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**SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY**

# Generalized Ulam-Hyers Stability of a General Mixed AQCQ-functional Equation in Multi-Banach Spaces: a Fixed Point Approach

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**Abstract.** Using the fixed point method, we investigate the generalized Hyers-Ulam stability of the general mixed additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} f(x+ny)+f(x-ny) &= n^2 f(x+y)+n^2 f(x-y)+2(1-n^2)f(x) \\ &+ \frac{n^4-n^2}{12}[f(2y)+f(-2y)-4f(y)-4f(-y)] \end{aligned}$$

for fixed integers $n$ with $n \neq 0, \pm 1$ in multi-Banach spaces.

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## 1. Introduction

The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [35] in 1940 and affirmatively solved by Hyers [17]. The result of Hyers was generalized by Rassias [28] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x,y) = f(x+y) - [f(x)+f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [11], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$. In addition, J. M. Rassias et al.([29]-[32], [37]-[39]) generalized the Hyers stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product-sum of powers of norms, respectively. Recently, several further interesting discussions, modifications, extensions, and generalizations of the original problem of Ulam have been proposed (see, e.g., [2]-[3], [6], [8]-[16], [18], [20]-[27], [33], [36]-[42] and the references therein).

The historical background and many important results for the Ulam-Hyers stability of various functional equations are surveyed in [4] (see also [19]. There are applications in actuarial and financial mathematics, sociology and psychology, as well as in algebra and geometry [1, 4, 19]. In addition, the motivation for studying these functional equations came from the fact that recently polynomial equations have found applications in approximate checking, self-testing, and self-correcting of computer programs that compute polynomials. The interested reader should refer to [34] and [40] and references therein.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

is said to be a quadratic functional equation because the quadratic function $f(x) = x^2$ is a solution of the functional equation (1). Every solution of the quadratic functional equation is said to be a quadratic mapping. A quadratic functional equation was used to characterize inner product spaces.

In 2001, J. M. Rassias [29] introduced the cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) = 6f(y) \tag{2}$$

and established the solution of the Ulam stability problem for these cubic mappings. It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (2), which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping. The quartic functional equation

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) + 6f(x) + 24f(y) \tag{3}$$

was introduced by J. M. Rassias [31]. It is easy to show that the function $f(x) = x^4$ is the solution of (3). Every solution of the quartic functional equation is said to be a quartic mapping. C. Park [25] proved the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation (briefly, AQCQ-functional equation)

$$f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4(-y) \tag{4}$$

in non-Archimedean normed spaces.

In [9, 33], the authors introduced a general mixed type functional equation

$$\begin{aligned} f(x+ny) &+ f(x-ny) = n^2 f(x+y) + n^2 f(x-y) + 2(1-n^2)f(x) \\ &+ \frac{n^4-n^2}{12}[f(2y)+f(-2y)-4f(y)-4f(-y)] \end{aligned} \tag{5}$$

which is a generalized form of the additive-quadratic-cubic-quartic (4) and obtained its general solution and generalized Hyers-Ulam stability for fixed integers $n$ with $n \neq 0, \pm 1$ in Banach spaces.

The notion of multi-normed space was introduced by H. G. Dales and M. E. Polyakov [5]. This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples were given in [5]. Also, the stability problems in multi-Banach spaces are studied by Dales and Moslehian [6], Moslehian et al. ([21]-[23]) and Wang et al. [36].

In 1996, Isac and Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. The stability problems of several various functional equations have been extensively investigated by a number of authors using the fixed point method (see [2]-[3], [6]-[7], [20], [25]-[27], [36], [38].

In this paper, we prove the generalized Hyers-Ulam stability of the general mixed AQCQ-functional equation (5) in multi-Banach spaces using the fixed point method.

## 2. Preliminaries

We recall some preliminaries concerning multi-Banach space (see [5]-[6], [21]-[23]).

Let $(E, \|\cdot\|)$ be a complex linear space, and let $k \in \mathbb{N}$. We denote by $E^k$ the linear space $E \oplus \cdots \oplus E$ consisting of $k$-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on $E^k$ are defined coordinate-wise. When we write $(0, \ldots, 0, x_i, 0, \ldots, 0)$ for an element in $E^k$, we understand that $x_i$ appears in the $i$th coordinate. The zero elements of either $E$ or $E^k$ are both denoted by 0 when there is no confusion. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by $\mathbb{B}_k$ the group of permutations on $\mathbb{N}_k$.

**Definition 1.** *A multi-norm on* $\{E^k, k \in \mathbb{N}\}$ *is a sequence* $(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$ *such that* $\|\cdot\|_k$ *is a norm on* $E^k$ *for each* $k \in \mathbb{N}$*, such that* $\|x\|_1 = \|x\|$ *for each* $x \in E$*, and the following axioms are satisfied for each* $k \in \mathbb{N}$ *with* $k \geq 2$*:*

*(A1)* $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1, \ldots, x_k)\|_k$ $(\sigma \in \mathbb{B}_k, x_1, \ldots, x_k \in E)$*;*

*(A2)* $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|)\|(x_1, \ldots, x_k)\|_k (x_i \in E, \alpha_i \in \mathbb{C}, i = 1, \ldots, k)$*;*

*(A3)* $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ $(x_1, \ldots, x_{k-1} \in E)$*;*

*(A4)* $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ $(x_1, \ldots, x_{k-1} \in E)$.

