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On Neat Reducts of Cylindric Algebras

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Abstract. Let $1 < n < m \leq \omega$. We investigate the following question: For which reducts of $\mathbf{CA}_m$ is the class of neat n - reducts (not) elementary. We also characterize the class of neat reducts using games.

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1. The Class of Neat Reducts

Neat reducts have been a central notion in algebraic logic since the very beginning, and the notion is still a versatile active field of research, [see e.g 18, 21, 34, 31, 43, 24, 45, 41, 42, 30, 33]. Indeed, the consecutive problems 2.11, 2.12, 2.13 in the monograph [11] are on neat reducts. Problem 2.12 is solved by Hirsch Hodkinson and Maddux [10]. The authors of [10] show that the sequence $\langle S\mathfrak{Nr}_n\mathbf{CA}_{n+k} : k \in \omega\rangle$ is strictly decreasing for $\omega > n > 2$ with respect to inclusion. (Recall that we generalized this result to quasipolyadic equality algebras). The infinite dimensional case is settled by Pigozzi as reported in [11]. The main result in [10] strengthes Monk's classical result that for every finite $n > 2$ and any $k \in \omega$, $\mathbf{RCA}_n \subset S\mathfrak{Nr}_n\mathbf{CA}_{n+k}$. Taking $\mathfrak{A}_k \in S\mathfrak{Nr}_n\mathbf{CA}_{n+k} \sim \mathbf{RCA}_n$, and forming the ultraproduct $\prod \mathfrak{A}_k/F$ relative to a non-principal ultrafilter on ω, the resulting structure will be representable, showing that $\mathbf{RCA}_n$, though, elementary (indeed a variety) is not finitely axiomatizable. Problem 2.13 is solved in [41]. Problem 2.11 which is relevant to our later discussion asks: For which pair of ordinals $\alpha < \beta$ is the class $\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$ closed under forming subalgebras and homomorphic images? Németi proves that for any $1 < \alpha < \beta$ the class $\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$ though closed under forming homomorphic images and products is not a variety, i.e., it is not closed under forming subalgebras [14]. The next natural question is whether this class is elementary, and in this particular case, since the class of neat reducts is closed under ultraproducts, this amounts to asking whether it is closed under elementary subalgebras? In [18] it is proved that for any $1 < \alpha < \beta$, the class $\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$ is not elementary answering problem 4.4 in [12]. In [30], it is shown that this class cannot be characterized by any $L_{\infty\omega}$ sentence. We know that $\mathfrak{Nr}_n\mathbf{CA}_\omega$

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is closed under products and homomorphic images, thus under ultraproducts. However, for $n > 1$, it is *not* closed under elementary subalgebras, equivalently, under ultraroots. (For $n \leq 1, \mathfrak{Nr}_n\mathbf{CA}_\omega = \mathbf{RCA}_n = \mathbf{CA}_n$; so this is a degenerate case which we ignore). For a class K, ELK denotes the elementary closure of K, that is the least elementary class containing K. UpK denotes the class of all ultraproducts of members of K and UrK denotes the class of all ultraroots of members of K. Recall that, by the celebrated Shelah - Keisler theorem, $ElK = UpUrK$.

Theorem 1. *Let $n > 1$. Then the class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ is pseudo-elementary, but is not elementary. Furthermore, $El\mathfrak{Nr}_n\mathbf{CA}_\omega \subset \mathbf{RCA}_n$, $EL\mathfrak{Nr}_n\mathbf{CA}_\omega$ is recursively enumerable, and for $n > 2$ is not finitely axiomatizable.*

Proof. The class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ is not elementary [18]. To show that it is pseudo-elementary, we use a three sorted defining theory, with one sort for a cylindric algebra of dimension $n(c)$, the second sort for the Boolean reduct of a cylindric algebra (b) and the third sort for a set of dimensions (δ). We use superscripts n, b, δ for variables and functions to indicate that the variable, or the returned value of the function, is of the sort of the cylindric algebra of dimension n, the Boolean part of the cylindric algebra or the dimension set, respectively. The signature includes dimension sort constants i^δ for each $i < \omega$ to represent the dimensions. The defining theory for $\mathfrak{Nr}_n\mathbf{CA}_\omega$ incudes sentences demanding that the constants i^δ for $i < \omega$ are distinct and that the last two sorts define a cylindric algebra of dimension ω. For example the sentence

$$\forall x^\delta, y^\delta, z^\delta(d^b(x^\delta, y^\delta) = c^b(z^\delta, d^b(x^\delta, z^\delta).d^b(z^\delta, y^\delta)))$$

represents the cylindric algebra axiom $\mathsf{d}_{ij} = \mathsf{c}_k(\mathsf{d}_{ik}.\mathsf{d}_{kj})$ for all $i, j, k < \omega$. We have have a function I^b from sort c to sort b and sentences requiring that I^b be injective and to respect the n dimensional cylindric operations as follows: for all x^r

$$I^b(\mathsf{d}_{ij}) = d^b(i^\delta, j^\delta)$$

$$I^b(\mathsf{c}_i x^r) = \mathsf{c}_i^b(I^b(x)).$$

Finally we require that I^b maps onto the set of n dimensional elements

$$\forall y^b((\forall z^\delta(z^\delta \neq 0^\delta, \dots (n-1)^\delta \to c^b(z^\delta, y^b) = y^b)) \longleftrightarrow \exists x^r(y^b = I^b(x^r))).$$

For $\mathfrak{A} \in \mathbf{CA}_n$, $\mathfrak{Rd}_3\mathfrak{A}$ denotes the $\mathbf{CA}_3$ obtained from $\mathfrak{A}$ by discarding all operations indexed by indices in $n \sim 3$. $\mathbf{Df}_n$ denotes the class of diagonal free cylindric algebras. $\mathfrak{Rd}_{df}\mathfrak{A}$ denotes the $\mathbf{Df}_n$ obtained from $\mathfrak{A}$ by deleting all diagonal elements. To prove the non-finite axiomatizability result we use Monk's algebras. For $3 \leq n, i < \omega$, with $n-1 \leq i, \mathbb{C}_{n,i}$ denotes the $\mathbf{CA}_n$ associated with the cylindric atom structure as defined on p. 95 of [11]. Then by [11, 3.2.79] for $3 \leq n$, and $j < \omega$, $\mathfrak{Rd}_3\mathbb{C}_{n,n+j}$ can be neatly embedded in a $\mathbf{CA}_{3+j+1}$.

(1) By [11, 3.2.84]) we have for every $j \in \omega$, there is an $3 \leq n$ such that $\mathfrak{Rd}_{df}\mathfrak{Rd}_3\mathscr{C}_{n,n+j}$ is a non-representable $\mathbf{Df}_3$.

(2) Now suppose $m \in \omega$. By (2), choose $j \in \omega \sim 3$ so that $\mathfrak{Rd}_{df}\mathfrak{Rd}_3\mathbb{C}_{j,j+m+n-4}$ is a non-representable $\mathbf{Df}_3$. By (1) we have $\mathfrak{Rd}_{df}\mathfrak{Rd}_3\mathbb{C}_{j,j+m+n-4} \subseteq \mathfrak{Nr}_3\mathfrak{B}_m$, for some $\mathfrak{B} \in \mathbf{CA}_{n+m}$. Put $\mathfrak{A}_m = \mathfrak{Nr}_n\mathfrak{B}_m$. $\mathfrak{Rd}_{df}\mathfrak{A}_m$ is not representable, a friotri, $\mathfrak{A}_m \notin \mathbf{RCA}_n$, for else its $\mathbf{Df}$ reduct would be representable. Therefore $\mathfrak{A}_m \notin EL\mathfrak{Nr}_n\mathbf{CA}_\omega$. Now let $\mathbb{C}_m$ be an algebra similar to $\mathbf{CA}_\omega$'s such that $\mathfrak{B}_m = \mathfrak{Rd}_{n+m}\mathbb{C}_m$. Then $\mathfrak{A}_m = \mathfrak{Nr}_n\mathbb{C}_m$. Let F be a non-principal ultrafilter on ω. Then

$$\prod_{m\in\omega}\mathfrak{A}_m/F = \prod_{m\in\omega}(\mathfrak{Nr}_n\mathbb{C}_m)/F = \mathfrak{Nr}_n(\prod_{m\in\omega}\mathbb{C}_m/F)$$

But $\prod_{m\in\omega}\mathbb{C}_m/F \in \mathbf{CA}_\omega$. Hence $\mathbf{CA}_n \sim El\mathfrak{Nr}_n\mathbf{CA}_\omega$ is not closed under ultraproducts. It follows that the latter class is not finitely axiomatizable. In [18] it is proved that for $1 < \alpha < \beta$, $El\mathfrak{Nr}_\alpha\mathbf{CA}_\beta \subset S\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$.

From the above proof it follows that

Corollary 1. *Let K be any class such that $\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq K \subseteq \mathbf{RCA}_n$. Then ELK is not finitely axiomatizable*

For $n > 2$ the addition of finitely many *first order definable operations* does not remedy the non-finite axiomatizability result for $\mathbf{RCA}_n$, as proved by Biro. First order definable operations are those operations that can be defined using spare dimensions, and hence the notion of neat reducts are appropriate for handing them. A non-trivial question that involves the class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ in an essential way, is whether we can expand the signature of cylindric algebras by extra natural operations on n-ary relations so that if $\mathfrak{A} \in \mathbf{Cs}_n$ and is closed under these operations then this forces $\mathfrak{A}$ to be in the class $\mathfrak{Nr}_n\mathbf{CA}_\omega$. (For example, the polyadic operations are not enough.) The class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ contains all first order definable operations, so the question can be reformulated as to whether one can capture all first order definable operations using a *finite* set of operations. This is strongly related to the Finitizability Problem [48] in algebraic logic. Next we characterize the class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ using games. Since games go deeper into the analysis, they could shed light on the possible choice of such operations. For that, we need some preparations. We use "cylindric algebra" games that are analogues to certain "relation algebra" games used by Robin Hirsch in [7]. In [7] Robin Hirsch studies quite extensively the class $\mathfrak{Ra}\mathbf{CA}_n$ of relation algebra reducts of cylindric algebras of dimension n. This class was studied by many authors, to mention a few, Maddux, Simon and Nemeti. References for their work can be found in the most recent reference [7]. Our treatment in this part follows very closely [7].

Definition 1. *Let n be an ordinal. An s word is a finite string of substitutions (s_i^j), a c word is a finite string of cylindrifications (c_k). An sc word is a finite string of substitutions and cylindrifications Any sc word w induces a partial map $\hat{w} : n \to n$ by*

- $\hat{\epsilon} = Id$
- $\widehat{w_j^i} = \hat{w} \circ [i|j]$
- $\widehat{w\mathsf{c}_i} = \hat{w} \upharpoonright (n \sim \{i\}$

If $\bar{a} \in {}^{<n-1}n$, we write $\mathsf{s}_{\bar{a}}$, or more frequently $\mathsf{s}_{a_0 \ldots a_{k-1}}$, where $k = |\bar{a}|$, for an an arbitrary chosen sc word w such that $\hat{w} = \bar{a}$. w exists and does not depend on w by [9, definition 5.23 lemma 13.29]. We can, and will assume [9, Lemma 13.29] that $w = s\mathsf{c}_{n-1}\mathsf{c}_n$. [In the notation of [9, definition 5.23, lemma 13.29], $\widehat{s_{ijk}}$ for example is the function $n \to n$ taking 0 to i, 1 to j and 2 to k, and fixing all $l \in n \setminus \{i, j, k\}$.] Let δ be a map. Then $\delta[i \to d]$ is defined as follows. $\delta[i \to d](x) = \delta(x)$ if $x \neq i$ and $\delta[i \to d](i) = d$. We write δ_i^j for $\delta[i \to \delta_j]$.

Definition 2. *From now on let* $2 \leq n < \omega$. *Let* $\mathbb{C}$ *be an atomic* $\mathbf{CA}_n$. *An* atomic network *over* $\mathbb{C}$ *is a map*

$$N : {}^n\Delta \to At\mathscr{C}$$

such that the following hold for each $i, j < n$, $\delta \in {}^n\Delta$ *and* $d \in \Delta$*:*

- $N(\delta_j^i) \leq \mathsf{d}_{ij}$
- $N(\delta[i \to d]) \leq \mathsf{c}_i N(\delta)$

Note than N can be viewed as a hypergraph with set of nodes Δ and each hyperedge in ${}^{\mu}\Delta$ is labeled with an atom from $\mathbb{C}$. We call such hyperedges atomic hyperedges. We write $\mathsf{nodes}(N)$ for Δ. But it can happen let N stand for the set of nodes as well as for the function and the network itself. Context will help.

Define $x \sim y$ if there exists $\bar{z}$ such that $N(x, y, \bar{z}) \leq \mathsf{d}_{01}$. Define an equivalence relation $\sim$ over the set of all finite sequences over $\mathsf{nodes}(N)$ by $\bar{x} \sim \bar{y}$ iff $|\bar{x}| = |\bar{y}|$ and $x_i \sim y_i$ for all $i < |\bar{x}|$.

(3) A *hypernetwork* $N = (N^a, N^h)$ over $\mathscr{C}$ consists of a network N^a together with a labelling function for hyperlabels $N^h : {}^{<\omega}\mathsf{nodes}(N) \to \Lambda$ (some arbitrary set of hyperlabels Λ) such that for $\bar{x}, \bar{y} \in {}^{<\omega}\mathsf{nodes}(N)$

IV. $\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y})$.

If $|\bar{x}| = k \in nats$ and $N^h(\bar{x}) = \lambda$ then we say that λ is a k-ary hyperlabel. $(\bar{x})$ is referred to a a k-ary hyperedge, or simply a hyperedge. (Note that we have atomic hyperedges and hyperedges) When there is no risk of ambiguity we may drop the superscripts a, h.

