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Complex Analysis Methods Related an Optimization Problem with Complex Variables

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Abstract. In this paper, we consider a nondifferentiable minimax fractional programming problem treated with complex variables. Duality problem in optimization theory plays an important role. The goal of this paper is to formulate the Wolfe type dual and Mond-Weir type dual problems. We aim to establish the duality problems, and prove that the duality theorems have no duality gap to the primal problem under some assumptions. The processes involves to show three theorems: the weak, strong and strictly converse duality theorem.

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1. Introduction

Complex programming problem was studied first by Levinson [10] (in 1966) who considered the linear programming in complex space. Short later Swarup and Sharma [14] studied linear fractional programming in complex space. Hence after complex fractional programming problems in the linear and the nonlinear cases were treated by numerous authors (e.g. Lai et al. [3-9], Parkash et al. [13], Ferrero [1] and references therein). In applications, many practical problems related to complex variables, for instance, in electrical engineering, filter

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theory, statistical signal processing, etc. For a complex fractional programming [cf. Lai et al. 7], one may maximize the equalizer output kurtosis as

$$K(z)=\frac{\left|E(|z|^4)-2\big(E(|z|^2)\big)^2-|E(z^2)|^2\right|}{E(|z|^2)^2}$$

where $E(\cdot)$ stands for expectation, and $|z|^2 = z\cdot\overline{z}$. While the minimax fractional complex variable problem, one can find an example in the book Haykin [2] that is a problem to evaluate the eigenvalues $\lambda_1,\ldots,\lambda_m$ of the correlation matrix A as follows

$$\lambda_k=\min_{\dim(S)=k}\ \max_{z\in S}\frac{z^HAz}{z^Hz},\ k=1,\ldots,m$$

where S is a subspace of $\mathbb{C}^m$, $\dim(S)$ denotes the dimension of subspace $S\subset\mathbb{C}^m$, and the maximum is taken the nonzero vector z over the subspace S. Note that A in the above expression is a positive semidefinite Hermitian matrix.

In this paper, we would study a more general minimax fractional programming problem with complex variables as in Lai and Huang [3] in which it has established the necessary and the sufficient optimality conditions. It is remarkable that the duality problem is also an important part in optimization theory, and the duality models are based on the sufficient optimality conditions established in [3], thus if once we have the optimality conditions, it is naturally to investigate the duality models, and proves its duality theorems with nonduality gap between the dual problem and its primary problem. Caused by the above reasons, in this paper we will constitute two duality models: the Wolfe type [cf. 15] and the Mond-Weir type dual [cf. Mond-Weir 12], and prove three theorems: weak, strong and strict converse duality theorem with respect to the given primal problem with zero duality gap in the duality theorems.

2. Minimax Fractional Programming Problem with Complex Variables

In this paper, we consider the following minimax fractional complex programming problem [see Lai et al. 3] as the following:

$$\text{(P)}\quad \min_{\zeta\in X}\ \max_{\eta\in Y}\ \frac{Re\ [f(\zeta,\eta)+(z^HAz)^{1/2}]}{Re\ [g(\zeta,\eta)-(z^HBz)^{1/2}]}$$
$$\text{s.t. } X=\left\{\zeta=(z,\overline{z})\in\mathbb{C}^{2n}\mid -h(\zeta)\in S\right\}\subset\mathbb{C}^{2n}$$

where Y is a compact subset of $\{\eta=(w,\overline{w})\mid w\in\mathbb{C}^m\}\subset\mathbb{C}^{2m}$; A and $B\in\mathbb{C}^{n\times n}$ are positive semidefinite Hermitian matrices; S is a polyhedral cone in $\mathbb{C}^p$; $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous functions, and for each $\eta\in Y$, $f(\cdot,\eta)$ and $g(\cdot,\eta):\mathbb{C}^{2n}\to\mathbb{C}$ are analytic, we assume further that $h(\cdot):\mathbb{C}^{2n}\to\mathbb{C}^p$ is an analytic map defined on $\zeta=(z,\overline{z})\in Q\subset\mathbb{C}^{2n}$.
This set $Q=\{(z,\overline{z})\mid z\in\mathbb{C}^n\}$ is a linear manifold over real field. Without loss of generality, it is assumed that $Re\ [f(\zeta,\eta)+(z^HAz)^{1/2}]\geq 0$ and $Re\ [g(\zeta,\eta)-(z^HBz)^{1/2}]>0$ for each $(\zeta,\eta)\in X\times Y$. This problem will be nonsmooth if there is a point $\zeta_0=(z_0,\overline{z_0})$ such that $z_0^HAz_0=0$ or $z_0^HBz_0=0$.

In complex programming problem, the analytic function $f(z,\overline{z})$ is defined on the set Q since a nonlinear analytic function can not have a convex real part in our requirement [cf. Ferrero 1]. By this reason in our programming problem (P), the complex variables are taken as the form $\zeta = (z,\overline{z}) \in \mathbb{C}^{2n}$.

In order to understand some problems studied as before in different view points that are the special cases of problem (P), we recall these special forms as the following:

(i) In problem (P), if Y vanishes and rewrite $\zeta = (z,\overline{z})$, then (P) is reduced to the following minimization problem [cf. Lai el al. 7]:

$$(\mathrm{P}_0) \quad \min_{\zeta=(z,\overline{z})\in X} \quad \frac{Re\ [f(z,\overline{z})+(z^H Az)^{1/2}]}{Re\ [g(z,\overline{z})-(z^H Bz)^{1/2}]}.$$

(ii) If $A=0$ and $B=0$ are zero matrices in problem (P), then (P) is reduced to (P_1) which was studied by Lai et al. [8].

$$(\mathrm{P}_1) \quad \min_{\zeta\in X} \ \max_{\eta\in Y} \quad \frac{Re\ [f(\zeta,\eta)]}{Re\ [g(\zeta,\eta)]}$$
$$\text{subject to} \qquad X = \left\{\zeta = (z,\overline{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S \subset \mathbb{C}^p\right\}.$$

where Y is a specified compact subset in $\mathbb{C}^{2m}$, and for each $\eta \in Y$, $f(\cdot,\eta)$ and $g(\cdot,\eta)$ are analytic functions.

(iii) If $B=0$, $g(\cdot,\cdot)\equiv 1$, then problem (P) is reduced to (P_2) which was investigated by Lai et al. [4, 9].

$$(\mathrm{P}_2) \quad \min_{\zeta\in X} \max_{\eta\in Y} \quad Re\left[f(\zeta,\eta)+(z^H Az)^{1/2}\right]$$
$$\text{subject to} \qquad \zeta \in X = \{\ \zeta = (z,\overline{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S\ \}$$

where Y is a specified compact subset in $\mathbb{C}^{2m}$.

(iv) If $A=0$, $B=0$ and $g(\cdot,\cdot)\equiv 1$, then problem (P) is reduced to (P_3) which was considered by Lai et al. [6].

$$(\mathrm{P}_3) \quad \min_{\zeta\in X} \max_{\eta\in Y} \quad Re\ f(\zeta,\eta)$$
$$\text{subject to} \qquad \zeta \in X = \left\{\zeta = (z,\overline{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S\ \right\}.$$

(v) If $\zeta = x \in \mathbb{R}^n$ and $\eta = y \in Y \subset \mathbb{R}^m$, then problem (P) is reduced to the real variable problem which was studied by Lai et al. [5].