In this case, we say that $((E^k, \|\cdot\|_k): k \in \mathbb{N})$ is a multi-normed space.

Suppose that $((E^k, \|\cdot\|_k): k \in \mathbb{N})$ is a multi-normed space and take $k \in \mathbb{N}$. It is easy to show that

(a) $\|(x, \ldots, x)\|_k = \|x\|$ $(x \in E)$;

(b) $\max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\|$ $(x_1, \ldots, x_k \in E)$.

It follows from (b) that if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case $((E^k, \|\cdot\|_k): k \in \mathbb{N})$ is said to be a multi-Banach space.

Now we state two important examples of multi-norms for arbitrary normed space $E$ (see [5]-[6], [21]-[23]).

**Example 1.** *Let $E$ be an arbitrary normed space. The sequence $(\|\cdot\|_k: k \in \mathbb{N})$ on $\{E^k: k \in \mathbb{N}\}$ defined by*

$$\|(x_1, \ldots, x_k)\|_k := \max_{i \in \mathbb{N}_k} \|x_i\| \ (x_1, \ldots, x_k \in E)$$

*is a multi-norm called the minimum multi-norm. The terminology minimum is justified by property (b).*

**Example 2.** *Let $E$ be an arbitrary normed space and let $\{(\|\cdot\|_k^\alpha: k \in \mathbb{N}): \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{E^k: k \in \mathbb{N}\}$. For $k \in \mathbb{N}$, consider*

$$|||(x_1, \ldots, x_k)|||_k := \sup_{\alpha \in A} \|(x_1, \ldots, x_k)\|_k^\alpha \ (x_1, \ldots, x_k \in E).$$

*Then $(|||\cdot|||_k: k \in \mathbb{N})$ is a multi-norm on $\{E^k: k \in \mathbb{N}\}$, called the maximum multi-norm.*

**Definition 2.** *Let $((E^k, \|\cdot\|_k): k \in \mathbb{N})$ be a multi-normed space. A sequence $\{x_n\}$ in $E$ is a multi-null sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\sup_{k \in \mathbb{N}} \|(x_n, \ldots, x_{n+k-1})\|_k < \varepsilon (n \geq n_0).$$

*Let $x \in E$. We say that the sequence $\{x_n\}$ is multi-convergent to $x$ in $E$ if $\{x_n - x\}$ is a multi-null sequence. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$.*

For explicitly later use, We recall a fundamental result in fixed point theory.

Let $X$ be a set. A function $d: X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies: (1) $d(x,y) = 0$ if and only if $x = y$; (2) $d(x,y) = d(y,x)$ for all $x, y \in X$; (3) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

**Theorem 1** (The fixed point alternative theorem, see [2, 7, 20, 25, 38])**.** *Let $(\Omega, d)$ be a complete generalized metric space and $J: \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 \leq L < 1$, that is*

$$d(Jx, Jy) \leq Ld(x,y) \text{ for all } x \in X.$$

*Then, for each given $x \in \Omega$, either*

$$d(J^m x, J^{m+1}x) = \infty \text{ for all } m \geq 0,$$

*or*

$$d(J^m x, J^{m+1}x) < \infty \text{ for all } m \geq m_0,$$

*for some nonnegative integer $m_0$. Actually, if the second alternative holds, then the sequence $\{J^m x\}$ converges to a fixed point $y^*$ of $J$ and*

*(i) $y^*$ is the unique fixed point of $J$ in the set $\Delta = \{y \in \Omega : \ d(J^{m_0}x, y) < \infty\}$;*

*(ii) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Delta$.*

## 3. Generalized Hyers-Ulam stability of the functional equation

In this section, we investigate the stability of the mixed type functional equation (5) in multi-Banach spaces. For convenience, we use the following abbreviation for a given mapping $f : E \to F$:

$$\begin{aligned} Df(x,y) \ &:= \ f(x+ny) + f(x-ny) - n^2 f(x+y) - n^2 f(x-y) - 2(1-n^2)f(x) \\ &- \ \frac{n^4 - n^2}{12}[f(2y) + f(-2y) - 4f(y) - 4f(-y)] \end{aligned}$$

for all $x, y \in X$.