The following notation is defined for hypernetworks, but applies equally to networks.

(4) If N is a hypernetwork and S is any set then $N{\restriction_S}$ is the n-dimensional hypernetwork defined by restricting N to the set of nodes $S \cap \mathsf{nodes}(N)$. For hypernetworks M, N if there is a set S such that $M = N{\restriction_S}$ then we write $M \subseteq N$. If $N_0 \subseteq N_1 \subseteq \ldots$ is a nested sequence of hypernetworks then we let the *limit* $N = \bigcup_{i<\omega} N_i$ be the hypernetwork defined by $\mathsf{nodes}(N) = \bigcup_{i<\omega} \mathsf{nodes}(N_i)$, $N^a(x_0, \ldots x_{n-1}) = N_i^a(x_0, \ldots x_{n-1})$ if $x_0 \ldots x_{\mu-1} \in \mathsf{nodes}(N_i)$, and $N^h(\bar{x}) = N_i^h(\bar{x})$ if $\mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(N_i)$. This is well-defined since the hypernetworks are nested and since hyperedges $\bar{x} \in {}^{<\omega}\mathsf{nodes}(N)$ are only finitely long.

For hypernetworks M, N and any set S, we write $M \equiv^S N$ if $N{\restriction_S} = M{\restriction_S}$. For hypernetworks M, N, and any set S, we write $M \equiv_S N$ if the symmetric difference $\Delta(\mathsf{nodes}(M), \mathsf{nodes}(N)) \subseteq S$ and $M \equiv^{(\mathsf{nodes}(M) \cup \mathsf{nodes}(N)) \setminus S} N$. We write $M \equiv_k N$ for $M \equiv_{\{k\}} N$.

Let N be a network and let θ be any function. The network $N\theta$ is a complete labeled graph with nodes $\theta^{-1}(\mathsf{nodes}(N)) = \{x \in \mathsf{dom}(\theta) : \theta(x) \in \mathsf{nodes}(N)\}$, and labeling defined by

$(N\theta)(i_0,\dots i_{\mu-1}) = N(\theta(i_0),\theta(i_1),\theta(i_{\mu-1}))$, for $i_0,\dots i_{\mu-1} \in \theta^{-1}(\mathsf{nodes}(N))$. Similarly, for a hypernetwork $N = (N^a, N^h)$, we define $N\theta$ to be the hypernetwork $(N^a\theta, N^h\theta)$ with hyperlabeling defined by $N^h\theta(x_0,x_1,\dots) = N^h(\theta(x_0),\theta(x_1),\dots)$ for $(x_0,x_1,\dots) \in {}^{<\omega}\theta^{-1}(\mathsf{nodes}(N))$.

Let M,N be hypernetworks. A *partial isomorphism* $\theta : M \to N$ is a partial map $\theta : \mathsf{nodes}(M) \to \mathsf{nodes}(N)$ such that for any $i_i \dots i_{\mu-1} \in \mathsf{dom}(\theta) \subseteq \mathsf{nodes}(M)$ we have $M^a(i_1,\dots i_{\mu-1}) = N^a(\theta(i),\dots \theta(i_{\mu-1}))$ and for any finite sequence $\bar{x} \in {}^{<\omega}\mathsf{dom}(\theta)$ we have $M^h(\bar{x}) = N^h\theta(\bar{x})$. If $M = N$ we may call θ a partial isomorphism of N.

Definition 3. *Let $2 \le n < \omega$. For any* $\mathbf{CA}_n$ *atom structure α, and $n \le m \le \omega$, we define two-player games $F_n^m(\alpha)$, and $H_n(\alpha)$, each with ω rounds, and for $m < \omega$ we define $H_{m,n}(\alpha)$ with n rounds.*

- *Let $m \le \omega$. In a play of $F_n^m(\alpha)$ the two players construct a sequence of networks $N_0, N_1, \dots$ where $\mathsf{nodes}(N_i)$ is a finite subset of $m = \{j : j < m\}$, for each i. In the initial round of this game $\forall$ picks any atom $a \in \alpha$ and $\exists$ must play a finite network N_0 with $\mathsf{nodes}(N_0) \subseteq n$, such that $N_0(\bar{d}) = a$ for some $\bar{d} \in {}^{\mu}\mathsf{nodes}(N_0)$. In a subsequent round of a play of $F_n^m(\alpha)$ $\forall$ can pick a previously played network N an index $\text{ł} < n$, a “face” $F = \langle f_0,\dots f_{n-2}\rangle \in {}^{n-2}\mathsf{nodes}(N)$, $k \in m \setminus \{f_0,\dots f_{n-2}\}$, and an atom $b \in \alpha$ such that $b \le \mathsf{c}_l N(f_0,\dots f_i, x,\dots f_{n-2})$. (the choice of x here is arbitrary, as the second part of the definition of an atomic network together with the fact that $\mathsf{c}_i(\mathsf{c}_i x) = \mathsf{c}_i x$ ensures that the right hand side does not depend on x). This move is called a* cylindrifier move *and is denoted $(N, \langle f_0,\dots f_{\mu-2}\rangle, k, b, l)$ or simply (N,F,k,b,l). In order to make a legal response, $\exists$ must play a network $M \supseteq N$ such that $M(f_0,\dots f_{i-1},k,f_i,\dots f_{n-2})) = b$ and* $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$.

  $\exists$ wins $F_n^m(\alpha)$ if she responds with a legal move in each of the ω rounds. If she fails to make a legal response in any round then $\forall$ wins.

- *Fix some hyperlabel λ_0. $H_n(\alpha)$ is a game the play of which consists of a sequence of λ_0-neat hypernetworks $N_0, N_1, \dots$ where $\mathsf{nodes}(N_i)$ is a finite subset of ω, for each $i < \omega$. In the initial round $\forall$ picks $a \in \alpha$ and $\exists$ must play a λ_0-neat hypernetwork N_0 with nodes contained in μ and $N_0(\bar{d}) = a$ for some nodes $\bar{d} \in {}^{\mu}N_0$. At a later stage $\forall$ can make any cylindrifier move (N,F,k,b,l) by picking a previously played hypernetwork N and $F \in {}^{n-2}\mathsf{nodes}(N)$, $l < n, k \in \omega \setminus \mathsf{nodes}(N)$ and $b \le \mathsf{c}_l N(f_0, f_{l-1}, x, f_{n-2})$. [In H_n we require that $\forall$ chooses k as a 'new node', i.e. not in $\mathsf{nodes}(N)$, whereas in F_n^m for finite m it was necessary to allow $\forall$ to 'reuse old nodes'. This makes the game easior as far as $\forall$ is concerned.) For a legal response, $\exists$ must play a λ_0-neat hypernetwork $M \equiv_k N$ where* $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$ *and $M(f_0, f_{i-1}, k, f_{n-2}) = b$. Alternatively, $\forall$ can play a* transformation move *by picking a previously played hypernetwork N and a partial, finite surjection $\theta : \omega \to \mathsf{nodes}(N)$, this move is denoted (N,θ). $\exists$ must respond with $N\theta$. Finally, $\forall$ can play an* amalgamation move *by picking previously played hypernetworks M,N such that $M \equiv^{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} N$ and $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) \neq \emptyset$. This move is denoted (M,N). To make a legal response, $\exists$ must play a λ_0-neat hypernetwork L extending M and N, where $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$.*

  Again, $\exists$ wins $H_n(\alpha)$ if she responds legally in each of the ω rounds, otherwise $\forall$ wins.

- *For $m < \omega$ the game $H_{m,n}(\alpha)$ is similar to $H_n(\alpha)$ but play ends after m rounds, so a play of $H_{m,n}(\alpha)$ could be*

$$N_0, N_1, \ldots, N_m$$

If $\exists$ responds legally in each of these m rounds she wins, otherwise $\forall$ wins.

Definition 4. *For $m \geq 5$ and $\mathscr{C} \in \mathbf{CA}_m$, if $\mathfrak{A} \subseteq \mathfrak{Nr}_n(\mathbb{C})$ is an atomic cylindric algebra and N is an $\mathfrak{A}$-network then we define $\widehat{N} \in \mathbb{C}$ by*

$$\widehat{N} = \prod_{i_0,\ldots i_{n-1} \in \mathsf{nodes}(N)} \mathsf{s}_{i_0,\ldots i_{n-1}} N(i_0 \ldots i_{n-1})$$

$\widehat{N} \in \mathbb{C}$ depends implicitly on $\mathbb{C}$.

We write $\mathfrak{A} \subseteq_c \mathfrak{B}$ if $\mathfrak{A} \in S_c\{\mathfrak{B}\}$.

Lemma 1. *Let $n < m$ and let $\mathfrak{A}$ be an atomic $\mathbf{CA}_n$, $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n \mathbb{C}$ for some $\mathbb{C} \in \mathbf{CA}_m$. For all $x \in \mathbb{C} \setminus \{0\}$ and all $i_0, \ldots i_{n-1} < m$ there is $a \in \mathsf{At}(\mathfrak{A})$ such that $\mathsf{s}_{i_0 \ldots i_{n-1}} a \,.\, x \neq 0$.*

Proof. We can assume, see definition 1, that $\mathsf{s}_{i_0,\ldots i_{n-1}}$ consists only of substitutions, since $\mathsf{c}_m \ldots \mathsf{c}_{m-1} \ldots \mathsf{c}_n x = x$ for every $x \in \mathfrak{A}$.We have s^i_j is a completely additive operator (any i, j), hence $\mathsf{s}_{i_0,\ldots i_{\mu-1}}$ is too (see definition 1). So $\sum\{\mathsf{s}_{i_0\ldots i_{n-1}} a : a \in \mathsf{At}(\mathfrak{A})\} = \mathsf{s}_{i_0\ldots i_{n-1}} \sum \mathsf{At}(\mathfrak{A}) = \mathsf{s}_{i_0\ldots i_{n-1}} 1 = 1$, for any $i_0, \ldots i_{n-1} < n$. Let $x \in \mathbb{C} \setminus \{0\}$. It is impossible that $\mathsf{s}_{i_0\ldots i_{n-1}} \,.\, x = 0$ for all $a \in \mathsf{At}(\mathscr{A})$ because this would imply that $1 - x$ was an upper bound for $\{\mathsf{s}_{i_0\ldots i_{n-1}} a : a \in \mathsf{At}(\mathfrak{A})\}$, contradicting $\sum\{\mathsf{s}_{i_0\ldots i_{n-1}} a : a \in \mathsf{At}(\mathscr{A})\} = 1$.

Lemma 2. *Let $n < m$ and let $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n \mathbb{C}$ be an atomic $\mathbf{CA}_n$*

1. *For any $x \in \mathbb{C} \setminus \{0\}$ and any finite set $I \subseteq m$ there is a network N such that $\mathsf{nodes}(N) = I$ and $x \,.\, \widehat{N} \neq 0$.*
2. *For any networks M, N if $\widehat{M} \,.\, \widehat{N} \neq 0$ then $M \equiv^{\mathsf{nodes}(M) \cap \mathsf{nodes}(N)} N$.*

Proof. The proof of the first part is based on repeated use of lemma 1. We define the edge labeling of N one edge at a time. Initially no hyperedges are labeled. Suppose $E \subseteq \mathsf{nodes}(N) \times \mathsf{nodes}(N) \ldots \times \mathsf{nodes}(N)$ is the set of labeled hyper edges of N (initially $E = \emptyset$) and $x \,.\, \prod_{\bar{c} \in E} \mathsf{s}_{\bar{c}} N(\bar{c}) \neq 0$. Pick $\bar{d}$ such that $\bar{d} \notin E$. By lemma 1 there is $a \in \mathsf{At}(\mathscr{A})$ such that $x \,.\, \prod_{\bar{c} \in E} \mathsf{s}_{\bar{c}} N(\bar{c}) \,.\, \mathsf{s}_{\bar{d}} a \neq 0$. Include the edge $\bar{d}$ in E. Eventually, all edges will be labeled, so we obtain a completely labeled graph N with $\widehat{N} \neq 0$. it is easily checked that N is a network. For the second part, if it is not true that $M \equiv^{\mathsf{nodes}(M) \cap \mathsf{nodes}(N)} N$ then there are is $\bar{c} \in^{n-1} \mathsf{nodes}(M) \cap \mathsf{nodes}(N)$ such that $M(\bar{c}) \neq N(\bar{c})$. Since edges are labeled by atoms we have $M(\bar{c}) \cdot N(\bar{c}) = 0$, so $0 = \mathsf{s}_{\bar{c}} 0 = \mathsf{s}_{\bar{c}} M(\bar{c}) \,.\, \mathsf{s}_{\bar{c}} N(\bar{c}) \geq \widehat{M} \,.\, \widehat{N}$.

Lemma 3. *Let Let $m > n$. Let $\mathbb{C} \in \mathbf{CA}_m$ and let $\mathfrak{A} \subseteq \mathfrak{Nr}_n(\mathbb{C})$ be atomic. Let N be a network over $\mathscr{A}$ and $i, j < n$.*

1. *If $i \notin \mathsf{nodes}(N)$ then $\mathsf{c}_i \widehat{N} = \widehat{N}$.*

2. $\widehat{NId_{-j}} \geq \widehat{N}$.