3. Notations

At first we describe briefly for some Notations and Definitions that are used in Lai et al. [3] as follows.

Let $S = \left\{\xi \in \mathbb{C}^p \mid Re(K\xi) \geq 0\right\} \subset \mathbb{C}^p$ be a polyhedral cone where $K \in \mathbb{C}^{k\times p}$ is a $k \times p$ matrix. The dual cone S^* of S is defined by

$$S^* = \left\{\mu \in \mathbb{C}^p \mid Re\langle \xi,\mu\rangle \geq 0,\ \text{for}\ \xi \in S\right\}.$$

For $s_0 \in S$, the set $S(s_0)$ is the intersection of those closed half spaces that include s_0 in their boundaries. Thus if $s_0 \in int(S)$, then $S(s_0)$ is the whole space $\mathbb{C}^p$. We say that problem (P) has the **constraint qualification** at a point $\zeta_0 = (z_0, \overline{z_0})$ if for any nonzero $\mu \in S^* \subset \mathbb{C}^p$, we have

$$Re\langle h'(\zeta_0)(\zeta - \zeta_0), \mu\rangle \neq 0 \quad \text{for } \zeta \neq \zeta_0.$$

Generalized convexity is an important role in optimization theory. Thus, for convenience, we recall the generalized convexities of complex functions as follows [cf. Lai et al. 7, 8].

Definition 1. *The real part of an analytic function $f(\cdot)$ from $\mathbb{C}^{2n}$ to $\mathbb{R}$ is called, respectively,*

(i) **convex (strictly)** *at $\zeta = \zeta_0 \in Q \subset \mathbb{C}^{2n}$ if*

$$Re\left[f(\zeta) - f(\zeta_0)\right] \underset{(>)}{\geq} Re\left[f'_\zeta(\zeta_0)(\zeta - \zeta_0)\right],$$

(ii) **pseudoconvex (strictly)** *at $\zeta = \zeta_0 \in Q$ if*

$$Re\left[f'_\zeta(\zeta_0)(\zeta - \zeta_0)\right] \geq 0 \Rightarrow Re\left[f(\zeta) - f(\zeta_0)\right] \underset{(>0)}{\geq 0},$$

(iii) **quasiconvex** *at $\zeta = \zeta_0 \in Q$ if*

$$Re\left[f(\zeta) - f(\zeta_0)\right] \leq 0 \Rightarrow Re\left[f'_\zeta(\zeta_0)(\zeta - \zeta_0)\right] \leq 0.$$

Definition 2. *An analytic mapping $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ is called, respectively,*

(i) **convex** *at $\zeta = \zeta_0 \in Q$ with respect to (w.r.t.) a polyhedral cone S in $\mathbb{C}^p$ if there is a nonzero $\mu \in S^* (\subset \mathbb{C}^p)$, the dual cone of S, such that*

$$Re\langle h(\zeta) - h(\zeta_0), \mu\rangle \geq Re\langle h'(\zeta_0)(\zeta - \zeta_0), \mu\rangle.$$

Here $\langle \cdot, \cdot \rangle$ stands for the inner product in complex spaces.

(ii) **pseudoconvex (strictly)** *at $\zeta = \zeta_0 \in Q$ w.r.t. S if there is a nonzero $\mu \in S^* (\subset \mathbb{C}^p)$ the dual cone of S, such that*

$$Re\langle h'(\zeta_0)(\zeta - \zeta_0), \mu\rangle \geq 0 \Rightarrow Re\langle h(\zeta) - h(\zeta_0), \mu\rangle \underset{(>0)}{\geq 0},$$

(iii) **quasiconvex** *at $\zeta = \zeta_0 \in Q$ w.r.t. S if there is a nonzero $\mu \in S^* (\subset \mathbb{C}^p)$ such that*

$$Re\langle h(\zeta) - h(\zeta_0), \mu\rangle \leq 0 \Rightarrow Re\langle h'(\zeta_0)(\zeta - \zeta_0), \mu\rangle \leq 0.$$

Throughout this paper, X is a subset of $\mathbb{C}^{2n}$, and for $\zeta = (z, \overline{z}) \in X$, $f(\zeta, \cdot)$ and $g(\zeta, \cdot)$ are continuous on the compact set Y. Thus we can denote

$$Y(\zeta) = \left\{ \eta \in Y \;\middle|\; \frac{Re[f(\zeta,\eta) + (z^H Az)^{1/2}]}{Re[g(\zeta,\eta) - (z^H Bz)^{1/2}]} = \max_{\nu \in Y} \frac{Re[f(\zeta,\nu) + (z^H Az)^{1/2}]}{Re[g(\zeta,\nu) - (z^H Bz)^{1/2}]} \right\}$$

since Y is compact, the supremum in the above $\nu \in Y$ is attained. This set $Y(\zeta)$ is also a compact subset of Y.

We need use the differential of a complex function by the gradient symbols ∇_z and $\nabla_{\overline{z}}$ as the following [cf. 4]:

For each $\eta \in Y \subset \mathbb{C}^{2m}$, $w \in \mathbb{C}^n$ and $\zeta = (z, \overline{z}) \in Q \subset \mathbb{C}^{2n}$, suppose that the function $\Phi(\zeta) = f(\zeta, \eta) + z^H Aw + \langle h(\zeta), \mu \rangle$ is differentiable at $\zeta_0 = (z_0, \overline{z_0})$. Then

$$\begin{aligned} Re[\Phi'(\zeta_0)(\zeta - \zeta_0)] = Re\Bigg[\Big\langle\, & z - z_0,\ \overline{\nabla_z f(\zeta_0, \eta)} + \nabla_{\overline{z}} f(\zeta_0, \eta) \\ & + Aw + \mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0) \,\Big\rangle \Bigg]. \end{aligned}$$

The generalized Schwarz inequality in complex space can be as the inequality:

$$Re(z^H Au) \leq (z^H Az)^{1/2} (u^H Au)^{1/2}. \tag{1}$$

4. Necessary and Sufficient Optimality Conditions

In Lai et al. [3], the authors have established the optimality conditions. For convenient, we restate the necessary optimality conditions as follows.