**Theorem 2.** *Let $E$ be a linear space and let $((F^k, \|\cdot\|_k) : \ k \in \mathbb{N})$ be a multi-Banach space. Suppose that $\varepsilon \geq 0$ and $f : E \to F$ is an odd mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1,y_1), \ldots, Df(x_k,y_k))\|_k \leq \varepsilon \tag{6}$$

*for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exists a unique additive mapping $A : E \to F$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) - 8f(x_1) - A(x_1), \ldots, f(2x_k) - 8f(x_k) - A(x_k))\|_k \leq \frac{9n^2+4}{n^4-n^2}\varepsilon \tag{7}$$

*for all $x_1, \ldots, x_k \in E$.*

*Proof.* Let $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Using the oddness of $f$ and (6), we have

$$\begin{aligned} \sup_{k\in\mathbb{N}} \|(&f(x_1+ny_1) + f(x_1-ny_1) - n^2 f(x_1+y_1) - n^2 f(x_1-y_1) \\ &-2(1-n^2)f(x_1), \ldots, f(x_k+ny_k) + f(x_k-ny_k) - n^2 f(x_k+y_k) \\ &-n^2 f(x_k-y_k) - 2(1-n^2)f(x_k))\|_k \leq \varepsilon. \end{aligned} \tag{8}$$

Replacing $y_i$ by $x_i (i \in \mathbb{N}_k)$ in (8), we get

$$\begin{aligned} \sup_{k\in\mathbb{N}} \|(&f((1+n)x_1) + f((1-n)x_1) - n^2 f(2x_1) - 2(1-n^2)f(x_1), \ldots, \\ &f((1+n)x_k) + f((1-n)x_k) - n^2 f(2x_k) - 2(1-n^2)f(x_k))\|_k \leq \varepsilon. \end{aligned} \tag{9}$$

Replacing $x_i$ by $2x_i (i \in \mathbb{N}_k)$ in (9), we get

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(2(1+n)x_1)+f(2(1-n)x_1)-n^2f(4x_1)-2(1-n^2)f(2x_1),\ldots,\\ &\quad f(2(1+n)x_k)+f(2(1-n)x_k)-n^2f(4x_k)-2(1-n^2)f(2x_k))\|_k \le \varepsilon.\end{aligned} \tag{10}$$

Replacing $x_i$ and $y_i$ by $2x_i$ and $x_i (i \in \mathbb{N}_k)$ in (8), respectively, we get

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f((2+n)x_1)+f((2-n)x_1)-n^2f(3x_1)-n^2f(x_1)-2(1-k^2)f(2x_1),\ldots,\\ &\quad f((2+n)x_k)+f((2-n)x_k)-n^2f(3x_k)-n^2f(x_k)-2(1-k^2)f(2x_k))\|_k \le \varepsilon.\end{aligned} \tag{11}$$

Replacing $y_i$ by $2x_i (i \in \mathbb{N}_k)$ in (8), we get

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f((1+2n)x_1)+f((1-2n)x_1)-n^2f(3x_1)+n^2f(x_1)-2(1-n^2)f(x_1),\ldots,\\ &\quad f((1+2n)x_k)+f((1-2n)x_k)-n^2f(3x_k)+n^2f(x_k)-2(1-n^2)f(x_k))\|_k \le \varepsilon.\end{aligned} \tag{12}$$

Replacing $y_i$ by $3x_i (i \in \mathbb{N}_k)$ in (8), we get

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f((1+3n)x_1)+f((1-3n)x_1)-n^2f(4x_1)+n^2f(2x_1)-2(1-n^2)f(x_1),\ldots,\\ &\quad f((1+3n)x_k)+f((1-3n)x_k)-n^2f(4x_k)+n^2f(2x_k)-2(1-n^2)f(x_k))\|_k \le \varepsilon.\end{aligned} \tag{13}$$

Replacing $x_i$ and $y_i$ by $(1+n)x_i$ and $x_i (i \in \mathbb{N}_k)$ in (8), respectively, we have

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f((1+2n)x_1)+f(x_1)-n^2f((2+n)x_1)-n^2f(nx_1)\\ &\quad -2(1-n^2)f((1+n)x_1),\ldots,f((1+2n)x_k)+f(x_k)\\ &\quad -n^2f((2+n)x_k)-n^2f(nx_k)-2(1-n^2)f((1+n)x_k))\|_k \le \varepsilon.\end{aligned} \tag{14}$$

Again replacing $x_i$ and $y_i$ by $(1-n)x_i$ and $x_i (i \in \mathbb{N}_k)$ in (8), respectively, we have

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f(x_1)+f((1-2n)x_1)-n^2f((2-n)x_1)+n^2f(nx_1)\\ &\quad -2(1-n^2)f((1-n)x_1),\ldots,f(x_k)+f((1-2n)x_k)\\ &\quad -n^2f((2-n)x_k)+n^2f(nx_k)-2(1-n^2)f((1-n)x_k))\|_k \le \varepsilon.\end{aligned} \tag{15}$$

By (14) and (15), we have

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f((1+2n)x_1)+f((1-2n)x_1)+2f(x_1)-n^2f((2+n)x_1)\\ &\quad -2(1-n^2)f((1+n)x_1)-n^2f((2-n)x_1)-2(1-n^2)f((1-n)x_1),\\ &\quad \ldots,f((1+2n)x_k)+f((1-2n)x_k)+2f(x_k)-n^2f((2+n)x_k)\\ &\quad -n^2f((2-n)x_k)-2(1-n^2)f((1+n)x_k)-2(1-n^2)f((1-n)x_k))\|_k \le 2\varepsilon.\end{aligned} \tag{16}$$