3. *If* $i \notin \mathsf{nodes}(N)$ *and* $j \in \mathsf{nodes}(N)$ *then* $\widehat{N} \neq 0 \to \widehat{N[i/j]} \neq 0$. *where* $N[i/j] = N \circ [i|j]$

4. *If* θ *is any partial, finite map* $n \to n$ *and if* $\mathsf{nodes}(N)$ *is a proper subset of* n*, then* $\widehat{N} \neq 0 \to \widehat{N\theta} \neq 0$.

Proof. The first part is easy. The second part is by definition of $\widehat{\ }$. For the third part suppose $\widehat{N} \neq 0$. Since $i \notin \mathsf{nodes}(N)$, by part 1, we have $\mathsf{c}_i\widehat{N} = \widehat{N}$. By cylindric algebra axioms it follows that $\widehat{N} \,.\, d_{ij} \neq 0$. By lemma 2 there is a network M where $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{i\}$ such that $\widehat{M} \,.\widehat{N} \,.\, d_{ij} \neq 0$. By lemma 2 we have $M \supseteq N$ and $M(i,j) \leq 1'$. It follows that $M = N[i/j]$. Hence $\widehat{N[i/j]} \neq 0$. For the final part (cf. [9, lemma 13.29]), since there is $k \in n \setminus \mathsf{nodes}(N)$, θ can be expressed as a product $\sigma_0\sigma_1 \ldots \sigma_t$ of maps such that, for $s \leq t$, we have either $\sigma_s = Id_{-i}$ for some $i < n$ or $\sigma_s = [i/j]$ for some $i, j < n$ and where $i \notin \mathsf{nodes}(N\sigma_0 \ldots \sigma_{s-1})$. Now apply the previous parts of the lemma.

We now prove two Theorems relating neat embeddings to the games we defined:

Theorem 2. *Let* $n < m$*, and let* $\mathfrak{A}$ *be a* $\mathbf{CA}_m$*. If* $\mathfrak{A} \in \mathbf{S_c}\mathfrak{Nr}_n\mathbf{CA}_m$*, then* $\exists$ *has a winning strategy in* $F^m(\mathsf{At}\mathfrak{A})$*.*

Proof. If $\mathfrak{A} \subseteq \mathfrak{Nr}_n\mathbb{C}$ for some $\mathbb{C} \in \mathbf{CA}_m$ then $\exists$ always plays hypernetworks N with $\mathsf{nodes}(N) \subseteq n$ such that $\widehat{N} \neq 0$. In more detail, in the initial round , let $\forall$ play $a \in \mathsf{At}\mathscr{A}$. $\exists$ play a network N with $N(0, \ldots n-1) = a$. Then $\widehat{N} = a \neq 0$. At a later stage suppose $\forall$ plays the cylindrifier move $(N, \langle f_0, \ldots f_{\mu-2}\rangle, k, b, l)$ by picking a previously played hypernetwork N and $f_i \in \mathsf{nodes}(N)$, $l < \mu, k \notin \{f_i : i < n-2\}$, and $b \leq \mathsf{c}_l N(f_0, \ldots f_{i-1}, x, f_{n-2})$. Let $\bar{a} = \langle f_0 \ldots f_{l-1}, k \ldots f_{n-2}\rangle$. Then $\mathsf{c}_k\widehat{N} \cdot \mathsf{s}_{\bar{a}} b \neq 0$. By 1 there is a network M such that $\widehat{M}.\widehat{\mathsf{c}_k N} \cdot \mathsf{s}_{\bar{a}} b \neq 0$. Hence $M(f_0, k, f_{n-2}) = b$.

Theorem 3. *Let* α *be a countable* $\mathbf{CA}_n$ *atom structure. If* $\exists$ *has a winning strategy in* $H_n(\alpha)$ *then there is a representable cylindric algebra* $\mathbb{C}$ *of dimension* ω *such that* $\mathfrak{Nr}_n\mathscr{C}$ *is atomic and* $\mathsf{At}\mathfrak{Nr}_n\mathbb{C} \cong \alpha$*.*

Proof. Suppose $\exists$ has a winning strategy in $H(\alpha)$. Fix some $a \in \alpha$. We can define a nested sequence $N_0 \subseteq N_1 \ldots$ of hypernetworks where N_0 is $\exists$'s response to the initial $\forall$-move a, requiring that

1. If N_r is in the sequence and and $b \leq \mathsf{c}_l N_r(\langle f_0, f_{n-2}\rangle \ldots, x, f_{n-2})$. then there is $s \geq r$ and $d \in \mathsf{nodes}(N_s)$ such that $N_s(f_0, f_{i-1}, d, f_{n-2}) = b$.

2. If N_r is in the sequence and θ is any partial isomorphism of N_r then there is $s \geq r$ and a partial isomorphism θ^+ of N_s extending θ such that $\mathsf{rng}(\theta^+) \supseteq \mathsf{nodes}(N_r)$.

Since α is countable there are countably many requirements to extend. Since the sequence of networks is nested , these requirements to extend remain in all subsequent rounds. So that we can schedule these requirements to extend so that eventually, every requirement gets dealt with. If we are required to find k and $N_{r+1} \supset N_r$ such that $N_{r+1}(f_0, k, f_{n-2}) = b$ then let $k \in \omega \setminus \mathsf{nodes}(N_r)$ be least possible for definiteness, and let N_{r+1} be ∃'s response using her winning strategy, to the ∀move $N_r, (f_0, \dots f_{n-1}), k, b, l)$. For an extension of type 2, let τ be a partial isomorphism of N_r and let θ be any finite surjection onto a partial isomorphism of N_r such that $dom(\theta) \cap nodes(N_r) = dom\tau$. ∃'s response to ∀'s move (N_r, θ) is necessarily $N\theta$. Let N_{r+1} be her response , using her wining strategy, to the subsequent ∀move $(N_r, N_r\theta)$.

Now let N_a be the limit of this sequence. This limit is well-defined since the hypernetworks are nested. Note, for $b \in \alpha$, that

$$(\exists i_0, \dots I_{\mu-1} \in \mathsf{nodes}(N_a),\ N_a(i_0 \dots, i_{\mu-1}) = b) \iff b \sim a \tag{1}$$

Let θ be any finite partial isomorphism of N_a and let X be any finite subset of $\mathsf{nodes}(N_a)$. Since θ, X are finite, there is $i < \omega$ such that $\mathsf{nodes}(N_i) \supseteq X \cup \mathsf{dom}(\theta)$. There is a bijection $\theta^+ \supseteq \theta$ onto $\mathsf{nodes}(N_i)$ and $j \geq i$ such that $N_j \supseteq N_i, N_i\theta^+$. Then θ^+ is a partial isomorphism of N_j and $\mathsf{rng}(\theta^+) = \mathsf{nodes}(N_i) \supseteq X$. Hence, if θ is any finite partial isomorphism of N_a and X is any finite subset of $\mathsf{nodes}(N_a)$ then

$$\exists \text{ a partial isomorphism } \theta^+ \supseteq \theta \text{ of } N_a \text{ where } \mathsf{rng}(\theta^+) \supseteq X \tag{2}$$

and by considering its inverse we can extend a partial isomorphism so as to include an arbitrary finite subset of $\mathsf{nodes}(N_a)$ within its domain. Let L be the signature with one μ -ary predicate symbol (b) for each $b \in \alpha$, and one k-ary predicate symbol (λ) for each k-ary hyperlabel λ. [Notational point: if λ is k-ary and l-ary for $k \neq l$ then make one k-ary predicate symbol λ and one l-ary predicate symbol λ', so that every predicate symbol has a unique arity.] The set of variables for L-formulas is $\{x_i : i < \omega\}$. We also have equality. Pick $f_a \in {}^{\omega}\mathsf{nodes}(N_a)$. Let $U_a = \{f \in {}^{\omega}\mathsf{nodes}(N_a) : \{i < \omega : g(i) \neq f_a(i)\} \text{ is finite}\}$. We can make U_a into the base of an L-structure $\mathcal{N}_a$ and evaluate L-formulas at $f \in U_a$ as follow. For $b \in \alpha,\ l_0, \dots l_{\mu-1}, i_0 \dots, i_{k-1} < \omega,\ k$-ary hyperlabels λ, and all L-formulas ϕ, ψ, let

$$\begin{aligned}
\mathcal{N}_a, f \models b(x_{l_0} \dots x_{n-1}) &\iff N_a(f(l_0), \dots f(l_{n-1})) = b \\
\mathcal{N}_a, f \models \lambda(x_{i_0}, \dots, x_{i_{k-1}}) &\iff N_a(f(i_0), \dots, f(i_{k-1})) = \lambda \\
\mathcal{N}_a, f \models \neg\phi &\iff \mathcal{N}_a, f \not\models \phi \\
\mathcal{N}_a, f \models (\phi \vee \psi) &\iff \mathcal{N}_a, f \models \phi \text{ or } \mathcal{N}_a, f \models \psi \\
\mathcal{N}_a, f \models \exists x_i\phi &\iff \mathcal{N}_a, f[i/m] \models \phi, \text{ some } m \in \mathsf{nodes}(N_a)
\end{aligned}$$

For any L-formula ϕ, write $\phi^{\mathcal{N}_a}$ for $\{f \in {}^{\omega}\mathsf{nodes}(N_a) : \mathcal{N}_a, f \models \phi\}$. Let $Form^{\mathcal{N}_a} = \{\phi^{\mathcal{N}_a} : \phi \text{ is an } L\text{-formula}\}$ and define a cylindric algebra

$$\mathcal{D}_a = (Form^{\mathcal{N}_a}, \cup, \sim, \mathsf{D}_{ij}, \mathsf{C}_i, i, j < \omega)$$

where $\mathsf{D}_{ij} = (x_i = x_j)^{\mathcal{N}_a}$, $\mathsf{C}_i(\phi^{\mathcal{N}_a}) = (\exists x_i\phi)^{\mathcal{N}_a}$. Observe that $\top^{\mathcal{N}_a} = U_a$, $(\phi \vee \psi)^{\mathcal{N}_a} = \phi^{\mathcal{N}_a} \cup \psi^{\mathcal{N}_a}$, etc. Note also that $\mathcal{D}$ is a subalgebra of the ω-dimensional cylindric set algebra on the base $\mathsf{nodes}(N_a)$, hence $\mathcal{D} \in \mathbf{RCA}_\omega$.

Let $\phi(x_{i_0}, x_{i_1}, \dots, x_{i_k})$ be an arbitrary L-formula using only variables belonging to $\{x_{i_0}, \dots, x_{i_k}\}$. Let $f, g \in U_a$ (some $a \in \alpha$) and suppose is a partial isomorphism of N_a. We can prove by induction over the quantifier depth of ϕ and using (2), that

$$\mathcal{N}_a, f \models \phi \iff \mathcal{N}_a, g \models \phi \tag{3}$$

Let $\mathscr{C} = \prod_{a\in\alpha} D_a$. Then $\mathscr{C} \in \mathbf{RCA}_\omega$. An element x of $\mathscr{C}$ has the form $(x_a : a \in \alpha)$, where $x_a \in \mathscr{D}_a$. For $b \in \alpha$ let $\pi_b : \mathscr{C} \to D_b$ be the projection defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : \mathscr{D}_a \to \mathscr{C}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Evidently $\pi_b(\iota_b(y)) = y$ for $y \in \mathscr{D}_b$ and $\pi_b(\iota_a(y)) = 0$ if $a \neq b$.

Suppose $x \in \mathfrak{Nr}_\mu \mathscr{C} \setminus \{0\}$. Since $x \neq 0$, it must have a non-zero component $\pi_a(x) \in \mathscr{D}_a$, for some $a \in \alpha$. Say $\emptyset \neq \phi(x_{i_0}, \dots, x_{i_k})^{\mathscr{D}_a} = \pi_a(x)$ for some L-formula $\phi(x_{i_0}, \dots, x_{i_k})$. We have $\phi(x_{i_0}, \dots, x_{i_k})^{\mathscr{D}_a} \in \mathfrak{Nr}_\mu \mathscr{D}_a)$. Pick $f \in \phi(x_{i_0}, \dots, x_{i_k})^{\mathscr{D}_a}$ and let $b = N_a(f(0), f(1), \dots f_{n-1}) \in \alpha$. We will show that $b(x_0, x_1, \dots x_{n-1})^{\mathscr{D}_a} \subseteq \phi(x_{i_0}, \dots, x_{i_k})^{\mathscr{D}_a}$. Take any $g \in b(x_0, x_1 \dots x_{n-1})^{\mathscr{D}_a}$, so $N_a(g(0), g(1) \dots g(n-1)) = b$. The map $\{(f(0), g(0)), (f(1), g(1)) \dots (f(n-1), g(n-1))\}$ is a partial isomorphism of N_a. By (2) this extends to a finite partial isomorphism θ of N_a whose domain includes $f(i_0), \dots, f(i_k)$. Let $g' \in U_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \mathrm{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

By (3), $\mathcal{N}_a, g' \models \phi(x_{i_0}, \dots, x_{i_k})$. Observe that $g'(0) = \theta(0) = g(0)$ and similarly $g'(n-1) = g(n-1)$, so g is identical to g' over μ and it differs from g' on only a finite set of coordinates. Since $\phi(x_{i_0}, \dots, x_{i_k})^{D_a} \in \mathfrak{Nr}_\mu(\mathscr{C})$ we deduce $\mathcal{N}_a, g \models \phi(x_{i_0}, \dots, x_{i_k})$, so $g \in \phi(x_{i_0}, \dots, x_{i_k})^{\mathscr{D}_a}$. This proves that $b(x_0, x_1 \dots x_{\mu-1})^{\mathscr{D}_a} \subseteq \phi(x_{i_0}, \dots, x_{i_k})^{\mathscr{D}_a} = \pi_a(x)$, and so $\iota_a(b(x_0, x_1, \dots x_{n-1})^{D_a}) \leq \iota_a(\phi(x_{i_0}, \dots, x_{i_k})^{\mathscr{D}_a}) \leq x \in \mathscr{C} \setminus \{0\}$. Hence every non-zero element x of $\mathfrak{Nr}_n \mathscr{C}$ is above a non-zero element $\iota_a(b(x_0, x_1 \dots n_1)^{\mathscr{D}_a})$ (some $a, b \in \alpha$) and these latter elements are the atoms of $\mathfrak{Nr}_n \mathscr{C}$. So $\mathfrak{Nr}_n \mathscr{C}$ is atomic and $\alpha \cong \mathsf{At}\mathfrak{Nr}_n \mathscr{C}$ — the isomorphism is $b \mapsto (b(x_0, x_1, \dots x_{n-1})^{\mathscr{D}_a} : a \in A)$.