Theorem 1. *(Necessary Optimality Conditions, [cf. 3, Theorem 2]) Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a (P)-optimal with optimal value ν^*. Suppose that the problem (P) satisfies the constraint qualification at ζ_0 with assumptions $z_0^H Az_0 = \langle Az_0, z_0 \rangle > 0$ and $z_0^H Bz_0 = \langle Bz_0, z_0 \rangle > 0$. Then there exist $0 \neq \mu \in S^* \subset \mathbb{C}^p$, $u_1, u_2 \in \mathbb{C}^n$ and positive integer k with the following properties (as $Y(\zeta_0) \subset Y$ is provided a compact subset in $\mathbb{C}^{2m}$):*

- *(i) finite points $\eta_i \in Y(\zeta_0)$ for $i = 1, \cdots, k$;*
- *(ii) for $i = 1, \cdots, k$, multipliers $\lambda_i > 0$ and $\sum_{i=1}^k \lambda_i = 1$*

such that $\sum_{i=1}^k \lambda_i [f(\zeta, \eta_i) - \nu^* g(\zeta, \eta_i)] + \langle h(\zeta), \mu \rangle + \langle Az, z \rangle^{1/2} + \nu^* \langle Bz, z \rangle^{1/2}$ satisfies the following conditions

$$\begin{aligned} \sum_{i=1}^k \lambda_i \Big\{ \Big[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\overline{z}} f(\zeta_0, \eta_i) \Big] - \nu^* \Big[\overline{\nabla_z g(\zeta_0, \eta_i)} + \nabla_{\overline{z}} g(\zeta_0, \eta_i) \Big] \Big\} & \\ + \Big(\mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0) \Big) + \big(Au_1 + \nu^* Bu_2 \big) = 0; & \end{aligned} \tag{2}$$

$$Re\langle\, h(\zeta_0), \mu \,\rangle = 0; \tag{3}$$

$$u_1^H A u_1 \le 1,\ (z_0^H A z_0)^{1/2} = Re(z_0^H A u_1); \tag{4}$$

$$u_2^H B u_2 \le 1,\ (z_0^H B z_0)^{1/2} = Re(z_0^H B u_2). \tag{5}$$

In order to get the necessary optimality conditions of (P) for nonsmooth situation at $\zeta_0 = (z_0, \overline{z_0}) \in Q$, that is, if $z_0^H A z_0 = 0$ or $z_0^H B z_0 = 0$, the problem (P) is a complex nondifferentiable minimax programming. In this case, we define a set:

$$Z_{\widetilde{\eta}}(\zeta_0) = \Big\{ \zeta \in \mathbb{C}^{2n} \ \Big|\ -h'_{\zeta}(\zeta_0)\zeta \in S(-h(\zeta_0)),\ \zeta = (z, \overline{z}) \in Q \text{ with any one of the next conditions (i), (ii)and (iii) holds} \Big\}.$$

(i) $Re\Big\{ \sum_{i=1}^k \lambda_i \Big[f'_{\zeta}(\zeta_0, \eta_i) - v^* g'_{\zeta}(\zeta_0, \eta_i) \Big] \zeta + \frac{\langle A z_0, z\rangle}{\langle A z_0, z_0\rangle^{1/2}} + \langle (v^*)^2 B z, z\rangle^{1/2} \Big\} < 0,$ if $z_0^H A z_0 > 0$ and $z_0^H B z_0 = 0$;

(ii) $Re\Big\{ \sum_{i=1}^k \lambda_i \Big[f'_{\zeta}(\zeta_0, \eta_i) - v^* g'_{\zeta}(\zeta_0, \eta_i) \Big] \zeta + \langle A z, z\rangle^{1/2} + \frac{\langle v^* B z_0, z\rangle}{\langle B z_0, z_0\rangle^{1/2}} \Big\} < 0,$ if $z_0^H A z_0 = 0$ and $z_0^H B z_0 > 0$;

(iii) $Re\Big\{ \sum_{i=1}^k \lambda_i \Big[f'_{\zeta}(\zeta_0, \eta_i) - v^* g'_{\zeta}(\zeta_0, \eta_i) \Big] \zeta + \langle [A + (v^*)^2 B] z, z\rangle^{1/2} \Big\} < 0,$ if $z_0^H A z_0 = 0$ and $z_0^H B z_0 = 0$.

This set $Z_{\widetilde{\eta}}(\zeta_0)$ plays an important role for the cases either $\langle A z_0, z_0\rangle = 0$ or $\langle B z_0, z_0\rangle = 0$. If the set $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$, then we can obtain the necessary optimality conditions of problem (P) as the following.

Theorem 2. *(Necessary Optimality Conditions, [cf. 3, Theorem 3]) Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be (P)-optimal with optimal value v^*. Suppose that problem (P) possesses constraint qualification at ζ_0 and $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$. Then there exist a nonzero $\mu \in S^* \subset \mathbb{C}^p$ and vectors $u_1, u_2 \in \mathbb{C}^n$ such that the conditions (2)∼(5) hold.*

We know that the sufficient optimality conditions for problem (P) follows from the converse of necessary optimality conditions with extra assumptions, thus the sufficient optimality conditions are various. The additional assumptions are convex as well as the generalized convexities, for instance, we can state the sufficient optimality conditions of (P) as follows [cf. 3, Theorem 4].

Theorem 3 (Sufficient Optimality Conditions)**.** *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a feasible solution of (P). Suppose that there exist a positive integer $k > 0$, $v^* \in \mathbb{R}^+$, for $i = 1, \cdots, k$, $\lambda_i > 0$, $\eta_i \in Y(\zeta_0)$ with $\sum_{i=1}^k \lambda_i = 1$, and that $0 \ne \mu \in S^* \subset \mathbb{C}^p$, u_1, $u_2 \in \mathbb{C}^n$ satisfying conditions (2)∼(5) of Theorem 1 for $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$. Assume that any one of the following conditions (i), (ii) and (iii) holds:*

(i) $Re\Big\{ \sum_{i=1}^k \lambda_i \Big[\big(f(\zeta, \eta_i) + z^H A u_1 \big) - v^* \big(g(\zeta, \eta_i) - z^H B u_2 \big) \Big] \Big\}$ is pseudoconvex on $\zeta = (z, \overline{z}) \in Q$, and $h(\zeta)$ is quasiconvex on Q w.r.t. the polyhedral cone $S \subset \mathbb{C}^p$;

(ii) $Re\left\{\sum_{i=1}^{k}\lambda_i\Big[\big(f(\zeta,\eta_i)+z^H Au_1\big)-\nu^*\big(g(\zeta,\eta_i)-z^H Bu_2\big)\Big]\right\}$ *is quasiconvex on* $\zeta=(z,\overline{z})\in Q$, *and* $h(\zeta)$ *is strictly pseudoconvex on* Q *w.r.t.* $S\subset\mathbb{C}^p$;

(iii) $Re\left\{\sum_{i=1}^{k}\lambda_i\Big[\big(f(\zeta,\eta_i)+z^H Au_1\big)-\nu^*\big(g(\zeta,\eta_i)-z^H Bu_2\big)\Big]+\langle h(\zeta),\mu\rangle\right\}$ *is pseudoconvex on* $\zeta=(z,\overline{z})\in Q$.

Then $\zeta_0=(z_0,\overline{z_0})$ *is an optimal solution of (P).*

5. Wolfe Type Dual Model

In order to construct a duality problems respect to the primal problem (P), we take some preparation.

