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Quasi-Hadamard Product of Certain Meromorphic P-Valent Analytic Functions

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Abstract. In this paper, we establish certain results concerning the quasi-Hadamard product for the classes related to meromorphic p-valent analytic functions with positive coefficients.

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1. Introduction

Throughout this paper, let $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and the functions of the form :

$$\varphi(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad \left(a_p > 0; a_{p+n} \geq 0\right),$$

$$\psi(z) = b_p z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \qquad \left(a_p > 0; b_{p+n} \geq 0\right),$$

be *analytic* and *p-valent* in the unit disc $\Delta = \{z : |z| < 1\}$. Also, let

$$f(z) = \frac{a_{p-1}}{z^p} + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} \ \left(a_p > 0; a_{p+n} \geq 0\right), \tag{1}$$

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$$f_i(z) = \frac{a_{p-1,i}}{z^p} + \sum_{n=1}^{\infty} a_{n+p-1,i} z^{n+p-1} \ \left(a_{p,i} > 0; a_{p+n,i} \geq 0\right), \tag{2}$$

$$g(z) = \frac{b_{p-1}}{z^p} + \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1} \ \left(b_p > 0; b_{p+n} \geq 0\right), \tag{3}$$

and

$$g_i(z) = \frac{b_{p-1,i}}{z^p} + \sum_{n=1}^{\infty} b_{n+p-1,i} z^{n+p-1} \ \left(b_{p,i} > 0; b_{p+n,i} \geq 0\right), \tag{4}$$

be *analytic* and *p-valent* in the punctured disc $\Delta^* = \{z : 0 < |z| < 1\}$.

Let $\sum \mathscr{S}\mathscr{T}_0^*(p, \alpha)$ denote the class of functions $f(z)$ defined by (1) and satisfy the condition

$$-\mathrm{Re}\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \Delta^*) \tag{5}$$

and $\sum \mathscr{C}_0^*(p, \alpha)$ denote the class of functions $f(z)$ defined by (1) and satisfy the condition

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \Delta^*) \tag{6}$$

where $0 \leq \alpha < p$.

The quasi-Hadamard product of two or more functions has recently been defined and used by Kumar ([7],[8], and [9]), Aouf et al. [3], Hossen [6], Darwish [4] and Sekine [12]. Accordingly, the quasi-Hadamard product of two functions $\varphi(z)$ and $\psi(z)$ is defined by

$$(\varphi * \psi)(z) = a_p b_p z^p - \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} \tag{7}$$

Aouf [1] defined the Hadamard product of two meromorphic p-valent functions $f(z)$ and $g(z)$ by

$$(f * g)(z) = \frac{a_{p-1} b_{p-1}}{z^p} + \sum_{n=1}^{\infty} a_{n+p-1} b_{n+p-1} z^{n+p-1} \tag{8}$$

Similarly, we can define the Hadamard product of more than two meromorphic $p-valent$ functions.

Let $\lambda(z)$ be a fixed function of the form

$$\lambda(z) = \frac{c_{p-1}}{z^p} + \sum_{n=1}^{\infty} c_{n+p-1} z^{n+p-1} \qquad \left(c_p > 0; c_{p+n} \geq 0\right), \tag{9}$$

Using the function defined by (9), we now define the following new classes

Definition 1. *A function* $f(z) \in \sum \mathscr{M}_{\lambda}^{0}(c_{n+p-1}, \delta)$ $(c_{n+p-1} \geq c_p > 0; n \geq 2)$ *if and only if*

$$\sum_{n=1}^{\infty} c_{n+p-1} a_{n+p-1} \leq \delta a_{p-1} \tag{10}$$

where $\delta > 0$.

Definition 2. *A function* $f(z) \in \sum \mathscr{B}_{\lambda}^{k}(c_{n+p-1}, \delta)$ $(c_{n+p-1} \geq c_p > 0; n \geq 2)$ *if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n+p-1}{p}\right)^{k} c_{n+p-1} a_{n+p-1} \leq \delta a_{p-1} \tag{11}$$

where $\delta > 0$.

It is easy to check that various subclasses of meromorphic and multivalent functions can be (studied by various authors) represented as $\sum \mathscr{B}_{\lambda}^{k}(c_{n+p-1}, \delta)$ for suitable choices of c_n, δ and k. For example:

(1) $\sum \mathscr{B}_{\lambda}^{k}((n+2p-1)+\beta(n+2\alpha-1), 2\beta(p-\alpha)) \equiv \sum_{k}^{*}(p, \alpha, \beta)$

(2) $\sum \mathscr{B}_{\lambda}^{0}((n+2p-1)+\beta(n+2\alpha-1), 2\beta(p-\alpha)) \equiv \sum S_{0}^{*}(p, \alpha, \beta)$

(3) $\sum \mathscr{B}_{\lambda}^{1}((n+2p-1)+\beta(n+2\alpha-1), 2\beta(p-\alpha)) \equiv \sum C_{0}^{*}(p, \alpha, \beta)$

(4) $\sum \mathscr{B}_{\lambda}^{k}((n(1+\beta)+(2\alpha-1)\beta+1,\ 2\beta(1-\alpha)) \equiv \sum S_{0}^{*}(k, \alpha, \beta)$ for p = 1

(5) $\sum \mathscr{B}_{\lambda}^{k}(n(n(1+\beta)+(2\alpha-1)\beta+1),\ 2\beta(1-\alpha)) \equiv \sum C_{0}^{*}(k, \alpha, \beta)$ for p = 1

The classes $\sum_{k}^{*}(p, \alpha, \beta)$, $\sum S_{0}^{*}(p, \alpha, \beta)$ and $\sum C_{0}^{*}(p, \alpha, \beta)$ have been studied by Aouf [1] and the classes $\sum S_{0}^{*}(k, \alpha, \beta)$ and $\sum C_{0}^{*}(k, \alpha, \beta)$ have been studied by El-Ashwah and Aouf [5].