Replacing $x_i$ and $y_i$ by $(1+2n)x_i$ and $x_i (i \in \mathbb{N}_k)$ in (8), respectively, we have

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(f((1+3n)x_1)+f((1+n)x_1)-n^2f(2(1+n)x_1)-n^2f(2nx_1)\\ &\quad -2(1-n^2)f((1+2n)x_1),\ldots,f((1+3n)x_k)+f((1+n)x_k)\\ &\quad -n^2f(2(1+n)x_k)-n^2f(2nx_k)-2(1-n^2)f((1+2n)x_k))\|_k \le \varepsilon.\end{aligned} \tag{17}$$

Again replacing $x_i$ and $y_i$ by $(1-2n)x_i$ and $x_i (i \in \mathbb{N}_k)$ in (8), respectively, we obtain

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|(& f((1-3n)x_1) + f((1-n)x_1) - n^2 f(2(1-n)x_1) + n^2 f(2nx_1)\\ & -2(1-n^2) f((1-2n)x_1), \ldots, f((1-3n)x_k) + f((1-n)x_k)\\ & -n^2 f(2(1-n)x_k) + n^2 f(2nx_k) - 2(1-n^2) f((1-2n)x_k))\|_k \le \varepsilon.\end{aligned} \tag{18}$$

By (17) and (18), we have

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|(& f((1+3n)x_1) + f((1-3n)x_1) + f((1+n)x_1) + f((1-n)x_1)\\ & -n^2 f(2(1+n)x_1) - n^2 f(2(1-n)x_1) - 2(1-n^2) f((1+2n)x_1)\\ & -2(1-n^2) f((1-2n)x_1), \ldots, f((1+3n)x_k) + f((1-3n)x_k)\\ & +f((1+n)x_k) + f((1-n)x_k) - n^2 f(2(1+n)x_k) - n^2 f(2(1-n)x_k)\\ & -2(1-n^2) f((1+2n)x_k) - 2(1-n^2) f((1-2n)x_k))\|_k \le 2\varepsilon.\end{aligned} \tag{19}$$

By (9), (11), (12) and (16), we get

$$\sup_{k\in\mathbb{N}} \|(f(3x_1) - 4f(2x_1) + 5f(x_1), \ldots, f(3x_k) - 4f(2x_k) + 5f(x_k))\|_k \le \frac{3n^2+1}{n^4-n^2}\varepsilon. \tag{20}$$

By (9), (10), (12), (13) and (19), we get

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|(& f(4x_1) - 2f(3x_1) - 2f(2x_1) + 6f(x_1), \ldots, f(4x_k) - 2f(3x_k)\\ & -2f(2x_k) + 6f(x_k))\|_k \le \tfrac{3n^2+2}{n^4-n^2}\varepsilon.\end{aligned} \tag{21}$$

By (20) and (21), we get

$$\sup_{k\in\mathbb{N}} \|(f(4x_1) - 10f(2x_1) + 16f(x_1), \ldots, f(4x_k) - 10f(2x_k) + 16f(x_k))\|_k \le \frac{9n^2+4}{n^4-n^2}\varepsilon. \tag{22}$$

Consider the set $\Omega := \{g \mid g : E \to F, g(0) = 0\}$ and introduce the generalized metric on $\Omega$,

$$d(g,h) = \inf\{\alpha > 0 | \sup_{k\in\mathbb{N}} \|(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k))\|_k \le \alpha, \forall x_1, \ldots, x_k \in E, k \in \mathbb{N}\}.$$

It is easy to show that $(\Omega, d)$ is a generalized complete metric space [see 20, Lemma 2.1].

Define $J : \Omega \to \Omega$ by $Jg(x) = g(2x)/2$ for all $x \in E$. Let $g, h \in \Omega$ be given such that $d(g,h) < \beta$, by the definition,

$$\sup_{k\in\mathbb{N}} \|(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k))\|_k \le \beta \text{ for all } x_1, \ldots, x_k \in E, k \in \mathbb{N}.$$

Hence

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(Jg(x_1) - Jh(x_1), \ldots, Jg(x_k) - Jh(x_k))\|_k\\ &\quad \le \tfrac{1}{2} \sup_{k\in\mathbb{N}} \|(g(2x_1) - h(2x_1), \ldots, g(2x_k) - h(2x_k))\|_k \le \tfrac{\beta}{2}\end{aligned}$$

for all $x_1,\dots,x_k \in E, k \in \mathbb{N}$. By definition, $d(Jg,Jh) \le \beta/2$. Therefore, $d(Jg,Jh) \le \frac{1}{2}d(g,h)$ for all $g,h \in \Omega$. This means that $J$ is a strictly contractive self-mapping of $\Omega$ with Lipschitz constant 1/2.