Let $\zeta=(z,\overline{z})\in Q\subset\mathbb{C}^{2n}$ be any feasible solution of problem (P). By the compactness of Y in (P), the closed subset $Y(\zeta)$ is also compact which is the set of points in Y maximizing the fractional function

$$\max_{\eta\in Y}\frac{Re\left[f(\zeta,\eta)+(z^H Az)^{1/2}\right]}{Re\left[g(\zeta,\eta)-(z^H Bz)^{1/2}\right]}\quad \text{at } \eta_1,\ \eta_2,\ldots,\ \eta_k \text{ for some } k\in\mathbb{N}.$$

Since for each $\zeta=(z,\overline{z})\in Q$, the functions $f(\zeta,\cdot)$ and $g(\zeta,\cdot)$ are continuous on Y. Thus one can show easily that the fractional function:

$$\varphi(\zeta)\equiv\max_{\eta\in Y}\frac{Re\left[f(\zeta,\eta)+(z^H Az)^{1/2}\right]}{Re\left[g(\zeta,\eta)-(z^H Bz)^{1/2}\right]}=\frac{\sum_{i=1}^{k}\lambda_i Re\left[f(\zeta,\eta_i)+(z^H Az)^{1/2}\right]}{\sum_{i=1}^{k}\lambda_i Re\left[g(\zeta,\eta_i)-(z^H Bz)^{1/2}\right]}\tag{6}$$

and so the problem (P) become

$$\text{(P)}\quad \min_{\zeta\in X}\varphi(\zeta).$$

Based on the optimality conditions (2)∼(5) in Theorem 1 as well as in Theorem 2, the existence of optimal solution for problem (P) even (P) is a nondifferentiable minimax programming problem with complex variables under some generalized convexities has established by Theorem 3.

By using the optimality conditions (2)∼(5) and the existence for optimal solutions of problem (P), one may consider the duality model with respect to the primal problem (P). In this section, we would construct the Wolfe type dual in fractional programming problem (WD) by considering the objective from the original fractional functional added the constraints of (P) with a multiplier $\mu\in S^*$ into the numerator of the fractional functional in (P). Precisely it likes

$$\Phi(\zeta)=\frac{\sum_{i=1}^{k}\lambda_i Re\left[f(\zeta,\eta_i)+(z^H Az)^{1/2}+\langle h(\zeta),\mu\rangle\right]}{\sum_{i=1}^{k}\lambda_i Re\left[g(\zeta,\eta_i)-(z^H Bz)^{1/2}\right]},\tag{7}$$

and then maximize $\Phi(\zeta)$ under suitable constraints.

In order to distinguish the feasible variable $\zeta = (z, \overline{z}) \in Q$ in (P) from the dual problem, we replace the variable in the dual problem (WD) by ξ. Of course this $\xi = (\alpha, \overline{\alpha}) \in Q \subset \mathbb{C}^{2n}$ still plays as a feasible solution of (P) and also assumes to satisfy the necessary conditions (2)∼(5). Then we could constitute the Wolfe type dual as the dual problem of (P) as the following form:

$$\text{(WD)} \quad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \ \max_{(\xi,\mu,w_1,w_2)\in X_1(k,\widetilde{\lambda},\widetilde{\eta})} \ \Phi(\xi)$$

where $\Phi(\xi)$ defines a fractional functional as the expression (7) replace ζ by ξ.
Here

(i) $K(\xi)$ stands for a set of the triplet points $(k, \overline{\lambda}, \overline{\eta})$ satisfying the optimality conditions of problem (P) for any given feasible solutions $\xi = (\alpha, \overline{\alpha}) \in Q$, then there exist a nonzero multiplier $\mu \in S^* \subset \mathbb{C}^p$, the dual cone of the polyhedral cone S in $\mathbb{C}^p$ such that $\langle v, \mu \rangle \geq 0$ for any $v \in S$. Thus $\langle h(\xi), \mu \rangle \leq 0$ as $-h(\xi) \in S \subset \mathbb{C}^p$.

(ii) the new constraint set $X_1(k, \widetilde{\lambda}, \widetilde{\eta})$ is the set of all feasible solution (ξ, μ, w_1, w_2) of (WD).

Consequently, the constraints of (WD) are as the following expression:
For $\xi = (\alpha, \overline{\alpha}) \in Q \subset \mathbb{C}^{2n}$,

$$\begin{aligned} &\left\{ \sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z f(\xi, \eta_i)} + \nabla_{\overline{z}} f(\xi, \eta_i) \right] + A w_1 + \mu^T \ \overline{\nabla_z h(\xi)} + \mu^H \nabla_{\overline{z}} h(\xi) \right\} \times \\ &\left(\sum_{i=1}^{k} \lambda_i \left[g(\xi, \eta_i) - (\alpha^H B \alpha)^{1/2} \right] \right) - \left(\sum_{i=1}^{k} \lambda_i \left[f(\xi, \eta_i) + (\alpha^H A \alpha)^{1/2} + \langle h(\xi), \mu \rangle \right] \right) \times \\ &\left\{ \sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z g(\xi, \eta_i)} + \nabla_{\overline{z}} g(\xi, \eta_i) \right] - B w_2 \right\} = 0, \end{aligned} \tag{8}$$

$$Re\langle h(\xi), \mu \rangle \geq 0, \quad \mu \neq 0 \text{ in } S^*, \tag{9}$$

$$w_1^H A w_1 \leq 1, \ (\alpha^H A \alpha)^{1/2} = Re(\alpha^H A w_1), \tag{10}$$

$$w_2^H B w_2 \leq 1, \ (\alpha^H B \alpha)^{1/2} = Re(\alpha^H B w_2), \tag{11}$$

If for a triplet $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\xi)$, the set $X_1(k, \widetilde{\lambda}, \widetilde{\eta}) = \emptyset$, then we define the supremum over $X_1(k, \widetilde{\lambda}, \widetilde{\mu})$ to be $-\infty$ for non exception in the formulation of (WD).

Without loss of generality, we may assume that

$$\sum_{i=1}^{k} \lambda_i Re \left[f(\xi, \eta_i) + (\alpha^H A \alpha)^{1/2} + \langle h(\xi), \mu \rangle \right] \geq 0$$

and

$$\sum_{i=1}^{k} \lambda_i Re \left[g(\xi, \eta_i) - (\alpha^H B \alpha)^{1/2} \right] > 0$$

for each $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\xi)$, $(\xi, \mu, w_1, w_2) \in X_1(k, \widetilde{\lambda}, \widetilde{\eta})$.

How we can approve that problem (WD) is really a dual problem of the problem (P)? To confirm the problems (WD) and (P) are surely in duality relation. The next three theorems must be established for non duality gap under extra assumptions.

Now for simplicity, we denote the function

$$\begin{aligned}\Phi_1(\bullet) &= \left(\sum_{i=1}^{k}\lambda_i Re\left[f(\bullet,\eta_i)+(\cdot)^H A w_1+\langle h(\bullet),\mu\rangle\right]\right)\times\left(\sum_{i=1}^{k}\lambda_i Re\left[g(\xi,\eta_i)-\alpha^H B w_2\right]\right)\\ &- \left(\sum_{i=1}^{k}\lambda_i Re\left[f(\xi,\eta_i)+\alpha^H A w_1+\langle h(\xi),\mu\rangle\right]\right)\times\left(\sum_{i=1}^{k}\lambda_i Re\left[g(\bullet,\eta_i)-(\cdot)^H B w_2\right]\right)\end{aligned}$$

for $\bullet = (\cdot, \bar{\cdot}) \in Q \subset \mathbb{C}^{2n}$.