In [36], we use such games to show that for $n \geq 3$, there is a representable $\mathfrak{A} \in \mathbf{CA}_n$ with atom structure α such that $\forall$ can win the game $F^{n+2}(\alpha)$. However $\exists$ has a winning strategy in $H_n(\alpha)$, for any $n < \omega$. It will follow that there a countable cylindric algebra $\mathscr{A}'$ such that $\mathscr{A}' \equiv \mathscr{A}$ and $\exists$ has a winning strategy in $H(\mathscr{A}')$. So let K be any class such that $\mathfrak{Nr}_n \mathbf{CA}_\omega \subseteq K \subseteq S_c \mathfrak{Nr}_n \mathbf{CA}_{n+2}$. $\mathscr{A}'$ must belong to $\mathfrak{Nr}_n(\mathbf{RCA}_\omega)$, hence $\mathscr{A}' \in K$. But $\mathscr{A} \notin K$ and $\mathscr{A} \preceq \mathscr{A}'$. Thus K is not elementary. From this it easily follows that the class of completely representable cylindric algebras is not elementary, and that the class $\mathfrak{Nr}_n \mathbf{CA}_{n+k}$ for any $k \geq 0$ is not elementary either. Furthermore the constructions works for many variants of cylindric algebras like Halmos' polyadic equality algebras and Pinter's substitution algebras.

Theorem 4. *Let $3 \leq n < \omega$. Then the following hold:*

(i) Any K such that $\mathfrak{Nr}_n \mathbf{CA}_\omega \subseteq K \subseteq S_c \mathfrak{Nr}_n \mathbf{CA}_{n+2}$ is not elementary.

(ii) The inclusions $\mathfrak{Nr}_n \mathbf{CA}_\omega \subseteq S_c \mathfrak{Nr}_n \mathbf{CA}_\omega \subseteq S \mathfrak{Nr}_n \mathbf{CA}_\omega$ are all proper

Proof. (i) is already mentioned. While for (ii), for the first inclusion [18], and for the second [8].

2. Other algebras

Now we turn our attention for other algebras for which the notion of neat reducts make sense. $\mathbf{SC}_n, \mathbf{CA}_n, \mathbf{QA}_n$ and $\mathbf{QEA}_n$ abbreviate the classes of substitution, cylindric, quasipolyadic, and quasipolyadic equality algebras, of dimension n, respectively. Such algebras are studied in e.g. [45, 34, 21, 22, 24, 2, 15, 29, 33]. $\mathbf{Df}_n$ stands for the class of diagonal free cylindric algebras. It is known, and indeed easy to show, that for $1 < n < m$, the class $\mathbf{N}r_n\mathbf{Df}_m$ of neat n-reducts of $\mathbf{Df}_m$ is a variety. In fact, it is equal to $\mathbf{Df}_n$ [12][5.1.2]. In particular, it is an elementary class. On the other hand, it is known [18] that for $1 < n < m$ the class $\mathfrak{Nr}_n\mathbf{CA}_m$ is *not* an elementary class. It is also known [29], [34] that $\mathfrak{Nr}_n\mathbf{QA}_m$ and $\mathfrak{Nr}_n\mathbf{QEA}_m$ are not elementary classes. It is proved in Op.cit that such classes are not closed under ultraroots. So what about reducts, i.e algebras "in between" $\mathbf{Df}$ and $\mathbf{CA}$. By "in between" we mean a class $\mathbf{K}_m$ that is a reduct of $\mathbf{CA}_m$ and an expansion of $\mathbf{Df}_m$. A typical example is the class $\mathbf{SC}_m$ [15]. We define another reduct $r\mathbf{SC}_m$ (class of algebras of dimension m) which is a (proper) reduct of $\mathbf{SC}_m$, which in turn is a reduct of $\mathbf{CA}_m$, $\mathbf{QA}_m$ $\mathbf{QEA}_m$.

Definition 5. *Let m be an ordinal. $\mathfrak{A} \in r\mathbf{SC}_m$, is defined to be an algebra*

$$\mathfrak{A} = \langle A, +, ., -0, 1, \mathsf{c}_i, \mathsf{s}_i^j \rangle_{i,j \in m}$$

obeying the following axioms for $x, y \in A$ and $i, j, k, l < m$:

(E_0) $\langle A, +, ., -, 0, 1\rangle$ *is a boolean algebra*

(E_1) $\mathsf{c}_j 0 = 0$, $x \leq \mathsf{c}_i x$, $\mathsf{c}_i(x \mathsf{c}_i y) = \mathsf{c}_i x . \mathsf{c}_i y$, *and* $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$, *and* $\mathsf{s}_i^i x = x$

In other words the c_is are complemented closure operators and c_i,c_j commute.

(E_2) $\mathsf{s}_i^i x = x$

(E_3) s_j^i *are boolean endomorphisms.*

(E_4) $\mathsf{s}_j^i \mathsf{c}_i x = \mathsf{c}_i x$

(E_5) $\mathsf{c}_i \mathsf{s}_j^i x = \mathsf{s}_j^i x$ *whenever* $i \neq j$

(E_6) $\mathsf{s}_j^i \mathsf{c}_k x = \mathsf{c}_k \mathsf{s}_j^i x$, *whenever* $k \notin \{i, j\}$

(E_7) $\mathsf{c}_i \mathsf{s}_i^j x = \mathsf{c}_j \mathsf{s}_j^i x$

(E_8) $\mathsf{s}_j^i \mathsf{s}_i^k \mathsf{c}_i x = \mathsf{s}_j^k \mathsf{c}_i x$

(E_9) $\mathsf{s}_i^j \mathsf{s}_k^l x = \mathsf{s}_k^l \mathsf{s}_i^j x$ *when* $|\{i, j, k, l\}| = 4$

Definition 6. *(i) Let $n < m$ be ordinals. Let $\mathfrak{B} \in r\mathbf{SC}_m$ Then the neat n-reduct of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_n\mathfrak{B}$ is the $r\mathbf{SC}_n$ with universe $Nr_nB = \{b \in B : \mathsf{c}_i b = b \text{ for all } n \leq i < m\}$, and whose operations are those of the similarity type of $\mathbf{SC}_m$ (evaluated in $\mathfrak{B}$ and) restricted to Nr_nB.*

(ii) For a given class $\mathbf{M} \subseteq r\mathbf{SC}_m$, we let $\mathfrak{Nr}_n\mathbf{M}$ denote the class obtained by forming the neat n-reduct of algebras in $\mathbf{M}$, that is

$$\mathfrak{Nr}_n\mathbf{M} = \{\mathfrak{Nr}_n\mathfrak{B} : \mathfrak{B} \in \mathbf{M}\}.$$

The definition of neat reducts for $\mathbf{SC}_m$ is the same.

We now prove:

Theorem 5. *Let $1 < n$ and $n+1 < m \leq \omega$. Then $\mathfrak{Nr}_n r\mathbf{SC}_m$ and $\mathfrak{Nr}_n\mathbf{SC}_m$ are not elementary.*

We do not know whether $\mathfrak{Nr}_n Ł_{n+1}$ for $Ł \in \{\mathbf{SC}, r\mathbf{SC}\}$ is elementary or not. But why is it of interest to settle such questions on neat reducts. There are (at least) three possible answers to this question. First there are aesthetic reasons. Motivated by intellectual curiosity, the investigation of such questions is likely to lead to nice mathematics. The second reason concerns definability or classification. Now that we have the class of neat reducts in front of us, the most pressing need is to try to classify it. Classifying is a kind of defining. Most mathematical classification is by axioms (preferably first order) or, even better, equations (if the class in question is a variety.) It is known (and indeed not difficult to show) that the class $\mathfrak{Nr}_n\mathbf{CA}_m$ is closed under products and homomorphic images for all $n < m$ [45]. However, it is not closed under forming (elementary) subalgebras [18], that is, it is not axiomatizable, a priori not a variety. Studying neat reducts of reducts of **CA**'s and for that matter expansions [34], [29], clarifies the properties of neat reducts. (This is similar to the situation with representability [15] where axiomatizations of representable algebras are better understood by passing to reducts or expansions.) Now we come to the third reason, where neat reducts are not treated on its own but rather in its interaction with algebraic properties like representability, amalgamation and complete representations. This in turn is related to completeness, interpolation and omittting types for variants of first order logic, be it reducts or expansions [24, 22], [33]. Indeed the old but venerable notion of neat reducts has turned to be central notion in the theory of cylindric like algebras of relations, [23, 32, 30, 16].

We shall need the following Lemma on substitutions:

Lemma 4. *For any $k, l, u < n$ and $\mathfrak{A} \in r\mathbf{SC}_n$, set*

$$ {}_u\mathsf{s}(k,l)x = \mathsf{s}^u_k\mathsf{s}^k_l\mathsf{s}^l_u x. $$

Then

(i) If k, l, u and v are distinct, then

$$ {}_u\mathsf{s}(k,l)\mathsf{c}_u\mathsf{c}_v x = {}_u\mathsf{s}(l,k)\mathsf{c}_u\mathsf{c}_v x $$

(ii) With the same condition in (i), we have

$$ {}_u\mathsf{s}(k,l)_u\mathsf{s}(k,l)\mathsf{c}_u\mathsf{c}_v x = \mathsf{c}_u\mathsf{c}_v x. $$

The proof is tedious, but fairly straight forward. We use the axiomatization $(E_1 - E_9)$.
Proof.

$$ \mathsf{s}^l_u\mathsf{s}^u_k\mathsf{s}^k_l\mathsf{s}^l_u\mathsf{c}_u\mathsf{c}_v x = \text{ (by } E_8\text{) } \mathsf{s}^l_u\mathsf{s}^u_k\mathsf{s}^k_l\mathsf{s}^v_u\mathsf{s}^l_v\mathsf{c}_v\mathsf{c}_u x $$
$$ = \text{ (by } E_9\text{)}\mathsf{s}^l_u\mathsf{s}^u_k\mathsf{s}^v_u\mathsf{s}^k_l\mathsf{s}^l_v\mathsf{c}_u\mathsf{c}_v x \text{ (by } E_6\text{) } = \mathsf{s}^l_u\mathsf{s}^u_k\mathsf{s}^v_u\mathsf{s}^k_l\mathsf{c}_u\mathsf{s}^l_v\mathsf{c}_v x $$
$$ \text{(by } E_6\text{) } = \mathsf{s}^l_u\mathsf{s}^u_k\mathsf{s}^v_u\mathsf{c}_u\mathsf{s}^k_l\mathsf{s}^l_v\mathsf{c}_v x = \text{(by } E_8\text{) } \mathsf{s}^l_u\mathsf{s}^v_k\mathsf{c}_u\mathsf{s}^k_l\mathsf{s}^l_v\mathsf{c}_u\mathsf{c}_v x. $$

Now

$$\mathsf{s}_u^l\mathsf{s}_k^v\mathsf{c}_u\mathsf{s}_l^k\mathsf{s}_v^l\mathsf{c}_u\mathsf{c}_v x(\text{ by } E_8) = \mathsf{s}_u^l\mathsf{s}_k^v\mathsf{s}_l^k\mathsf{s}_v^l\mathsf{c}_u\mathsf{c}_v x$$

$$= (\text{ by } E_9)\ \mathsf{s}_k^v\mathsf{s}_u^l\mathsf{s}_l^k\mathsf{s}_v^l\mathsf{c}_u\mathsf{c}_v x = (\text{ by } E_5)\mathsf{s}_k^v\mathsf{s}_u^l\mathsf{s}_l^k\mathsf{c}_l\mathsf{s}_v^l\mathsf{c}_u\mathsf{c}_v x =$$

$$(\text{by } E_8)\ \mathsf{s}_k^v\mathsf{s}_u^k\mathsf{c}_l\mathsf{s}_v^l\mathsf{c}_u\mathsf{c}_v x = (\text{by } E_5)\mathsf{s}_k^v\mathsf{s}_u^k\mathsf{s}_v^l\mathsf{c}_u\mathsf{c}_v x$$

$$(\text{by } E_9) = \mathsf{s}_k^v\mathsf{s}_v^l\mathsf{s}_u^k\mathsf{c}_u\mathsf{c}_v x = (\text{ by } E_6)\ \mathsf{s}_k^v\mathsf{s}_v^l\mathsf{c}_v\mathsf{s}_u^k\mathsf{c}_u x = (\text{ by } E_8)\mathsf{s}_k^l\mathsf{s}_u^k\mathsf{c}_u\mathsf{c}_v x.$$