Now, let $\tilde{f} : E \to F$ be the mapping defined by $\tilde{f}(x) := f(2x) - 8f(x)$ for each $x \in E$. By (22), we get

$$\sup_{k\in\mathbb{N}} \|(\tilde{f}(2x_1) - 2\tilde{f}(x_1), \dots, \tilde{f}(2x_k) - 2\tilde{f}(x_k))\|_k \le \frac{9n^2+4}{n^4-n^2}\varepsilon. \tag{23}$$

Multiplying (23) by 1/2, we obtain

$$\sup_{k\in\mathbb{N}} \|(J\tilde{f}(x_1) - \tilde{f}(x_1), \dots, J\tilde{f}(x_k) - \tilde{f}(x_k))\|_k \le \frac{9n^2+4}{2(n^4-n^2)}\varepsilon. \tag{24}$$

Then $d(J\tilde{f},\tilde{f}) \le (9n^2+4)/(2(n^4-n^2))\varepsilon$ and therefore, by Theorem 1, $J$ has a unique fixed point $A : E \to F$ in the set $\Delta = \{h \in \Omega : \ d(\tilde{f},h) < \infty\}$. This implies that $A(2x) = 2A(x)$ and

$$A(x) = \lim_{m\to\infty} J^m \tilde{f}(x) = \lim_{m\to\infty} \frac{1}{2^m}\tilde{f}(2^m x) \tag{25}$$

for all $x \in E$. Since $\tilde{f} : E \to F$ is odd, $A : E \to F$ is an odd mapping. Moreover,

$$d(\tilde{f},A) \le \frac{1}{1-L} d(\tilde{f},J\tilde{f}) \le \frac{9n^2+4}{n^4-n^2}\varepsilon.$$

This implies that the inequality (7) holds. Also we have

$$\|DA(x,y)\| = \lim_{m\to\infty} \frac{1}{2^m}\|Df(2^{m+1}x, 2^{m+1}y) - 8Df(2^m x, 2^m y)\| \le \lim_{m\to\infty} \frac{9\varepsilon}{2^m} = 0,$$

and $A$ satisfies (5). By Theorem 2.2 of [33], the function $x \to A(2x) - 8A(x)$ is additive. Hence $A(2x) = 2A(x)$ implies that $A$ is an additive mapping.

If $T$ is another additive mapping satisfying (7). Then $T$ is a fixed point of $J$ in $\Delta$. However, by Theorem 1, $J$ has only one fixed point in $\Delta$, hence $A = T$. This completes the proof.

**Theorem 3.** *Let $E$ be a linear space and let $((F^k, \|\cdot\|_k) : \ k \in \mathbb{N})$ be a multi-Banach space. Suppose that $\varepsilon \ge 0$ and $f : E \to F$ is an odd mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1,y_1), \dots, Df(x_k,y_k))\|_k \le \varepsilon$$

*for all $x_1,\dots,x_k,y_1,\dots,y_k \in E$. Then there exists a unique cubic mapping $C : E \to F$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) - 2f(x_1) - C(x_1), \dots, f(2x_k) - 2f(x_k) - C(x_k))\|_k \le \frac{9n^2+4}{7(n^4-n^2)}\varepsilon$$

*for all $x_1,\dots,x_k \in E$.*

*Proof.* The proof is similar to that of Theorem 2.

**Theorem 4.** *Let $E$ be a linear space and let $((F^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach space. Suppose that $\varepsilon \geq 0$ and $f : E \to F$ is an even mapping with $f(0) = 0$, satisfying condition*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1, y_1), \ldots, Df(x_k, y_k))\|_k \leq \varepsilon \tag{26}$$

*for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exists a unique quadratic mapping $B : E \to F$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) - 16f(x_1) - B(x_1), \ldots, f(2x_k) - 16f(x_k) - B(x_k))\|_k \leq \frac{8n^2 + 2}{n^4 - n^2}\varepsilon \tag{27}$$

*for all $x_1, \ldots, x_k \in E$.*

*Proof.* Let $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Using the evenness of $f$ and from (26), we have

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \|(f(x_1 + ny_1) + f(x_1 - ny_1) - n^2 f(x_1 + y_1) - n^2 f(x_1 - y_1) - 2(1 - n^2)f(x_1) \\ &\quad -\tfrac{n^4-n^2}{12}[2f(2y_1) - 8f(y_1)], \ldots, f(x_k + ny_k) + f(x_k - ny_k) - n^2 f(x_k + y_k) \\ &\quad -n^2 f(x_k - y_k) - 2(1 - n^2)f(x_k) - \tfrac{n^4-n^2}{12}[2f(2y_k) - 8f(y_k)])\|_k \leq \varepsilon. \end{aligned} \tag{28}$$