Employing the necessary optimality conditions (2)∼(5) with some generalized convexity, we can prove three theorems: the weak, strong and strict converse duality theorem of problem (WD) as follows.

Theorem 4. *[Weak Duality] Let $\zeta = (z, \overline{z})$ be (P)-feasible, and $(k, \widetilde{\lambda}, \widetilde{\eta}, \xi, \mu, w_1, w_2)$ be (WD)-feasible. If $\Phi_1(\xi)$ is pseudoconvex on Q, then*

$$\max_{\eta\in Y}\frac{Re\left[f(\zeta,\eta)+(z^H Az)^{1/2}\right]}{Re\left[g(\zeta,\eta)-(z^H Bz)^{1/2}\right]}\geq\Phi(\xi).$$

Proof. Suppose on the contrary that

$$\max_{\eta\in Y}\frac{Re\left[f(\zeta,\eta)+(z^H Az)^{1/2}\right]}{Re\left[g(\zeta,\eta)-(z^H Bz)^{1/2}\right]}<\Phi(\xi)=\frac{\sum_{i=1}^{k}\lambda_i Re\left[f(\xi,\eta_i)+(\alpha^H A\alpha)^{1/2}+\langle h(\xi),\mu\rangle\right]}{\sum_{i=1}^{k}\lambda_i Re\left[g(\xi,\eta_i)-(\alpha^H B\alpha)^{1/2}\right]}.$$

Then for each $\eta \in Y$, we get

$$\begin{aligned}&\left(Re\left[f(\zeta,\eta)+(z^H Az)^{1/2}\right]\right)\times\left(\sum_{i=1}^{k}\lambda_i Re\left[g(\xi,\eta_i)-(\alpha^H B\alpha)^{1/2}\right]\right)\\ &\quad<\left(Re\left[g(\zeta,\eta)-(z^H Bz)^{1/2}\right]\right)\times\left(\sum_{i=1}^{k}\lambda_i Re\left[f(\xi,\eta_i)+(\alpha^H A\alpha)^{1/2}+\langle h(\xi),\mu\rangle\right]\right).\end{aligned}$$

Now we are replaced η by η_i, multiplies λ_i (with $\sum_{i=1}^{k}\lambda_i = 1$). Then it deduce to

$$\begin{aligned}&\left(\sum_{i=1}^{k}\lambda_i\, Re\left[f(\zeta,\eta_i)+(z^H Az)^{1/2}\right]\right)\times\left(\sum_{i=1}^{k}\lambda_i Re\left[g(\xi,\eta_i)-(\alpha^H B\alpha)^{1/2}\right]\right)\\ &\quad-\left(\sum_{i=1}^{k}\lambda_i\, Re\left[g(\zeta,\eta_i)-(z^H Bz)^{1/2}\right]\right)\times\left(\sum_{i=1}^{k}\lambda_i Re\left[f(\xi,\eta_i)+(\alpha^H A\alpha)^{1/2}+\langle h(\xi),\mu\rangle\right]\right)\\ &\quad<0. \qquad (12)\end{aligned}$$

From inequalities (10), (11) and generalized Schwarz inequality (1), we obtain

$$Re(z^H Aw_1) \leq (z^H Az)^{1/2}(w_1^H Aw_1)^{1/2} \leq (z^H Az)^{1/2} \text{ and} \tag{13}$$

$$Re(z^H Bw_2) \leq (z^H Bz)^{1/2}(w_2^H Bw_2)^{1/2} \leq (z^H Bz)^{1/2}, \tag{14}$$

since $w_1^H Aw_1 \leq 1$ and $w_2^H Bw_2 \leq 1$.

From inequalities (12), (13) and (14), we obtain

$$\begin{aligned}
\Phi_1(\zeta) &= \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[f(\zeta,\eta_i) + z^H Aw_1 + \langle h(\zeta),\mu\rangle\big]\Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big) \\
&\quad - \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[f(\xi,\eta_i) + \alpha^H Aw_1 + \langle h(\xi),\mu\rangle\big]\Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[g(\zeta,\eta_i) - z^H Bw_2\big]\Big) \\
&< \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[f(\zeta,\eta_i) + (z^H Az)^{1/2} + \langle h(\zeta),\mu\rangle\big]\Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big) \\
&\quad - \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[f(\xi,\eta_i) + \alpha^H Aw_1 + \langle h(\xi),\mu\rangle\big]\Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[g(\zeta,\eta_i) - (z^H Bz)^{1/2}\big]\Big) \\
&< 0 + Re\ \langle h(\zeta),\mu\rangle \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \big[g(\zeta,\eta_i) - \alpha^H Bw_2\big]\Big).
\end{aligned}$$

Since $Re\ \langle h(\zeta),\mu\rangle < 0$ and $\Big(\sum_{i=1}^k \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H Bw_2\big]\Big) > 0$, the above inequality implies that

$$\Phi_1(\zeta) < 0 = \Phi_1(\xi).$$

By hypothesis Φ_1 is pseudoconvex and $\Phi_1(\zeta) - \Phi_1(\xi) < 0$, we get

$$\begin{aligned}
&\Big\{ \sum_{i=1}^k \lambda_i \big[\overline{\nabla_z f(\xi,\eta_i)} + \nabla_{\overline{z}} f(\xi,\eta_i)\big] + Aw_1 + \mu^T\ \overline{\nabla_z h(\xi)} + \mu^H \nabla_{\overline{z}} h(\xi)\Big\} \\
&\quad \cdot \Big(\sum_{i=1}^k \lambda_i\ \big[g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}\big]\Big) \\
&\quad - \Big(\sum_{i=1}^k \lambda_i\ \big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2} + \langle h(\xi),\mu\rangle\big]\Big) \\
&\quad \cdot \Big\{ \sum_{i=1}^k \lambda_i \big[\overline{\nabla_z g(\xi,\eta_i)} + \nabla_{\overline{z}} g(\xi,\eta_i)\big] - Bw_2 \Big\} < 0.
\end{aligned}$$

This contradicts the equality of (8). Hence the proof is complete. □

Theorem 5. *[Strong Duality] Let $\zeta_0 = (z_0, \overline{z_0})$ be an optimal solution of problem (P) satisfying the hypothesis of Theorem 1. Then there exist $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\zeta_0)$ and $(\zeta_0, \mu, w_1, w_2) \in X_1(k, \widetilde{\lambda}, \widetilde{\eta})$ such that $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is a feasible solution of the dual problem (WD). If the hypotheses of Theorem 4 are fulfilled, then $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is an optimal solution of (WD), and the two problems (P) and (WD) have the same optimal values.*