We have proved that

$$\mathsf{s}_u^l\mathsf{s}_k^u\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = \mathsf{s}_k^l\mathsf{s}_u^k\mathsf{c}_u\mathsf{c}_v x.$$

Now we apply s_l^u to both sides, we obtain, the right hand side is equal to

$$\mathsf{s}_l^u\mathsf{s}_u^l\mathsf{s}_k^u\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = (\text{ by } E_5)\mathsf{s}_l^u\mathsf{s}_u^l\mathsf{c}_u\mathsf{s}_k^u\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x$$

$$= (\text{by } E_8)\ \mathsf{s}_l^l\mathsf{c}_u\mathsf{s}_k^u\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = (\text{by } E_5)\ \mathsf{s}_k^u\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = {}_u\mathsf{s}(k,l)\mathsf{c}_u\mathsf{c}_v x$$

And by definition the left hand side is equal to ${}_u\mathsf{s}(l,k)\mathsf{c}_u\mathsf{c}_v x$. We have proved (i). We now prove (ii). From (i) we have

$${}_u\mathsf{s}(k,l)_u\mathsf{s}(k,l)\mathsf{c}_u\mathsf{c}_v x =_u \mathsf{s}(l,k)_u\mathsf{s}(k,l)\mathsf{c}_u\mathsf{c}_v x$$

$$= (\text{ by definition })\ \mathsf{s}_l^u\mathsf{s}_k^l\mathsf{s}_u^k\mathsf{s}_k^u\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = (\text{ by } E_5)\ \mathsf{s}_l^u\mathsf{s}_k^l\mathsf{s}_u^k\mathsf{s}_k^u\mathsf{c}_k\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x$$

$$= (\text{by } E_8)\ \mathsf{s}_l^u\mathsf{s}_k^l\mathsf{s}_u^u\mathsf{c}_k\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = (\text{ by } E_2)\ \mathsf{s}_l^u\mathsf{s}_k^l\mathsf{c}_k\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x$$

$$= (\text{ by } E_5)\ \mathsf{s}_l^u\mathsf{s}_k^l\mathsf{s}_l^k\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = (\text{by } E_5)\ \mathsf{s}_l^u\mathsf{s}_k^l\mathsf{s}_l^k\mathsf{c}_l\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x$$

$$= (\text{ by } E_8)\ \mathsf{s}_l^u\mathsf{s}_k^k\mathsf{c}_l\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x = (\text{ by } E_8)\ \mathsf{s}_l^u\mathsf{s}_u^l\mathsf{c}_u\mathsf{c}_v x =$$

$$(\text{ by } E_8)\ \mathsf{s}_l^l\mathsf{c}_u\mathsf{c}_v x = (\text{ by } E_2)\ \mathsf{c}_u\mathsf{c}_v x.$$

(i) and (ii) are sometimes called the merry-go-round identities [12]. Since our proof is model theoretic, we recall some notions and concepts from Model Theory. A good reference is [6]. (Our treatment will be self contained.)

3. Some Model-theoretic preparations

Definition 7. *Let L be a signature. By an unnested atomic formula of signature L we mean an atomic formula of one of the following forms:*

$$x = y, c = y, F(\bar{x}) = y \text{ and } R(\bar{x})$$

where c is a constant, F is a function symbol and R is a relation symbol.

Definition 8. *Let L and K be signatures, $\mathfrak{A}$ a K structure $\mathfrak{B}$ an L structure and n a positive integer. An n dimensional interpretation Γ of $\mathfrak{B}$ in $\mathfrak{A}$ is defined to consist of*

(1) *a formula $\partial_\Gamma(x_0, \ldots x_{n-1})$ of signature K,*

(2) for each unnested atomic formula $\phi_\Gamma(\bar{y_0}, \ldots \bar{y}_{m-1})$ a formula $\phi_\Gamma(\bar{x_0}, \ldots \bar{x}_{m-1})$ of signature K in which the x_i's are disjoint n tuples of distinct variables,

(3) a surjective map $f_\Gamma : \partial_\Gamma({}^nA) \to dom(\mathfrak{B})$, such that for all unnested atomic formula ϕ of L and $\bar{a_i} \in \partial_\Gamma({}^nA)$

$$\mathfrak{B} \models \phi(f_\Gamma \bar{a_0}, \ldots f_\Gamma a_{m-1}) \longleftrightarrow \mathfrak{A} \models \phi(\bar{a_0}, \ldots \bar{a_{m-1}}).$$

The formula ∂_Γ is the domain formula of Γ; the formula ∂_Γ and ϕ_T for all unnested atomic formula ϕ are the defining formulas of Γ. If Γ is an interpretation of an L structure $\mathfrak{B}$ in a K structure $\mathfrak{A}$, then there are certain sequences of signature K which must be true in $\mathfrak{A}$ just becuase Γ is an interpretation, regardless of what $\mathfrak{A}$ and $\mathfrak{B}$ are. These sentences say:

(i) Let $=_\Gamma$ be ϕ_Γ when ϕ is $y_0 = y_1$. Then $=_\Gamma$ is an equivalence relation.

(ii) for each unnested atomic formula ϕ of L, if $\mathfrak{A} \models \phi_\Gamma(\bar{a_0}, \ldots \bar{a_{n-1}})$ where $\bar{a_0}, \ldots \bar{a_{n-1}} \in \partial^n_\Gamma A$, then also $\mathfrak{A} \models \phi_\Gamma(\bar{b_0} \ldots \bar{b_{n-1}})$ where each $\bar{b_i}$ is an element of $\partial_\Gamma({}^nA)$ which is $= \Gamma$ equivalent to $\bar{a}_i$.

(iii) if $\phi(y_0)$ is a formula of L of the form $c = y_0$, then there is an $\bar{a}$ in $\partial_\Gamma({}^nA)$such that for all $\bar{b}$ in $\partial_\Gamma({}^nA)$, $\mathfrak{A} \models \phi_\Gamma \bar{b}$ if $\bar{b}$ is $=_\Gamma$ equivalent to $\bar{a}$.

(iv) a clause like (iii) for each function symbol.

For a signature L, $L_{\infty\omega}$ denotes the extension of the first order language of L by infinitary conjunctions and disjunctions. The following Lemma is more general than what we need (however the proof is the same):

Lemma 5. *Let $\mathfrak{A}$ be a K structure, $\mathfrak{B}$ an L structure and Γ an n interpretation of $\mathfrak{B}$ in $\mathfrak{A}$. Then for every formula $\phi(\bar{y})$ of the language $L_{\infty\omega}$ there is a formula $\phi_\Gamma(x)$ of the language $K_{\infty\omega}$ such that*

$$\mathfrak{B} \models \phi(f_\Gamma \bar{a}) \longleftrightarrow \mathfrak{A} \models \phi_\Gamma(\bar{a})$$

Proof. Every formula of $L_{\infty\omega}$ is equivalent to a formula in which all atomic subformulas are nested. We prove the theorem by induction on complexity of formulas, and Definition 6 takes care for the atomic formulas. For compound formulas, we define:

$$(\neg\phi)_\Gamma = \neg(\phi_\Gamma),$$

$$(\bigwedge \phi_i)_\Gamma = \bigwedge(\phi_i)_\Gamma \text{ and likewise with } \bigvee$$

$$(\exists y \phi)_\Gamma = \exists x_0 \ldots x_{n-1}(\partial_\Gamma(x_0, \ldots, x_{n-1}) \wedge \phi_\Gamma).$$

We need to show that elementary equivalence is preserved by taking products. For this purpose we devise a game between $\forall$ (male) and $\exists$(female). We imagine that $\forall$ wants to prove that $\mathfrak{A}$ is different from $\mathfrak{B}$ while $\exists$ tries to show that $\mathfrak{A}$ is the same as $\mathfrak{B}$. So their conversation has the form of a game. Player $\forall$ wins if he manages to find a difference between $\mathfrak{A}$ and $\mathfrak{B}$

before the play is over; otherwise $\exists$ wins. The game is played in $\mu \leq \omega$ steps. At the ith step of a play, player $\forall$ takes one of the structures $\mathfrak{A}$, $\mathfrak{B}$ and chooses an element this structure; then $\exists$ chooses an atom of the other structure. So between them they choose an element a_i of $\mathfrak{A}$ and an element b_i of $\mathfrak{B}$. Apart from the fact that player $\exists$ must choose from the other structure from player $\forall$ at each step, both players have complete freedom to choose as they please; in particular, either player can choose an element which was chosen at an earlier step. Player $\exists$ is allowed to see and remember all previous moves in the play. (As the game theorists would say, this is a game of perfect information.) At the end of the play sequences $\bar{a} = (a_i : i < \mu)$ and $\bar{b} = (b_i : i < \mu)$ have been chosen. The pair $(\bar{a}, \bar{b})$ is known as the play. We count the play $(\bar{a}, \bar{b})$ as a win for player $\exists$, and we say that $\exists$ wins the play, if for every unnested atomic formula ϕ of L

$$\mathfrak{A} \models \phi(\bar{a}) \longleftrightarrow \mathfrak{B} \models \phi(\bar{b})$$

Let us denote this game by $EF_\mu(\mathfrak{A}, \mathfrak{B})$. (It is an instance of an Ehrenfeuch-Fraisse game.) The more $\mathfrak{A}$ is like $\mathfrak{B}$, the better chance player $\exists$ has of winning these games. For example if player $\exists$ knows about an isomorphism $i : \mathfrak{A} \to \mathfrak{B}$ then she can be sure of winning every time. All she has to do to follow the rule is: Choose $i(a)$ whenever player $\forall$ has just chosen an element a of $\mathfrak{A}$ and $i^{-1}(b)$ whenever player $\forall$ has just chosen b from $\mathfrak{B}$. We write $\mathfrak{A} \sim_k \mathfrak{B}$ if $\exists$ can win $EF_k(\mathfrak{A}, \mathfrak{B})$.

Lemma 6. *Let L be a first order language with finite signature. Then for any two L structures $\mathfrak{A}$ and $\mathfrak{B}$ the following are equivalent*

(i) $\mathfrak{A} \equiv \mathfrak{B}$

(ii) $\mathfrak{A} \sim_k \mathfrak{B}$ *for all $k < \omega$.*

Proof. [Sketch] $\mathfrak{A}$ and $\mathfrak{B}$ agree on all unnested sentences of finite quantifier rank, so (i) implies (ii). The other direction follows from the fact that every first order sentence is equivalent to an unnested sentence of finite quantifier rank.

A strategy for a player in a game is a set of rules which tell the player exactly how to move, depending on what has happened earlier in the play. We say that the player uses the strategy σ in a play if each of his or her moves obeys the rules of σ. We say that σ is a winning strategy if the player wins every play in which he or she uses σ. We now have

Lemma 7. *Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{B}$ be boolean algebras. Assume that $\mathfrak{B}_1 \equiv \mathfrak{B}_2$, then $\mathfrak{B}_1 \times \mathfrak{B} \equiv \mathfrak{B}_2 \times \mathfrak{B}$.*

Proof. It suffices to show that if $k < \omega$ and $\mathfrak{B}_1 \sim_k \mathfrak{B}_2$ then $\mathfrak{B}_1 \times \mathfrak{B} \sim_k \mathfrak{B}_2 \times \mathfrak{B}$. Assume henceforth that $\mathfrak{B}_1 \sim_k \mathfrak{B}_2$. Then $\exists$ has a winning strategy σ for the game $EF_k(\mathfrak{B}_1, \mathfrak{B}_2)$. Let the two players play the game $EF_k(\mathfrak{B}_1 \times \mathfrak{B}, \mathfrak{B}_2 \times \mathfrak{B})$. $\exists$ guides her choices by the side game $EF_k(\mathfrak{B}_1, \mathfrak{B}_2)$. Whenever $\forall$ offers an element, say the element $a \in B_1 \times B$, player $\exists$ first splits it into a product $a = (g, h)$ with $g \in B_1$ and $h \in B$. Then she pretends that $\forall$ has chosen g in the side game. She uses her strategy σ to choose a reply g' of g in the side game. Her reply to the element a will be the element $b = (g', h) \in B_2 \times B$. At the end of the game let the play be $((g_0,h_0),\dots(g_{k-1},h_{k-1});(g'_0,h'_0),\dots(g'_{k-1},h'_{k-1}))$. Player $\exists$ has won the side game. Now the unnested atomic formulas of boolean algebras are of the form $x=y$, $1=x$, $0=x, x_0\wedge x_1=y, x_0\vee x_1=y$ and $-x=y$. So for $i,j,l<k$ we have

$$g_i=g_j \text{ iff } g'_i=g'_j$$

$$1=g_i \text{ iff } 1=g'_i$$

$$0=g_i \text{ iff } 0=g'_i$$

$$g_i\wedge g_j=g_l \text{ iff } g'_i\wedge g'_j=g'_l$$

$$g_i\vee g_i=g_l \text{ iff } g'_i\wedge g'_j=g'_l$$

$$-g_i=g_j \text{ iff } -g'_i=g'_j.$$

By the cartesian product for boolean algebras , this implies that for all $i,j,l<k$ we also have

$$(g_i,h_i)=(g_j,h_j) \text{ iff } (g'_i,h_i)=(g'_j,h_i)$$

$$1=(g_i,h_i) \text{ iff } 1=(g'_i,h_i)$$

same for 0

$$0=(g_i,h_i) \text{ iff } 0=(g'_i,h_i)$$

$$(g_i\wedge h_i,g_j\wedge h_j)=(g_l,h_l) \text{ iff } (g'_i\wedge h_i,g'_j\wedge h_j)=(g'_l,h_l)$$

$$(g_i\vee h_i,g_j\vee h_j)=(g_l,h_l) \text{ iff } (g'_i\vee h_i,g'_j\vee h_j)=(g'_l,h_l)$$

$$-(g_i,h_i)=(g_j,h_j) \text{ iff } -(g'_i,h_i)=(g'_j,h_j).$$

So $\exists$ wins the game, which proves the lemma..