Interchanging $x_i$ and $y_i (i \in \mathbb{N}_k)$ in (28), we get

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \|(f(nx_1 + y_1) + f(nx_1 - y_1) - n^2 f(x_1 + y_1) - n^2 f(x_1 - y_1) - 2(1 - n^2)f(y_1) \\ &\quad -\tfrac{n^4-n^2}{12}[2f(2x_1) - 8f(x_1)], \ldots, f(nx_k + y_k) + f(nx_k - y_k) - n^2 f(x_k + y_k) \\ &\quad -n^2 f(x_k - y_k) - 2(1 - n^2)f(y_k) - \tfrac{n^4-n^2}{12}[2f(2x_k) - 8f(x_k)])\|_k \leq \varepsilon. \end{aligned} \tag{29}$$

Letting $y_i = 0 (i \in \mathbb{N}_k)$ in (29), we get

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \|(2f(nx_1) - 2n^2 f(x_1) - \tfrac{n^4-n^2}{12}[2f(2x_1) - 8f(x_1)], \ldots, \\ &\quad 2f(nx_k) - 2n^2 f(x_k) - \tfrac{n^4-n^2}{12}[2f(2x_k) - 8f(x_k)])\|_k \leq \varepsilon. \end{aligned} \tag{30}$$

Putting $y_i = x_i (i \in \mathbb{N}_k)$ in (29), we have

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \|(f((n+1)x_1) + f((n-1)x_1) - n^2 f(2x_1) - 2(1 - n^2)f(x_1) \\ &\quad -\tfrac{n^4-n^2}{12}[2f(2x_1) - 8f(x_1)], \ldots, f((n+1)x_k) + f((n-1)x_k) \\ &\quad -n^2 f(2x_k) - 2(1 - n^2)f(x_k) - \tfrac{n^4-n^2}{12}[2f(2x_k) - 8f(x_k)])\|_k \leq \varepsilon. \end{aligned} \tag{31}$$

Replacing $x_i$ by $2x_i (i \in \mathbb{N}_k)$ in (30), we get

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \|(2f(2nx_1) - 2n^2 f(2x_1) - \tfrac{n^4-n^2}{12}[2f(4x_1) - 8f(2x_1)], \ldots, \\ &\quad 2f(2nx_k) - 2n^2 f(2x_k) - \tfrac{n^4-n^2}{12}[2f(4x_k) - 8f(2x_k)])\|_k \leq \varepsilon. \end{aligned} \tag{32}$$

Letting $y_i = nx_i (i \in \mathbb{N}_k)$ in (29), we get

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|(&f(2nx_1) - n^2 f((1+n)x_1) - n^2 f((1-n)x_1) - 2(1-n^2)f(nx_1) \\ &-\tfrac{n^4-n^2}{12}[2f(2x_1) - 8f(x_1)], \dots, f(2nx_k) - n^2 f((1+n)x_k) \\ &-n^2 f((1-n)x_k) - 2(1-n^2)f(nx_k) - \tfrac{n^4-n^2}{12}[2f(2x_k) - 8f(x_k)])\|_k \leq \varepsilon.\end{aligned} \tag{33}$$

By (30)-(33), we obtain

$$\sup_{k\in\mathbb{N}} \|(f(4x_1) - 20f(2x_1) + 64f(x_1), \dots, f(4x_1) - 20f(2x_1) + 64f(x_1))\|_k \leq \frac{24n^2+6}{n^4-n^2}\varepsilon. \tag{34}$$

Consider the set $\Omega := \{g \mid g : E \to F, g(0) = 0\}$ and introduce the generalized metric on $\Omega$,

$$d(g,h) = \inf\{\alpha > 0 | \sup_{k\in\mathbb{N}} \|(g(x_1) - h(x_1), \dots, g(x_k) - h(x_k))\|_k \leq \alpha, \forall x_1, \dots, x_k \in E, k \in \mathbb{N}\}.$$

It is easy to show that $(\Omega, d)$ is a generalized complete metric space [see 20, Lemma 2.1].

Define $J : \Omega \to \Omega$ by $Jg(x) = g(2x)/4$ for all $x \in E$. Let $g, h \in \Omega$ be given such that $d(g,h) < \beta$, by the definition,

$$\sup_{k\in\mathbb{N}} \|(g(x_1) - h(x_1), \dots, g(x_k) - h(x_k))\|_k \leq \beta \quad \text{for all } x_1, \dots, x_k \in E, k \in \mathbb{N}.$$

Hence

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(Jg(x_1) - Jh(x_1), \dots, Jg(x_k) - Jh(x_k))\|_k \\ &\quad \leq \tfrac{1}{4} \sup_{k\in\mathbb{N}} \|(g(2x_1) - h(2x_1), \dots, g(2x_k) - h(2x_k))\|_k \leq \tfrac{\beta}{4}\end{aligned}$$

for all $x_1, \dots, x_k \in E, k \in \mathbb{N}$. By definition, $d(Jg, Jh) \leq \beta/4$. Therefore, $d(Jg, Jh) \leq \frac{1}{4}d(g,h)$ for all $g, h \in \Omega$. This means that $J$ is a strictly contractive self-mapping of $\Omega$ with Lipschitz constant $1/4$.