Proof. If $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be an optimal solution of problem (P) with optimal value

$$v^* = \varphi(\zeta_0) = \frac{\sum_{i=1}^{k} \lambda_i Re[f(\zeta_0, \eta_i) + (z_0^H A z_0)^{1/2}]}{\sum_{i=1}^{k} \lambda_i Re[g(\zeta_0, \eta_i) - (z_0^H B z_0)^{1/2}]},$$

then by Theorem 1, there exist $0 \neq \mu \in S^* \subset \mathbb{C}^p$, $w_1, w_2 \in \mathbb{C}^n$ and positive integer k to satisfy the following equality:

$$\left\{ \sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\overline{z}} f(\zeta_0, \eta_i) \right] + A w_1 + \mu^T \ \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0) \right\} \times$$
$$\left(\sum_{i=1}^{k} \lambda_i \ [g(\zeta_0, \eta_i) - (z_0^H B z_0)^{1/2}] \right) - \left(\sum_{i=1}^{k} \lambda_i \ [f(\zeta_0, \eta_i) + (z_0^H A z_0)^{1/2} + \langle h(\zeta_0), \mu \rangle] \right) \times$$
$$\left\{ \sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z g(\zeta_0, \eta_i)} + \nabla_{\overline{z}} g(\zeta_0, \eta_i) \right] - B w_2 \right\} = 0.$$

It follows that $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\zeta_0)$ and $(\zeta_0, \mu, w_1, w_2) \in X(k, \widetilde{\lambda}, \widetilde{\eta})$ such that $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is a feasible solution of the dual problem (WD).

If the hypotheses of Theorem 4 are also fulfilled, then $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is an optimal solution of the dual problem (WD). □

Now we consider $\Phi_1(\bullet)$ as a strictly pseudoconvex on Q instead of pseudoconvex. Then we have the strict converse duality theorem as follows.

Theorem 6. *[Strict Converse Duality] Let* $\widehat{\zeta}$ *and* $(\widehat{k}, \widehat{\lambda}, \widehat{\eta}, \widehat{\xi}, \widehat{\mu}, \widehat{w_1}, , \widehat{w_1})$ *be optimal solutions of (P) and (WD), respectively, and assume that the assumptions of Theorem 5 are fulfilled. If* $\Phi_1(\bullet)$ *is strictly pseudoconvex on Q, then* $\widehat{\zeta} = \widehat{\xi}$*; and the optimal values of (P) and (WD) are equal.*

Proof. Assume that $(\widehat{z}, \overline{\widehat{z}}) = \widehat{\zeta} \neq \widehat{\xi} = (\widehat{\alpha}, \overline{\widehat{\alpha}})$, and reach a contradiction.

By Theorem 5, we know that

$$\max_{\eta \in Y} \frac{Re[f(\widehat{\zeta}, \eta) + (\widehat{z}^H A \widehat{z})^{1/2}]}{Re[g(\widehat{\zeta}, \eta) - (\widehat{z}^H B \widehat{z})^{1/2}]} = \frac{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re \ \left[f(\widehat{\xi}, \widehat{\eta}_i) + (\widehat{\alpha}^H A \widehat{\alpha})^{1/2} + \langle h(\widehat{\xi}), \widehat{\mu} \rangle \right]}{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re \ \left[g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B \widehat{\alpha})^{1/2} \right]}.$$

Then for each $\eta \in Y$,

$$\frac{Re[f(\widehat{\zeta}, \eta) + (\widehat{z}^H A \widehat{z})^{1/2}]}{Re[g(\widehat{\zeta}, \eta) - (\widehat{z}^H B \widehat{z})^{1/2}]} \leq \frac{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re \ \left[f(\widehat{\xi}, \widehat{\eta}_i) + (\widehat{\alpha}^H A \widehat{\alpha})^{1/2} + \langle h(\widehat{\xi}), \widehat{\mu} \rangle \right]}{\sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re \ \left[g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B \widehat{\alpha})^{1/2} \right]}.$$

That is, for each $\eta \in Y$,

$$\left(Re[f(\widehat{\zeta}, \eta) + (\widehat{z}^H A \widehat{z})^{1/2} \right) \times \left\{ \sum_{i=1}^{\widehat{k}} \widehat{\lambda}_i Re \ \left[g(\widehat{\xi}, \widehat{\eta}_i) - (\widehat{\alpha}^H B \widehat{\alpha})^{1/2} \right] \right\}$$

$$-\left(Re[g(\widehat{\zeta},\eta)-(\widehat{z}^H B\widehat{z})^{1/2}]\right)\times\left\{\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i Re\ [f(\widehat{\xi},\widehat{\eta}_i)+(\widehat{\alpha}^H A\widehat{\alpha})^{1/2}+\langle h(\widehat{\xi}),\widehat{\mu}\rangle]\right\}\leq 0.$$

It implies that

$$\begin{aligned}&\Big(\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i Re[f(\widehat{\zeta},\widehat{\eta}_i)+(\widehat{z}^H A\widehat{z})^{1/2}]\Big)\times\left\{\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i Re\ [g(\widehat{\xi},\widehat{\eta}_i)-(\widehat{\alpha}^H B\widehat{\alpha})^{1/2}]\right\}\\&-\Big(\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i Re[g(\widehat{\zeta},\widehat{\eta}_i)-(\widehat{z}^H B\widehat{z})^{1/2}]\Big)\times\left\{\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i Re\ [f(\widehat{\xi},\widehat{\eta}_i)+(\widehat{\alpha}^H A\widehat{\alpha})^{1/2}+\langle h(\widehat{\xi}),\widehat{\mu}\rangle]\right\}\\&\leq 0. \end{aligned}\tag{15}$$

From inequality (15), we use the same line as the proof of Theorem 4, one can easily obtain

$$\Phi_1(\widehat{\zeta})\leq\Phi_1(\widehat{\xi}).$$

Since hypothesis $\Phi_1(\bullet)$ is strictly pseudoconvex on Q, it implies that

$$\begin{aligned}&\left\{\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i\left[\overline{\nabla_z f(\widehat{\xi},\widehat{\eta}_i)}+\nabla_{\overline{z}} f(\widehat{\xi},\widehat{\eta}_i)\right]+A\widehat{w}_1+\widehat{\mu}^T\ \overline{\nabla_z h(\widehat{\xi})}+\widehat{\mu}^H\nabla_{\overline{z}}h(\widehat{\xi})\right\}\\&\cdot\Big(\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i\ [g(\widehat{\xi},\widehat{\eta}_i)-(\widehat{\alpha}^H B\widehat{\alpha})^{1/2}]\Big)-\Big(\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i\ [f(\widehat{\xi},\widehat{\eta}_i)+(\widehat{\alpha}^H A\widehat{\alpha})^{1/2}+\langle h(\widehat{\xi}),\widehat{\mu}\rangle]\Big)\\&\cdot\left\{\sum_{i=1}^{\widehat{k}}\widehat{\lambda}_i\left[\overline{\nabla_z g(\widehat{\xi},\widehat{\eta}_i)}+\nabla_{\overline{z}} g(\widehat{\xi},\widehat{\eta}_i)\right]-B\widehat{w}_2\right\}<0\end{aligned}$$

which contradicts the equality of (8). Hence the proof is complete. □

6. Mond-Weir Dual Problem and its Duality Theorems

The minimax fractional problem (P), actually is a minimization problem with objective function

$$\varphi(\zeta)=\frac{\sum_{i=1}^k\lambda_i Re\left[f(\zeta,\eta_i)+(z^H Az)^{1/2}\right]}{\sum_{i=1}^k\lambda_i Re\left[g(\zeta,\eta_i)-(z^H Bz)^{1/2}\right]}.$$

Thus the Mond-Weir type dual problem (D) can be regarded as a maximization problem with objective function

$$\varphi(\xi)=\frac{\sum_{i=1}^k\lambda_i Re\left[f(\xi,\eta_i)+(\alpha^H A\alpha)^{1/2}\right]}{\sum_{i=1}^k\lambda_i Re\left[g(\xi,\eta_i)-(\alpha^H B\alpha)^{1/2}\right]}$$

which is obtained by using the original variable $\zeta=(z,\overline{z})\in Q$ of $\varphi(\zeta)$ replaced by $\xi=(\alpha,\overline{\alpha})\in Q$ in the same fractional functional to be $\varphi(\xi)$.