Definition 9. *(1) Let L be a signature and $\mathfrak{D}$ an L structure. The age of $\mathfrak{D}$ is the class* **K** *of all finitely generated structures that can be embedded in $\mathfrak{D}$.*

(2) A class **K** *is the age of $\mathfrak{D}$ if the structures in* **K** *are up to isomorphism, exactly the finitely generated substructures of $\mathfrak{D}$.*

(3) Let **K** *be a class of structures.*

(4) **K** *has the Hereditary Property, HP for short. if whenever $\mathfrak{A}\in$* **K** *and $\mathfrak{B}$ is a finitely generated substructure of $\mathfrak{A}$ then $\mathfrak{B}$ is isomorphic to some structure in* **K***.*

(5) **K** *has the Joint Embedding Property, JEP for short if whenever $\mathfrak{A},\mathfrak{B}\in$* **K** *then there is a $\mathbb{C}\in$* **K** *such that both $\mathfrak{A}$ and $\mathfrak{B}$ are embeddable in $\mathbb{C}$.*

(6) **K** *has Amalgamation Property, or AP for short if $\mathfrak{A},\mathfrak{B},\mathbb{C}\in K$ and $e:\mathfrak{A}\to\mathfrak{B}$, $f:\mathfrak{A}\to\mathbb{C}$ are embeddings, then there are $\mathfrak{D}$ in* **K** *and embeddings $g:\mathfrak{B}\to\mathfrak{D}$ and $h:\mathbb{C}\to\mathfrak{D}$ such that $g\circ e=h\circ f$.*

(7) A structure $\mathfrak{D}$ is weakly homogeneous if it has the the following property if $\mathfrak{A}$, $\mathfrak{B}$ are finitely generated substructures of $\mathfrak{D}$, $A \subseteq B$ and $f : \mathfrak{A} \to \mathfrak{D}$ is an embedding, then there is an embedding $g : \mathfrak{B} \to \mathfrak{D}$ which extends f.

(8) We call a structure $\mathfrak{D}$ homogeneous if every isomorphism between finitely generated substructures extends to an automorphism of $\mathfrak{D}$.

Note that if $\mathfrak{D}$ is homogeneous, then it is weakly homogeneous. We recall from [6] Thm 7.1.2, a theorem of Fraisse that puts the above pieces together.

Theorem 6. *Let L be a countable signature and let* **K** *be a non-empty finite or countable set of finitely generated L-structures which has HP, JEP and AP. Then there is an L structure $\mathfrak{D}$, unique up to isomorphism, such that*

(1) $\mathfrak{D}$ has cardinality $\leq \omega$

(2) **K** *is the age of D, and*

(3) $\mathfrak{D}$ is homogeneous.

Following Hodges [6] we also refer to $\mathfrak{D}$ is as *Fraisse limit* of the class **K**. Our next theorem, gives a sufficient condition for when the Fraisse limit $\mathfrak{D}$ of a class **K** of finitely generated structures, has quantifier elimination. Recall that an L-structure **M** has *quantifier elimination* if every L formula $\phi(\bar{x})$ is equivalent in **M** to a boolean combination of quantifier free formulas, equivalently atomic formulas. A theory T is ω- categorical if all countable models of T are isomorphic.

Lemma 8. *Suppose that the signature L is finite and has no function symbols. Suppose that* **K** *is a countable set of finite L structures with HP, JEP and AP. Let* **M** *be the Fraisse limit of* **K**. *Let T be the first order theory $Th(\mathbf{M})$ of* **M**. *Then*

(i) T is ω-categorial.

(ii) **M** *has quantifier elimination*

Proof. The proof is taken from [6]. We include it for the sake of completeness. We note that the following hold: If $\mathfrak{A}$ is any finite L structure with n generators $\bar{a}$, then there is a quantifier free formula $\psi_{A,\bar{a}}(x_0 \dots x_{n-1})$ such that for any L structure $\mathfrak{B}$ and n-tuple $\bar{b}$ of elements of $\mathfrak{B}$,

(1) $\mathfrak{B} \models \phi[\bar{b}]$ if and only if there is an isomorphism from $\mathfrak{A}$ to $\langle b \rangle_B$ which takes $\bar{a}$ to $\bar{b}$. In fact $\psi_{A,\bar{a}}$ is a conjunction of literals satisfied by $\bar{a}$ in $\mathfrak{A}$.

Also or each $n < \omega$ there are only finitely many isomorphism types of structures in **K** with n generators.

Let U_0 be the set of all sentences of the form

$$(\forall \bar{x})(\psi_{A,\bar{a}}(\bar{x}) \implies \exists y \psi_{B,\bar{a}b}(\bar{x}, y)) \tag{4}$$

where $\mathfrak{B}$ is a structure in $\mathbf{K}$ generated by a tuple $\bar{a}b$ of distinct elements, and $\mathfrak{A}$ is the substructure generated by $\bar{a}$. let U_1 be the set of sentences of the form

$$(\forall x)\bigvee \psi_{\mathfrak{A},\bar{a}}(\bar{x}) \tag{5}$$

where the disjunction is over all pairs $\mathfrak{A}, \bar{a}$ such that $\mathfrak{A} \in \mathbf{K}$ and $\bar{a}$ is a tuple of the same length as $\bar{x}$ which generates $\mathfrak{A}$. Then this is a finite disjunction. Let $U = U_0 \cup U_1$. Then $\mathbf{M}$ is a model of U. Suppose that $\mathfrak{D}$ is any countable model of U. Then the sentences (1) say that if

(4) $\mathfrak{A}$, $\mathfrak{B}$ are finitely generated substructures of $\mathfrak{D}$ $A \subseteq B$, $\mathfrak{B}$ comes from $\mathfrak{A}$ by adding one more generator, and $f : \mathfrak{A} \to \mathfrak{D}$ is an embedding , then there is an embedding $g : \mathfrak{B} \to \mathfrak{D}$ which extends f. Using induction on the number of generators, imply that every structure in $\mathbf{K}$ is embeddable in $\mathfrak{D}$; so together with (3) this implies that the age of $\mathfrak{D}$ is exactly $\mathbf{K}$. Using (2) an induction on the size of $dom(\mathfrak{B}) \setminus dom(\mathfrak{A})$, tells us that $\mathfrak{D}$ is weakly homogeneous, so $\mathfrak{D}$ is isomorphic to $\mathbf{M}$. Hence U is ω categorical and U is a set of axioms for T. Suppose now that $\phi(\bar{x})$ is a formula of L, and let X be the set of all tuples $\bar{a}$ in $\mathbf{M}$ such that $\mathbf{M} \models \phi(\bar{a})$. If $\bar{a}$ is in X, and $\bar{b}$ is a tuple of elements such that there is an isomorphism $e : \langle \bar{a}_M \rangle \to \langle \bar{b}_M \rangle$ taking $\bar{a} \to \bar{b}$, then e extends to an automorphism of $\mathbf{M}$, so that $\bar{b}$ is in X too. It follows that ϕ is equivalent modulo T to the disjunction of all the formulas $\psi_{\langle \bar{a} \rangle, \bar{a}}(\bar{x})$ with $\bar{a} \in X$.This is a finite disjunction of quantifier free formulas. Finally if ϕ is a sentence of L then since T is complete, ϕ is equivalent to either $\top$ or $\perp$.

Notation . S_3 denotes the set of all permutations of 3. ${}^XY$ denotes the set of functions from X to Y. For $u, v \in {}^33$, $i < 3$ we write u_i for $u(i) < 3$, and we write $u \equiv_i v$ if u and v agree off i, i.e if $u_j = v_j$ for all $j \in 3 \setminus \{i\}$. For a symbol R of the signature of $\mathbf{M}$ we write $R^{\mathbf{M}}$ for the interpretation of R in $\mathbf{M}$.

Lemma 9. *Let L be a signature consisting of the unary relation symbols P_0, P_1, P_2 and uncountably many 3-ary predicate symbols. For $u \in {}^33$, let χ_u be the formula $\bigwedge_{i<3} P_{u_i}(x_i)$. Then there exists an L-structure $\mathbf{M}$ with the following properties:*

(1) $\mathbf{M}$ *has quantifier elimination, i.e. every L-formula is equivalent in* $\mathbf{M}$ *to a boolean combination of atomic formulas.*

(2) The sets $P_i^{\mathbf{M}}$ for $i < 3$ partition M,

(3) $\mathbf{M} \models \forall x_0 x_1 x_2 (R(x_0, x_1 x_2) \longrightarrow \bigvee_{u \in S_3} \chi_u)$*, for all $R \in L$,*

(4) $\mathbf{M} \models \exists x_0 x_1 x_2 (\chi_u \wedge R(x_0, x_1, x_2) \wedge \neg S(x_0, x_1, x_2))$ *for all distinct ternary $R, S \in L$, and $u \in S_3$,*

(5) For $u \in S_3$, $i < 3$, $\mathbf{M} \models \forall x_0 x_1 x_2 (\exists x_i \chi_u \longleftrightarrow \bigvee_{v \in {}^33, v \equiv_i u} \chi_v)$,

(6) For $u \in S_3$ and any L-formula $\phi(x_0, x_1, x_2)$, if $\mathbf{M} \models \exists x_0 x_1 x_2 (\chi_u \wedge \phi)$ *then* $\mathbf{M} \models \forall x_0 x_1 x_2 (\exists x_i \chi_u \longleftrightarrow \exists x_i (\chi_u \wedge \phi))$ *for all $i < 3$.*

Proof. Throughout the proof, we use the notation $\bar{x}, \bar{a}$ for finite sequences, or tuples $\langle x_0, \cdots x_{m-1}\rangle, \langle a_0, \cdots a_{m-1}\rangle$. Given a structure **M** and a tuple $\bar{a}$, we often write, with a slight abuse of notation, $\bar{a} \in M$ instead of $\bar{a} \in {}^m M$, where m is the arity of the tuple $\bar{a}$. The arity of tuples will be clear from context. Let $\mathscr{L}$ be the relational signature containing unary relation symbols $P_0, \ldots, P_3$ and a 4-ary relation symbol X. Let **K** be the class of all finite $\mathscr{L}$-structures $\mathfrak{D}$ satsfying

$$\text{The } P_i\text{'s are disjoint } : \forall x \bigvee_{i<j<4} (P_i(x) \wedge \bigwedge_{j \neq i} \neg P_j(x)). \tag{6}$$

$$\forall x_0 \cdots x_3 (X(x_0, \cdots, x_3) \longrightarrow P_3(x_3) \wedge \bigvee_{u \in \mathsf{s}_3} \chi_u). \tag{7}$$

Then **K** contains countably many isomorphism types, because for each $n \in \omega$, there are countably many isomorphism types of finite L structures (satifying (6) and (7)) having cardinality $\leq n$. Also it is easy to check that **K** is closed under substructures and that **K** has the *AP*. From the latter it follows that it has the JEP, since **K** contains the one element structure that is embeddable in any structure in **K**. * Then there is a countably infinite homogeneous $\mathscr{L}$-structure $\mathfrak{N}$ with age **K**. $\mathfrak{N}$ has quantifier elimination, and obviously, so does any elementary extension of $\mathfrak{N}$. **K** contains structures with arbitrarily large P_3-part, so $P_3^{\mathfrak{N}}$ is infinite. Let $\mathfrak{N}^*$ be an elementary extension of $\mathfrak{N}$ such that $|P_3^{\mathfrak{N}^*}| = |L|$, and fix a bijection $*$ from the set of ternary relation symbols of L to $P_3^{\mathfrak{N}^*}$. Define an L-structure **M** with domain $P_0^{\mathfrak{N}^*} \cup P_1^{\mathfrak{N}^*} \cup P_2^{\mathfrak{N}^*}$, by: $P_i^{\mathbf{M}} = P_i^{\mathfrak{N}^*}$ for $i < 3$ and for ternary $R \in L$,

$$\mathbf{M} \models R(a_0, a_1, a_2) \text{ iff } \mathfrak{N}^* \models X(a_0, a_1, a_2, R^*).$$

If $\phi(\bar{x})$ is any L-formula, let $\phi^*(\bar{x}, \bar{R})$ be the $\mathscr{L}$-formula with parameters $\bar{R}$ from $\mathfrak{N}^*$ obtained from ϕ by replacing each atomic subformula $R(x, y, z)$ by $X(x, y, z, R^*)$ and relativizing quantifiers to $\neg P_3$, that is replacing $(\exists x)\phi(x)$ and $(\forall x)\phi(x)$ by $(\exists x)(\neg P_3(x) \to \phi(x))$ and $(\forall x)(\neg P_3(x) \to \phi(x))$, respectively. A straightforward induction on complexity of formulas gives that for $\bar{a} \in \mathbf{M}$

$$\mathbf{M} \models \phi(\bar{a}) \text{ iff } \mathfrak{N}^* \models \phi^*(\bar{a}, \bar{R}).$$

We show that **M** is as required. For quantifier elimination, if $\phi(\bar{x})$ is an L-formula , then $\phi^*(\bar{x}, \bar{R}^*)$ is equivalent in $\mathfrak{N}^*$ to a quantifier free $\mathscr{L}$-formula $\psi(\bar{x}, \bar{R}^*)$. Then replacing ψ's atomic subformulas $X(x, y, z, R^*)$ by $R(x, y, z)$, replacing all $X(t_0, \cdots t_3)$ not of this form by $\perp$, replacing subformulas $P_3(x)$ by $\perp$, and $P_i(R^*)$ by $\perp$ if $i < 3$ and $\top$ if $i = 3$, gives a quantifier free L -formula ψ equivalent in **M** to ϕ.