Now, let $\tilde{f} : E \to F$ be the mapping defined by $\tilde{f}(x) := f(2x) - 16f(x)$ for each $x \in E$. By (34), we get

$$\sup_{k\in\mathbb{N}} \|(\tilde{f}(2x_1) - 4\tilde{f}(x_1), \dots, \tilde{f}(2x_k) - 4\tilde{f}(x_k))\|_k \leq \frac{24n^2+6}{n^4-n^2}\varepsilon. \tag{35}$$

Multiplying (35) by $1/4$, we obtain

$$\sup_{k\in\mathbb{N}} \|(J\tilde{f}(x_1) - \tilde{f}(x_1), \dots, J\tilde{f}(x_k) - \tilde{f}(x_k))\|_k \leq \frac{24n^2+6}{4(n^4-n^2)}\varepsilon. \tag{36}$$

Then $d(J\tilde{f}, \tilde{f}) \leq \varepsilon(24n^2+6)/(4(n^4-n^2))$ and therefore, by Theorem 1, $J$ has a unique fixed point $B : E \to F$ in the set $\Delta = \{h \in \Omega : d(\tilde{f}, h) < \infty\}$. This implies that $B(2x) = 4B(x)$ and

$$B(x) = \lim_{m\to\infty} J^m \tilde{f}(x) = \lim_{m\to\infty} \frac{1}{4^m}\tilde{f}(2^m x) \tag{37}$$

for all $x \in E$. Since $\tilde{f}: E \to F$ is even, $B: E \to F$ is an even mapping. Moreover,

$$d(\tilde{f}, A) \leq \frac{1}{1-L} d(\tilde{f}, J\tilde{f}) \leq \frac{8n^2+2}{n^4-n^2}\varepsilon.$$

This implies that the inequality (27) holds. Also we have

$$\begin{aligned} \|DB(x,y)\| &= \lim_{m\to\infty} \tfrac{1}{4^m}\|Df(2^{m+1}x, 2^{m+1}y) - 16Df(2^m x, 2^m y)\| \\ &\leq \lim_{m\to\infty} \tfrac{17\varepsilon}{4^m} = 0, \end{aligned}$$

and $B$ satisfies (5). By Theorem 2.1 of [33], the mapping $x \to B(2x) - 16B(x)$ is quadratic. Hence $B(2x) = 4B(x)$ implies that $B$ is a quadratic mapping.

The rest of the proof is similar to that of Theorem 2.

**Theorem 5.** *Let $E$ be a linear space and let $((F^k, \|\cdot\|_k): k \in \mathbb{N})$ be a multi-Banach space. Suppose that $\varepsilon \geq 0$ and $f: E \to F$ is an even mapping with $f(0) = 0$, satisfying condition*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1, y_1), \ldots, Df(x_k, y_k))\|_k \leq \varepsilon$$

*for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exists a unique quartic mapping $Q: E \to F$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) - 4f(x_1) - Q(x_1), \ldots, f(2x_k) - 4f(x_k) - Q(x_k))\|_k \leq \frac{8n^2+2}{5(n^4-n^2)}\varepsilon$$

*for all $x_1, \ldots, x_k \in E$.*

*Proof.* The proof is similar to that of Theorem 4.

**Theorem 6.** *Let $E$ be a linear space and let $((F^k, \|\cdot\|_k): k \in \mathbb{N})$ be a multi-Banach space. Suppose that $\varepsilon \geq 0$ and $f: E \to F$ is an odd mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1, y_1), \ldots, Df(x_k, y_k))\|_k \leq \varepsilon \tag{38}$$

*for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exist a unique additive mapping $A: E \to F$ and a unique cubic mapping $C: E \to F$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1) - C(x_1), \ldots, f(x_k) - A(x_k) - C(x_k))\|_k \leq \frac{4(9n^2+4)}{21(n^4-n^2)}\varepsilon \tag{39}$$

*for all $x_1, \ldots, x_k \in E$.*

*Proof.* By Theorems 2 and 3, there exist a unique additive mapping $A_0: E \to F$ and a unique cubic mapping $C_0: E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) - 8f(x_1) - A_0(x_1), \ldots, f(2x_k) - 8f(x_k) - A_0(x_k))\|_k \leq \frac{9n^2+4}{n^4-n^2}\varepsilon \tag{40}$$

and

$$\sup_{k\in\mathbb{N}} \|(f(2x_1)-2f(x_1)-C_0(x_1),\dots,f(2x_k)-2f(x_k)-C_0(x_k))\|_k \leq \frac{9n^2+4}{7(n^4-n^2)}\varepsilon \tag{41}$$

for all $x_1,\dots,x_k \in E$. Now from (40) and (41), one can see that

$$\sup_{k\in\mathbb{N}} \|(6f(x_1)+A_0(x_1)-C_0(x_1),\dots,6f(x_k)+A_0(x_k)-C_0(x_k))\|_k \leq \frac{8(9n^2+4)}{7(n^4-n^2)}\varepsilon$$

for all $x_1,\dots,x_k \in E$. Thus we obtain (39) by defining $A(x)=-A_0(x)/6$ and $C(x)=C_0(x)/6$. The uniqueness of $A$ and $C$ is easy to show.