The main task for the dual model (D) is to establish the constraints of (D) and search the conditions to approve the problem (D) is surely a dual problem with respect to the primal problem (P). Moreover to prove there are no duality gap between (D) and (P). That is, they have the same optimal values. In other word,

$$\min_{\zeta}(P) = \max_{\xi}(D) \text{ is approved.}$$

This is the main thought. Fortunately, from necessary optimality conditions (2)∼(5) is used to the assumptions for sufficient optimality conditions.
Consequently one can find the existence of optimal solution for problem (P), besides a feasible solution satisfying conditions (2)∼(5), it needs extra assumptions (for instance, generalized convexity) to obtain a sufficient optimality condition. Caused from this reason, we establish the Mond-Weir type dual problem (MWD) as the following form:

$$\text{(MWD)} \quad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \max_{(\xi,\mu,w_1,w_2)\in X_2(k,\widetilde{\lambda},\widetilde{\eta})} \varphi(\xi)$$

where $\xi = (\alpha, \overline{\alpha}) \in Q \subset \mathbb{C}^{2n}$ is given as any feasible point satisfying the conditions (2)∼(5) in theorem 1, it corresponds to $k \in \mathbb{N}$, $\widetilde{\lambda} = (\lambda_1, \cdots, \lambda_k)$, $\lambda_i > 0$ with $\sum_{i=1}^{k} \lambda_i = 1$ and $\widetilde{\eta} = (\eta_1, \cdots, \eta_k)$ of $\eta_i \in Y(\xi) \subset Y$. We use $K(\xi)$ to denote the set of all triplet points $(k, \widetilde{\lambda}, \widetilde{\eta})$ depending on ξ. Then by the triplet points $(k, \widetilde{\lambda}, \overline{\eta})$ corresponding to all points $(\xi, \mu, w_1, w_2) \in \mathbb{C}^{2n} \times \mathbb{C}^p \times \mathbb{C}^n \times \mathbb{C}^n$ in the fractional functional $\varphi(\xi)$ and maximizing the real number over ξ under the complex variables. We denote $X_2(k, \widetilde{\lambda}, \widetilde{\eta})$ as the set of all points (ξ, μ, w_1, w_2) in problem (MWD). Consequently, from the above preparation, the Mond-Weir type dual problem is then formulated by

$$\begin{aligned}
\text{(MWD)} \qquad & \max_{\xi\in X} \ \max_{\eta\in Y} \frac{Re[f(\xi,\eta)+(\alpha^H A\alpha)^{1/2}]}{Re[g(\xi,\eta)-(\alpha^H B\alpha)^{1/2}]} \\
\Bigg(= \ & \max_{\xi\in X} \ \varphi(\xi) \\
= \ & \max_{\xi\in X} \frac{\sum_{i=1}^{k} \lambda_i Re\left[f(\xi,\eta_i)+(\alpha^H A\alpha)^{1/2}\right]}{\sum_{i=1}^{k} \lambda_i Re\left[g(\xi,\eta_i)-(\alpha^H B\alpha)^{1/2}\right]} \\
= \ & \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)} \ \max_{(\xi,\mu,w_1,w_2)\in X_2(k,\widetilde{\lambda},\widetilde{\eta})} \varphi(\xi) \ \Bigg)
\end{aligned}$$

subject to $\xi = (\alpha, \overline{\alpha}) \in Q \subset \mathbb{C}^{2n}$ and

$$\begin{aligned}
&\left\{ \sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z f(\xi,\eta_i)} + \nabla_{\overline{z}} f(\xi,\eta_i)\right] + A w_1 \right\} \times \left(\sum_{i=1}^{k} \lambda_i \ Re \ [g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}] \right) \\
&\quad - \left(\sum_{i=1}^{k} \lambda_i Re \ [f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2}] \right) \times \left\{ \sum_{i=1}^{k} \lambda_i \left[\overline{\nabla_z g(\xi,\eta_i)} + \nabla_{\overline{z}} g(\xi,\eta_i)\right] - B w_2 \right\} \\
&\quad + \mu^T \ \overline{\nabla_z h(\xi)} + \mu^H \nabla_{\overline{z}} h(\xi) = 0, \qquad (16)
\end{aligned}$$

$$Re\langle h(\xi), \mu \rangle \geq 0, \quad \mu \neq 0 \text{ in } S^*, \qquad (17)$$

$$w_1^H A w_1 \leq 1,\ (\alpha^H A\alpha)^{1/2} = Re(\alpha^H A w_1), \tag{18}$$

$$w_2^H B w_2 \leq 1,\ (\alpha^H B\alpha)^{1/2} = Re(\alpha^H B w_2). \tag{19}$$

For convenient, we denote the function

$$\begin{aligned}\Phi_2(\bullet) \;&= \Big(\sum_{i=1}^k \lambda_i Re\ \big[f(\bullet,\eta_i) + (\cdot)^H A w_1\big]\Big) \times \Big(\sum_{i=1}^k \lambda_i Re\ \big[g(\xi,\eta_i) - \alpha^H B w_2\big]\Big)\\ &\quad - \Big(\sum_{i=1}^k \lambda_i Re\ \big[f(\xi,\eta_i) + \alpha^H A w_1\big]\Big) \times \Big(\sum_{i=1}^k \lambda_i Re\ \big[g(\bullet,\eta_i) - (\cdot)^H B w_2\big]\Big)\end{aligned}$$

for $\bullet = (\cdot,\bar{\cdot}) \in Q \subset \mathbb{C}^{2n}$.

In order to show that the problem (MWD) is a dual problem of (P), we need to establish the following duality theorems: weak, strong and strict converse duality theorem for problem (MWD), *mutatis mutandis*, the same as the proof of the weak, strong and strict converse duality theorem for problem (WD).