For (2), let

$$\sigma = \forall x(\neg P_3(x) \longrightarrow \bigvee_{i<3} (P_i(x) \wedge \bigwedge_{j \neq i} \neg P_j(x))).$$

Then $\mathbf{K} \models \sigma$, so $\mathfrak{N} \models \sigma$ and $\mathfrak{N}^* \models \sigma$. It follows from the definition that **M** satisfies (2); (3) is similar.

*It is not always true that *AP* implies JEP; think of fields.

For (4), let $u \in S_3$ and let $r,s \in P_3^{\mathbf{M}}$ be distinct. Take a finite $\mathscr{L}$-structure $\mathfrak{D}$ with points $a_i \in P_{u_i}^{\mathfrak{D}}(i < 3)$ and distinct $r', s' \in P_3^{\mathfrak{D}}$ with

$$\mathfrak{D} \models X(a_0, a_1, a_2, r') \wedge \neg X(a_0, a_1, a_2, s').$$

Then $\mathfrak{D} \in \mathbf{K}$, so D embeds into $\mathfrak{N}$. By homogeneity, we can assume that the embedding takes r' to r and s' to s. Therefore

$$\mathfrak{N} \models \exists \bar{x}(\chi_u \wedge X(\bar{x}, r) \wedge \neg X(\bar{x}, s)),$$

where $\bar{x} = \langle x_0, x_1, x_2 \rangle$. Since r,s were arbitrary and $\mathfrak{N}^*$ is an elementary extension of $\mathfrak{N}$, we get that

$$\mathfrak{N}^* \models \forall yz(P_3(y) \wedge P_3(z) \wedge y \neq z \longrightarrow \exists \bar{x}(\chi_u \wedge X(\bar{x}, y) \wedge \neg(X(\bar{x}, z))).$$

The result for $\mathbf{M}$ now follows.

Note that it follows from (3,4) that $P_i^{\mathbf{M}} \neq \emptyset$ for each $i < 3$. So it is clear that

$$\mathbf{M} \models \forall x_0 x_1 x_2 (\exists x_i \chi_u \longleftrightarrow \bigvee_{v \in {}^3 3, v \equiv_i u} \chi_v);$$

giving (5).

Finally consider (6). Clearly, it is enough to show that for any $\mathscr{L}$-formula $\phi(\bar{x})$ with parameters $\bar{r} \in P_3^{\mathbf{M}}, u \in S_3, i < 3$, we have

$$\mathfrak{N} \models \exists \bar{x}(\chi_u \wedge \phi) \longrightarrow \forall \bar{x}(\exists x_i(\chi_u \longrightarrow \exists x_i(\chi_u \wedge \phi)).$$

For simplicity of notation assume $i = 2$. Let $\bar{a}, \bar{b} \in \mathfrak{N}$ with

$$\mathfrak{N} \models (\chi_u \wedge \phi)(\bar{a}) \text{ and } \mathfrak{N} \models \exists x_2(\chi_u(\bar{b})).$$

We require

$$\mathfrak{N} \models \exists x_2(\chi_u \wedge \phi)(\bar{b}).$$

It follows from the assumptions that

$$\mathfrak{N} \models P_{u_0}(a_0) \wedge P_{u_1}(a_1) \wedge a_0 \neq a_1, \text{ and } \mathfrak{N} \models P_{u_0}(b_0) \wedge P_{u_1}(b_1) \wedge b_0 \neq b_1.$$

These are the only relations on $a_0 a_r \bar{r}$ and on $b_0 b_1 \bar{r}$ (cf. property (3) of Lemma 13), so

$$\theta^- = \{(a_0, b_0)(a_1, b_1)(r_l, r_l) : l < |\bar{r}|\}$$

is a partial isomorphism of $\mathfrak{N}$. By homogeneity, it is induced by an automorphism θ of $\mathfrak{N}$. Let $c = \theta(\bar{a}) = (b_0, b_1, \theta(a_2))$. Then $\mathfrak{N} \models (\chi_u \wedge \phi)(\bar{c})$. Since $\bar{c} \equiv_2 \bar{b}$, we have $\mathfrak{N} \models \exists x_2(\chi_u \wedge \phi)(\bar{b})$ as required.

Now we explain the idea behind the construction of such an $\mathbf{M}$, and in the process give an outline of the proof that the class of neat reducts is not elementary, that paves the way for a smooth (formal) proof of our main Theorem. Throughout fix $\mathbf{M}$ as in Lemma 9.

We will go through the conditions one by one. Condition 1 of quantifier elimination says that the set of atomic formulas

$$J = \{R(y_0, y_1, y_2) : \{y_0, y_1.y_2\} = \{x_0, x_1, x_2\} \text{ and } R \in L \text{ is a ternary relation}\}$$

$$\bigcup \{P_i(x_j) : i, j < 3\} \cup \{x_i = x_j : i, j < 3\}$$

is an elimination set for **M**, meaning that every formula $\phi \in L$ is equivalent in **M** to a boolean combination of formulas in J. This implies that the cylindric set algebra based on **M** using only the first three variables is a neat reduct. In more detail, for $\phi \in L$, let $\phi^{\mathbf{M}}$ be the set of all assignments satisfying ϕ in **M** i.e.

$$\phi^{\mathbf{M}} = \{s \in {}^{\omega}M : \mathbf{M} \models \phi[s]\}.$$

Cs_n denotes the class of cylindric set algebras of dimension n. Let $\mathfrak{A}_\omega$ be the Cs_ω with domain

$$\{\phi^M : \phi \in L\}$$

and operations (well-) defined by (cf.[12])†

$$\phi^{\mathbf{M}}.\psi^{\mathbf{M}} = \phi^{\mathbf{M}} \cap \psi^{\mathbf{M}} = (\phi \wedge \psi)^{\mathbf{M}};$$

$$-\phi^{\mathbf{M}} = (\neg\phi)^{\mathbf{M}};$$

and for $i, j < \omega$

$$d_{ij} = (x_i = x_j)^{\mathbf{M}};$$

and

$$\mathsf{c}_i(\phi^{\mathbf{M}}) = (\exists x_i \phi)^{\mathbf{M}}.$$

Now write L_3 for the set of all L-formulas using only the first three variables. Then a moment's reflection will show that condition 1 says that the Cs_3 $\mathfrak{A}$ with domain

$$\{\phi^{\mathbf{M}} : \phi \in L_3\}$$

is the same as the (possibly bigger) Cs_3 with domain

$$\{\phi^{\mathbf{M}} : \phi \in L \text{ and } \phi \text{ contains } x_0, x_1, x_2 \text{ as free variables}\},$$

with the operation defined, for both, as for $\mathfrak{A}_\omega$. But the latter, as easily checked, is isomorphic to $\mathfrak{Nr}_3\mathfrak{A}_\omega$, so condition 1 guarantees that $\mathfrak{A} \in \mathfrak{Nr}_3\mathbf{CA}_\omega$. The rest of the conditions are designed to extract an elementary subalgebra of $\mathfrak{A}$ such that its $r\mathbf{SC}$ reduct is not in $\mathfrak{Nr}_3(r\mathbf{SC}_5)$. But let us first understand the (abstract) structure of $\mathfrak{A}$ based on **M**. Condition (2), says that

$$\{\chi_u^{\mathbf{M}} : u \in {}^{3}3\}$$

†In [12], sec 4.3, cf. Definition 4.3.4 $\mathscr{A}_\omega$ would be denoted by $Cf_3^{\mathbf{M}}$, which is the set algebra based on **M**. In this connection we note that $\mathfrak{A}$ is a *regular locally* finite Cs_ω.

is a partition of ${}^3M$, the unit of $\mathfrak{A}$. That is

$$\bigcup_{u\in{}^33}(\chi_u)^{\mathbf{M}} = {}^3M,$$

and for distinct $u, v \in {}^33$ we have

$$(\chi_u)^{\mathbf{M}} \cap (\chi_v)^{\mathbf{M}} = \emptyset.$$

Conditions (3) and (4) single out the $\chi_u^{\mathbf{M}}$'s that are indexed by permutations $u \in S_3$. Note that for any such u, if $\langle a_0, a_1, a_2\rangle \in \chi_u^{\mathbf{M}}$, then the a_i's are distinct because $a_i \in P_{u_i}$ for $i < 3$ and by (2) these are disjoint. Condition (3) says that for any ternary $R(\bar{x}) \in L$, $R(\bar{x})^M \subseteq \bigcup_{u\in S_3}(\chi_u)^{\mathbf{M}}$ with $u \in S_3$, so this means that if $\langle a_0, a_1, a_2\rangle \in R(\bar{x})^{\mathbf{M}}$, then the a_i's must be distinct, too. Condition (4) says that below every such $(\chi_u)^{\mathbf{M}}$ with $u \in S_3$, there are *uncountably* many pairwise distinct non-empty elements, namely, the $R(\bar{x})^{\mathbf{M}} \cap (\chi_u)^{\mathbf{M}}$, for ternary $R \in L$. Condition (5) tells us how the $(\chi_u)^{\mathbf{M}}$'s behave with respect to cylindrifications. It simply says that for $u \in S_3$ and $i < 3$ we have

$$\mathsf{c}_i(\chi_u)^{\mathbf{M}} = \bigcup_{v\in{}^33, v\equiv_i u}(\chi_v)^{\mathbf{M}}.$$

A moment's reflection will reveal that this follows from (3) and (4.) Finally, condition (6) says that elements below $\chi_u^{\mathbf{M}}$ are *big*, as far as cylindrifications are concerned, that is for any ϕ such that

$$(\phi \cap \chi_u)^{\mathbf{M}} \neq \emptyset$$

and any $i < 3$, we have

$$\mathsf{c}_i(\phi^{\mathbf{M}} \cap \chi_u^{\mathbf{M}}) = \mathsf{c}_i(\chi_u^M) = \bigcup_{v\in{}^33, v\equiv_i u}(\chi_v)^{\mathbf{M}}.$$

Summarizing the above, let 1_u denote $\chi_u^{\mathbf{M}}$. Then by condition (2), we have $\{1_u : u \in {}^33\}$ is a partition of the unit of $\mathfrak{A}$. If $u \in S_3$, then below every 1_u, there are uncountably many pairwise distinct non empty elements, namely the $R(\bar{x})^{\mathbf{M}}$'s intersected with 1_u. (conditions (3), (4)). Such elements are big as far as the cylindrifications are concerned, that is for $i < 3$ we have (by conditions (5), (6))

$$\mathsf{c}_i(R(\bar{x})^{\mathbf{M}} \cap 1_u) = \mathsf{c}_i(1_u) = \bigcup_{v\equiv_i u} 1_u.$$

Having explained the idea behind the conditions of Lemma 13 we explain how we will go about extracting an elementary subalgebra of $\mathfrak{A}$ that is not a neat reduct.

For $u \in {}^33$, let A_u stand for the *relativisation* of $\mathfrak{A}$ to 1_u i.e.

$$A_u = \{x \in A : x \leq 1_u\}.$$

A_u is the domain of a boolean set algebra which we denote by $\mathfrak{A}_u$. Then for $u \in S_3$, $\mathfrak{A}_u$ is uncountable. Because $\{1_u : u \in {}^33\}$ is a partition of the unit of $\mathfrak{A}$, it follows that the boolean reduct of $\mathfrak{A}$ is isomorphic to the boolean product, $\prod_{u\in {}^33}\mathfrak{A}_u$. Moreover we can expand the language of boolean algebras by diagonal elements and the constants 1_u in such a way that the cylindric algebra $\mathfrak{A}$ becomes *interpretable* in this product. Then we are able to extract an elementary subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ by an infinite cardinality twist, that first order logic does not see. $\mathfrak{B}$ is simply obtained from $\mathfrak{A}$ by keeping only many countably elements below 1_{Id}, where Id is the identity function on 3, and throwing away the rest of the elements below 1_{Id}. In the product, this corresponds to replacing the component $\mathfrak{A}_{Id}$ by an arbitrary elementary countable boolean subalgebra $\mathfrak{B}_{Id}$ of $\mathfrak{A}_{Id}$ and giving the resulting algebra the interpretation given to the the boolean product $\prod_{u\in {}^33}\mathfrak{A}_u$. This will not be witnessed by first order logic, but will *enforce* that the resulting structure $\mathfrak{B}$, which is of course a $\mathbf{CA}_3$, is *not* a neat reduct. In fact, $\mathfrak{B}$ will not be even in $\mathfrak{Nr}_3\mathbf{CA}_4$ and its $r\mathbf{SC}$ reduct is not in $\mathfrak{Nr}_3 r\mathbf{SC}_5$. The idea is that had $\mathfrak{B}$ been a neat reduct then using a substitution term definable in extra dimensions, will give uncountably many elements in the component $\mathfrak{B}_{Id}$, which contradicts that the latter, by construction, is countable. Now we implement the details of the above sketch.