**Theorem 7.** *Let $E$ be a linear space and let $((F^k,\|\cdot\|_k): \quad k\in\mathbb{N})$ be a multi-Banach space. Suppose that $\varepsilon \geq 0$ and $f: E\to F$ is an even mapping with $f(0)=0$, satisfying condition*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1,y_1),\dots,Df(x_k,y_k))\|_k \leq \varepsilon \tag{42}$$

*for all $x_1,\dots,x_k,y_1,\dots,y_k \in E$. Then there exist a unique quadratic mapping $B: E\to F$ and a unique quartic mapping $Q: E\to F$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1)-B(x_1)-Q(x_1),\dots,f(x_k)-B(x_k)-Q(x_k))\|_k \leq \frac{4n^2+1}{5(n^4-n^2)}\varepsilon \tag{43}$$

*for all $x_1,\dots,x_k \in E$.*

*Proof.* By Theorems 4 and 5, there exist a unique quadratic mapping $B_0: E\to F$ and a unique quartic mapping $Q_0: E\to F$ such that

$$\sup_{k\in\mathbb{N}} \|(f(2x_1)-16f(x_1)-B_0(x_1),\dots,f(2x_k)-16f(x_k)-B_0(x_k))\|_k \leq \frac{8n^2+2}{n^4-n^2}\varepsilon \tag{44}$$

and

$$\sup_{k\in\mathbb{N}} \|(f(2x_1)-4f(x_1)-Q_0(x_1),\dots,f(2x_k)-4f(x_k)-Q_0(x_k))\|_k \leq \frac{8n^2+2}{5(n^4-n^2)}\varepsilon \tag{45}$$

for all $x_1,\dots,x_k \in E$. Now from (44) and (45), one can see that

$$\sup_{k\in\mathbb{N}} \|(12f(x_1)+B_0(x_1)-Q_0(x_1),\dots,12f(x_k)+B_0(x_k)-Q_0(x_k))\|_k \leq \frac{6(8n^2+2)}{5(n^4-n^2)}\varepsilon$$

for all $x_1,\dots,x_k \in E$. Thus we obtain (43) by defining $B(x)=-B_0(x)/12$ and $Q(x)=Q_0(x)/12$. The uniqueness of $B$ and $Q$ is easy to show.

**Theorem 8.** *Let $E$ be a linear space and let $((F^k, \|\cdot\|_k) : \quad k \in \mathbb{N})$ be a multi-Banach space. Suppose that $\varepsilon \geq 0$ and $f : E \to F$ is a mapping with $f(0) = 0$, satisfying condition*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1, y_1), \ldots, Df(x_k, y_k))\|_k \leq \varepsilon \tag{46}$$

*for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exist a unique additive mapping $A : E \to F$, a unique quadratic mapping $B : E \to F$, a unique cubic mapping $C : E \to F$, and a unique quartic mapping $Q : E \to F$ such that*

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1) - B(x_1) - C(x_1) - Q(x_1), \ldots, \\ f(x_k) - A(x_k) - B(x_k) - C(x_k) - Q(x_k))\|_k \leq \tfrac{164n^2+101}{105(n^4-n^2)}\varepsilon\end{aligned} \tag{47}$$

*for all $x_1, \ldots, x_k \in E$.*

*Proof.* Let $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in E$. Then $f_o(0) = 0$, $f_o(x) = -f_o(-x)$ . Hence

$$\sup_{k\in\mathbb{N}} \|(Df_o(x_1, y_1), \ldots, Df_o(x_k, y_k))\|_k \leq \varepsilon$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. By Theorem 6, there exist a unique additive mapping $A : E \to F$ and a unique cubic mapping $C : E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1) - C(x_1), \ldots, f(x_k) - A(x_k) - C(x_k))\|_k \leq \frac{4(9n^2+4)}{21(n^4-n^2)}\varepsilon \tag{48}$$

for all $x_1, \ldots, x_k \in E$. Let $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in E$. Then $f_e(0) = 0$, $f_e(x) = f_e(-x)$ and

$$\sup_{k\in\mathbb{N}} \|(Df_e(x_1, y_1), \ldots, Df_e(x_k, y_k))\|_k \leq \varepsilon$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. By Theorem 7, there exist a unique quadratic mapping $B : E \to F$ and a unique quartic mapping $Q : E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - B(x_1) - Q(x_1), \ldots, f(x_k) - B(x_k) - Q(x_k))\|_k \leq \frac{4n^2+1}{5(n^4-n^2)}\varepsilon \tag{49}$$

for all $x_1, \ldots, x_k \in E$. By (48) and (49), we get (47). This completes the proof.

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