Theorem 7 (Weak Duality). *Let $\zeta = (z,\bar{z})$ be (P)-feasible, and $(k,\tilde{\lambda},\tilde{\eta},\xi,\mu,w_1,w_2)$ be (WD)-feasible. Suppose that any one of the following conditions (i) and (ii) holds:*

(i) $\Phi_2(\bullet)$ is pseudoconvex on Q and $\langle h(\bullet),\mu\rangle$ is quasiconvex on Q,

(ii) $\Phi_2(\bullet)$ is quasiconvex on Q and $\langle h(\bullet),\mu\rangle$ is strictly pseudoconvex on Q,

Then

$$\max_{\eta\in Y} \frac{Re\big[f(\zeta,\eta) + (z^H A z)^{1/2}\big]}{Re\big[g(\zeta,\eta) - (z^H B z)^{1/2}\big]} \geq \varphi(\xi).$$

Proof. Suppose on the contrary that

$$\max_{\eta\in Y} \frac{Re\big[f(\zeta,\eta) + (z^H A z)^{1/2}\big]}{Re\big[g(\zeta,\eta) - (z^H B z)^{1/2}\big]} < \varphi(\xi) = \frac{\sum_{i=1}^k \lambda_i Re\ \big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2}\big]}{\sum_{i=1}^k \lambda_i Re\ \big[g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}\big]}.$$

Then for each $\eta \in Y$, we get

$$\begin{aligned}&\Big(Re\big[f(\zeta,\eta) + (z^H A z)^{1/2}\big]\Big) \times \Big(\sum_{i=1}^k \lambda_i Re\ \big[g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}\big]\Big)\\ &\quad < \Big(Re\big[g(\zeta,\eta) - (z^H B z)^{1/2}\big]\Big) \times \Big(\sum_{i=1}^k \lambda_i Re\ \big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2}\big]\Big).\end{aligned}$$

Now we are replaced η by η_i, multiplies λ_i (with $\sum_{i=1}^k \lambda_i = 1$). It reduces to

$$\begin{aligned}&\Big(\sum_{i=1}^k \lambda_i\ Re\big[f(\zeta,\eta_i) + (z^H A z)^{1/2}\big]\Big) \times \Big(\sum_{i=1}^k \lambda_i Re\ \big[g(\xi,\eta_i) - (\alpha^H B\alpha)^{1/2}\big]\Big)\\ &\quad - \Big(\sum_{i=1}^k \lambda_i\ Re\big[g(\zeta,\eta_i) - (z^H B z)^{1/2}\big]\Big) \times \Big(\sum_{i=1}^k \lambda_i Re\ \big[f(\xi,\eta_i) + (\alpha^H A\alpha)^{1/2}\big]\Big)\end{aligned}$$

$$< 0. \tag{20}$$

From inequality (18), (19) and generalized Schwarz inequality (1), we obtain

$$Re(z^H A w_1) \le (z^H A z)^{1/2} (w_1^H A w_1)^{1/2} \le (z^H A z)^{1/2} \text{ and} \tag{21}$$

$$Re(z^H B w_2) \le (z^H B z)^{1/2} (w_2^H B w_2)^{1/2} \le (z^H B z)^{1/2}. \tag{22}$$

From inequalities(19), (20), (21) and (22),

$$\begin{aligned}
\Phi_2(\zeta) &= \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[f(\zeta,\eta_i) + z^H A w_1 \right] \Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[g(\xi,\eta_i) - \alpha^H B w_2 \right] \Big) \\
&\quad - \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[f(\xi,\eta_i) + \alpha^H A w_1 \right] \Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[g(\zeta,\eta_i) - z^H B w_2 \right] \Big) \\
&< \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[f(\zeta,\eta_i) + (z^H A z)^{1/2} \right] \Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[g(\xi,\eta_i) - \alpha^H B w_2 \right] \Big) \\
&\quad - \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[f(\xi,\eta_i) + \alpha^H A w_1 \right] \Big) \times \Big(\textstyle\sum_{i=1}^k \lambda_i Re\ \left[g(\zeta,\eta_i) - (z^H B z)^{1/2} \right] \Big) \\
&< 0 = \Phi_2(\xi).
\end{aligned}$$

We obtain

$$\Phi_2(\zeta) < 0 = \Phi_2(\xi). \tag{23}$$

Since $\zeta = (z, \overline{z})$ and $\xi = (\alpha, \overline{\alpha})$ are feasible solutions of (P) and (MWD), we have

$$Re\langle h(\zeta), \mu \rangle \le 0 \le Re\langle h(\xi), \mu \rangle. \tag{24}$$

If hypothesis (i) holds, $\Phi_2(\bullet)$ is pseudoconvex at ξ and $\langle h(\bullet), \mu \rangle$ is quasiconvex at ξ, then by (23) and (24) , we have

$$Re[\Phi_2'(\xi)(\zeta - \xi)] < 0 \text{ and } \ Re\langle\ h'(\xi)(\zeta - \xi), \mu\ \rangle \le 0.$$

That is

$$\begin{aligned}
&\left\{ \sum_{i=1}^k \lambda_i \left[\overline{\nabla_z f(\xi,\eta_i)} + \nabla_{\overline{z}} f(\xi,\eta_i) \right] + A w_1 \right\} \cdot \left(\sum_{i=1}^k \lambda_i\ Re\ \left[g(\xi,\eta_i) - (\alpha^H B \alpha)^{1/2} \right] \right) \\
&- \ \left(\sum_{i=1}^k \lambda_i\ Re\ \left[f(\xi,\eta_i) + (\alpha^H A \alpha)^{1/2} \right] \right) \cdot \left\{ \sum_{i=1}^k \lambda_i \left[\overline{\nabla_z g(\xi,\eta_i)} + \nabla_{\overline{z}} g(\xi,\eta_i) \right] - B w_2 \right\} \\
&+ \mu^T\ \overline{\nabla_z h(\xi)} + \mu^H \nabla_{\overline{z}} h(\xi) \quad < 0.
\end{aligned}$$

This contradicts the equality of (16).

In hypothesis (ii), it follows by the same lines as the proof given for (i).

Hence the proof is complete. □

As for the strong and strict converse duality theorems of (MWD), *mutatis mutandis*, the same as the proof of duality theorems for (WD). Hence, we state directly the strong and strict converse duality theorems as in the following:

Theorem 8 (Strong Duality). *Let $\zeta_0 = (z_0, \overline{z_0})$ be an optimal solution of problem (P) satisfying the hypothesis of Theorem 1. Then there exist $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\zeta_0)$ and $(\zeta_0, \mu, w_1, w_2) \in X(k, \widetilde{\lambda}, \widetilde{\eta})$ such that $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is a feasible solution of the dual problem (MWD). If the hypotheses of Theorem 7 are fulfilled, then $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, w_1, w_2)$ is an optimal solution of (MWD), and the two problems (P) and (MWD) have the same optimal values.*

Theorem 9 (Strict Converse Duality). *Let $\widehat{\zeta}$ and $(\widehat{k}, \widehat{\lambda}, \widehat{\eta}, \widehat{\xi}, \widehat{\mu}, \widehat{w_1}, , \widehat{w_1})$ be the optimal solutions of (P) and (WD), respectively, and assume that the assumptions of Theorem 8 are fulfilled. If $\Phi_2(\bullet)$ is strictly pseudoconvex on Q and $\langle h(\bullet), \mu \rangle$ is quasiconvex on Q, then $\widehat{\zeta} = \widehat{\xi}$; and the optimal values of (P) and (WD) are equal.*

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