Proof. [main result] Fix L and $\mathbf{M}$ as in Lemma 9. Let $\mathfrak{A}_\omega$, $\mathfrak{A}$ be as specified above. That is $A_\omega = \{\phi^M : \phi \in L\}$ and $A = \{\phi^M : \phi \in L_3\}$. Then $\mathfrak{A} \cong \mathfrak{Nr}_3\mathfrak{A}_\omega$, the isomorphism is given by

$$\phi^{\mathbf{M}} \mapsto \phi^{\mathbf{M}}.$$

Quantifier elimination in M guarantees that this map is onto. For $u \in {}^33$, let $\mathfrak{A}_u$ denote the relativisation of $\mathfrak{A}$ to $\chi_u^{\mathbf{M}}$ i.e

$$\mathfrak{A}_u = \{x \in A : x \leq \chi_u^M\}.$$

$\mathfrak{A}_u$ is a boolean algebra. Also $\mathfrak{A}_u$ is uncountable for every $u \in S_3$ because by property (4) of Lemma 9 the sets $(\chi_u \wedge R(x_0, x_1, x_2))^{\mathbf{M}}$, for $R \in L$ are distinct elements of A_u. Define a map $f : \mathfrak{A} \to \prod_{u\in {}^33}(\mathfrak{A}_u)$, by

$$f(a) = \langle a.\chi_u\rangle_{u\in {}^33}.$$

We will expand the language of the boolean algebra $\prod_{u\in {}^33}\mathfrak{A}_u$ in such a way that the cylindric algebra $\mathfrak{A}$ becomes interpretable in the expanded structure. For this we need.

Definition 10. *Let ¶ denote the following structure for the signature of boolean algebras expanded by constant symbols 1_u for $u \in {}^33$ and d_{ij} for $i, j \in 3$:*

(1) The boolean part of ¶ is the boolean algebra $\prod_{u\in {}^33}\mathfrak{A}_u$,

(2) $1_u^{¶} = f(\chi_u^{\mathbf{M}}) = \langle 0, \cdots 0, 1, 0, \cdots\rangle$ (with the 1 in the u^{th} place) for each $u \in {}^33$,

(3) $\mathsf{d}_{ij}^{¶} = f(\mathsf{d}_{ij}^{\mathfrak{A}})$ for $i, j < 3$.

We now show that $\mathfrak{A}$ is interpretable in ¶. For this it is enough to show that f is one to one and that $Rng(f)$ (Range of f) and the f-images of the graphs of the cylindric algebra functions in $\mathfrak{A}$ are definable in ¶. Since the $\chi_u^{\mathbf{M}}$ partition the unit of $\mathfrak{A}$, each $a \in A$ has a unique expression in the form $\sum_{u\in {}^33}(a.\chi_u^{\mathbf{M}})$, and it follows that f is boolean isomorphism: $bool(\mathfrak{A}) \to \prod_{u\in {}^33}\mathfrak{A}_u$. So the f-images of the graphs of the boolean functions on $\mathfrak{A}$ are trivially definable. f is bijective so $Rng(f)$ is definable, by $x = x$. For the diagonals, $f(\mathsf{d}_{ij}^{\mathfrak{A}})$ is definable by $x = \mathsf{d}_{ij}$. Finally we consider cylindrifications. For $S \subseteq {}^33$, $i < 3$, let t_S be the closed term

$$\sum\{1_v : v \in {}^33, v \equiv_i u \text{ for some } u \in S\}.$$

Let

$$\eta_i(x,y) = \bigwedge_{S\subseteq {}^33}(\bigwedge_{u\in S} x.1_u \neq 0 \land \bigwedge_{u\in {}^33\smallsetminus S} x.1_u = 0 \longrightarrow y = t_S).$$

We claim that for all $a \in A$, $b \in P$, we have

$$\P \models \eta_i(f(a), b) \text{ iff } b = f(\mathsf{c}_i^{\mathfrak{A}}a).$$

To see this, let $f(a) = \langle a_u\rangle_{u\in {}^33}$, say. So in $\mathfrak{A}$ we have $a = \sum_u a_u$. Let u be given; a_u has the form $(\chi_i \land \phi)^{\mathbf{M}}$ for some $\phi \in L^3$, so $\mathsf{c}_i^A(a_u) = (\exists x_i(\chi_u \land \phi))^{\mathbf{M}}$. By property 6 of Lemma 9, if $a_u \neq 0$, this is $(\exists x_i\chi_u)^M$; by property 5, this is $(\bigvee_{v\in {}^33, v\equiv_i u}\chi_v)^{\mathbf{M}}$. Let $S = \{u \in {}^33 : a_u \neq 0\}$. By normality and additivity of cylindrifications we have,

$$\mathsf{c}_i^A(a) = \sum_{u\in {}^33}\mathsf{c}_i^A a_u = \sum_{u\in S}\mathsf{c}_i^A a_u = \sum_{u\in S}(\sum_{v\in {}^33, v\equiv_i u}\chi_v^{\mathbf{M}})$$

$$= \sum\{\chi_v^{\mathbf{M}} : v \in {}^33, v \equiv_i u \text{ for some } u \in S\}.$$

So $\P \models f(\mathsf{c}_i^{\mathfrak{A}}a) = t_S$. Hence $\P \models \eta_i(f(a), f(\mathsf{c}_i^{\mathfrak{A}}a))$. Conversely, if $\P \models \eta_i(f(a), b)$, we require $b = f(\mathsf{c}_i a)$. Now S is the unique subset of ^{33} such that

$$\P \models \bigwedge_{u\in S} f(a).1_u \neq 0 \land \bigwedge_{u\in {}^33\smallsetminus S} f(a).1_u = 0.$$

So we obtain

$$b = t_S = f(\mathsf{c}_i^A a).$$

We have proved that $\P$ is interpretable in $\mathfrak{A}$. Furthermore it is easy to see that the interpretation is one dimensional and quantifier free. Next we extract an algebra $\mathfrak{B}$ elementary equivalent to $\mathfrak{A}$ that is not a neat reduct i.e. not in $\mathfrak{Nr}_3\mathbf{CA}_4$. Also $Rd_{r\mathbf{SC}}\mathfrak{B} \notin \mathfrak{Nr}_3 r\mathbf{SC}_5$. Let $Id \in {}^33$ be the identity map on 3. Choose any countable boolean elementary subalgebra of $\mathfrak{A}_{Id}$, $\mathfrak{B}_{Id}$ say. Thus $\mathfrak{B}_{Id} \preceq \mathfrak{A}_{Id}$. By lemma 9

$$Q = ((B_{Id} \times \prod_{u\in {}^33\smallsetminus Id}\mathfrak{A}_u), 1_u, d_{ij})_{u\in {}^33, i,j<3} \equiv$$

$$(\prod_{u\in {}^33}\mathfrak{A}_u)), 1_u, \mathsf{d}_{ij})_{u\in {}^33, i,j<3} = P.$$

(Note that the Id^th coordinate of each constant is 0 or 1, so the constants do lie in Q.) Let $\mathfrak{B}$ be the result of applying the interpretation given above to Q. Then $\mathfrak{B} \equiv \mathfrak{A}$ as cylindric algebras. Now we show that $\mathfrak{B}$ cannot be a neat reduct, in fact we show that $\mathfrak{B} \notin \mathfrak{Nr}_3\mathbf{CA}_\beta$

for any $\beta > 3$, while $Rd_{r\mathbf{SC}}\mathfrak{B} \notin \mathfrak{Nr}_3 r\mathbf{SC}_\beta$ for $\beta > 4$. We settle first the cylindric case. Assume for contradiction that $\mathfrak{B} = \mathfrak{Nr}_3\mathfrak{D}$ for some $\mathfrak{D} \in \mathbf{CA}_\beta$; with $\beta > 3$. Note that $\mathfrak{D}$ may not be representable. It is only here that we deal with possibly non-representable algebras. Now $\chi_u^M \in B$ for each $u \in {}^33$. Identifying functions with sequences we let $v = \langle 1, 0, 2\rangle \in {}^33$. Let $t(x)$ be the $\mathbf{CA}_2$ term $\mathsf{s}_1^0\mathsf{c}_1 x.\mathsf{s}_0^1\mathsf{c}_0 x$, where $\mathsf{s}_i^j(x) = \mathsf{c}_i(\mathsf{d}_{ij}.x)$, for $i \neq j$. Then we claim that $t^B(\chi_v^{\mathbf{M}}) = \chi_{Id}^{\mathbf{M}}$. For the sake of brevity, denote $\chi_v^{\mathbf{M}}$ by 1_{10} and $\chi_{Id}^{\mathbf{M}}$ by 1_{01}. Then, by definition, we have

$$t^B(1_{01}) = \mathsf{c}_0(\mathsf{d}_{01}.\mathsf{c}_1 1_{10}).\mathsf{c}_1(\mathsf{d}_{01}.\mathsf{c}_0 1_{10}).$$

Computing we get

$$\begin{aligned}\mathsf{c}_0(\mathsf{d}_{01}.\mathsf{c}_1 1_{10}) &= \mathsf{c}_0(\mathsf{d}_{01}.(\sum\{1_u : u \equiv_1 1_{10}\})\\ &= \mathsf{c}_0(\mathsf{d}_{01}.1_{112}) = 1_{01} + 1_{112}.\end{aligned}$$

Here 1_{112} denotes $\chi_{\langle 1,1,2\rangle}$. Note that we are using that the evaluation of the term $\mathsf{c}_1 1_{10}$ in $\mathfrak{B}$ is equal to its value in $\mathfrak{A}$. This is so, because $\mathfrak{B}$ inherits the interpretation given to $\prod A_u$. A similar computation gives

$$\mathsf{c}_1(\mathsf{d}_{01}.\mathsf{c}_0 1_{01}) = 1_{002} + 1_{01},$$

where 1_{002} denotes $\chi_{\langle 0,0,2\rangle}$. Therefore as claimed

$$t^B(1_{10}) = 1_{01}.$$

Now let ${}_3s(0,1)$ be the unary substitution term as defined in [11] 1.5.12, that is

$${}_3\mathsf{s}(0,1)x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1(x).$$

Then for any $\beta > 3$ we have

$$\mathbf{CA}_\beta \models {}_3s(0,1)\mathsf{c}_3 x \leq t(\mathsf{c}_3 x).$$

Indeed by [11] 1.5.12, 1.5.8 and 1.5.10 (ii), we get

$$\begin{aligned}{}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leq {}_3s(0,1)\mathsf{c}_1\mathsf{c}_3 x &= \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1\mathsf{c}_1\mathsf{c}_3 x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_3 x\\ &= \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_3\mathsf{c}_1 x = \mathsf{s}_0^3\mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1 x = \mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1 x = \mathsf{s}_0^1\mathsf{c}_1\mathsf{c}_3 x.\end{aligned}$$

Similarly

$${}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leq \mathsf{s}_1^0\mathsf{c}_0\mathsf{c}_3 x$$

Therefore

$${}_3\mathsf{s}(0,1)\mathsf{c}_3 x \leq t(\mathsf{c}_3 x).$$

It thus follows that

$$\mathfrak{D} \models {}_3s(0,1)(\chi_u^M) \leq \mathsf{s}_1^0\mathsf{c}_1(\chi_u^M).\mathsf{s}_0^1\mathsf{c}_0(\chi_u^M) = \chi_{Id}^M.$$

Now ${}_3\mathsf{s}(0,1)$ preserves $\leq$ and is one to one $\mathfrak{Nr}_3\mathfrak{D}$. By [11], 1.5.12 and 1.5.1, we have:

$${}_3\mathsf{s}(0,1)\mathsf{c}_3 x = \mathsf{s}_0^n\mathsf{s}_1^0\mathsf{s}_3^1\mathsf{c}_3 x = \mathsf{c}_3(\mathsf{d}_{30} \cap \mathsf{c}_0(\mathsf{d}_{01} \cap \mathsf{c}_1(\mathsf{d}_{01} \cap \mathsf{c}_1(\mathsf{d}_{13} \cap \mathsf{c}_3 x))).$$

By [11], 1.3.8, $0 < x$, implies $0 < \mathsf{d}_{ij} \cap \mathsf{c}_j x$, for all $i, j \in \beta$.

We have shown that if $x > 0 \in Nr_3 D$, then ${}_3\mathsf{s}(0,1)x > 0$, i.e that ${}_3\mathsf{s}(0,1)$, being a boolean endomorphism, is one to one. Since $B_v = A_v$ it follows (by condition (4) in Lemma 13) that $B_v = \{b \in B : b \leq \chi_v^M\}$ is uncountable. Since ${}_3\mathsf{s}(0,1)$ is one to one, it follows that ${}_3\mathsf{s}(0,1)B_u$ is also uncountable. But by the above we have

$$ {}_3\mathsf{s}(0,1)B_u \subseteq B_{Id} = \{b \in B : b \leq \chi_{Id}^B\}, $$

and so B_{Id} is also uncountable. But by construction, we have $B_{Id} = \{b \in B : b \leq \chi_{Id}^M\}$ is countable. This contradiction shows that $\mathfrak{B} \notin \mathfrak{Nr}_3\mathbf{CA}_\beta$ for any $\beta > 3$. The $r\mathbf{SC}$ is the same by using the axiomatization $(E_1 - E_9)$ and noting that ${}_3\mathsf{s}(0,1)$ is a permutation of $\mathfrak{Nr}_3\mathfrak{D}$ when $\mathfrak{D} \in r\mathbf{SC}_5$. Same reasoning for $\mathbf{SC}_m$.

Finally we should mention that the proof presented herein is substantially different than the proofs in [21], [18], [34], [29], since it uses genuine model theoretic arguments.

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