ideal of R and $x \in P$. Then there exists $y \in R \setminus P$ such that $x \wedge y = 0$. By (2), there exists $a \in B$ and a complement a' of a such that $a \wedge x = y \wedge (a \wedge x) = 0$ and $a' \wedge y = x \wedge (a' \wedge y) = 0$. So that $a' \wedge y \in P$ and hence $a' \in P$. Now, $x = (a \vee a') \wedge x = (a \wedge x) \vee (a' \wedge x) = a' \wedge x$. Therefore P is $B-$ideal of R and hence P is a $B-$maximal ideal of R, since B is a relatively complemented ADL. Similarly, we get that $(2') \Rightarrow (3')$.

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