Evidently, $\sum \mathscr{B}_{\lambda}^{0}(c_{n+p-1}, \delta) \equiv \sum \mathscr{M}_{\lambda}^{0}(c_{n+p-1}, \delta)$. Further, $\sum \mathscr{B}_{\lambda}^{k}(c_{n+p-1}, \delta) \subset \sum \mathscr{B}_{\lambda}^{h}(c_{n+p-1}, \delta)$ if $k > h \geq 0$, the containment being proper. Moreover, for any positive integer k we have the following inclusion relation

$$\sum \mathscr{B}_{\lambda}^{k}(c_{n+p-1}, \delta) \subset \sum \mathscr{B}_{\lambda}^{k-1}(c_{n+p-1}, \delta) \subset \ldots \subset \sum \mathscr{M}_{\lambda}^{0}(c_{n+p-1}, \delta) \subset \sum \mathscr{C}_{0}^{*}(p, \alpha) \subset \sum \mathscr{S}_{0}^{*}(p, \alpha).$$

We also note that for every nonnegative real number k, the class $\sum \mathscr{B}_{\lambda}^{k}(c_{n+p-1}, \delta)$ is nonempty as the functions of the form

$$f(z) = \frac{a_{p-1}}{z^p} + \sum_{n=1}^{\infty} \left(\frac{p}{n+p-1}\right)^{k} \frac{\delta a_{p+n-1}}{c_{p+n-1}} \mu_{n+p-1} z^{n+p-1} \qquad \left(a_p > 0; a_{p+n} \geq 0\right),$$

where $a_{p-1} > 0$, $\mu_{n+p-1} \geq 0$ and $\sum_{n=1}^{\infty} \mu_{n+p-1} \leq 1$, satisfy the inequality (11).

In this paper we establish a theorem concerning the quasi-Hadamard product of functions in the classes $\sum \mathscr{M}_{\lambda}^{0}(c_{n+p-1}, \delta)$ and $\sum \mathscr{B}_{\lambda}^{k}(c_{n+p-1}, \delta)$. The theorem and its applications generalize the results obtained by Aouf [1], Mogra [11] and El-Ashwah and Aouf [5].

2. Main Theorem

Theorem 1. *Let the functions $f_i(z)$ defined by (2) belong to the class $\sum \mathscr{B}_\lambda^k(c_{n+p-1},\delta)$ for every $i=1,2,\dots,m$, and let the functions $g_j(z)$ defined by (4) belong to the class $\sum \mathscr{M}_\lambda^0(c_{n+p-1},\delta)$ for every $j=1,2,\dots,q$. If $c_{n+p-1}\geq \left(\frac{n+p-1}{p}\right)\delta$, then the quasi-Hadamard product $f_1 * f_2 * \dots * f_m * g_1 * g_2 * \dots * g_q(z)$ belongs to the class $\sum \mathscr{B}_\lambda^{(k+1)m+q-1}(c_{n+p-1},\delta)$.*

Proof. Let $h(z) := f_1 * f_2 * \dots * f_m * g_1 * g_2 * \dots * g_q(z)$, then

$$h(z)=\frac{\left\{\prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q} b_{p-1,j}\right\}}{z^p}+\sum_{n=1}^{\infty}\left\{\prod_{i=1}^{m} a_{n+p-1,i}\prod_{j=1}^{q} b_{n+p-1,j}\right\}z^{n+p-1}. \tag{12}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{n+p-1}{p}\right)^{m(k+1)+q-1}\prod_{i=1}^{m} a_{n+p-1,i}\prod_{j=1}^{q} b_{n+p-1,j}\leq \delta\prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q} b_{p-1,j} \tag{13}$$

Since $f_i(z)\in \sum \mathscr{B}_\lambda^k(c_{n+p-1},\delta)$, we have

$$\sum_{n=1}^{\infty}\left(\frac{n+p-1}{p}\right)^{k} c_{n+p-1}a_{n+p-1,i}\leq \delta a_{p-1,i} \tag{14}$$

for every $i=1,2,\dots,m$. Therefore,

$$a_{n+p-1,i}\leq \left(\frac{n+p-1}{p}\right)^{-k}\left(\frac{\delta}{c_{n+p-1}}\right)a_{p-1,i} \tag{15}$$

which by virtue of the condition (given with the theorem) implies that

$$a_{n+p-1,i}\leq \left(\frac{n+p-1}{p}\right)^{-k-1} a_{p-1,i} \tag{16}$$

for every $i=1,2,\dots,m$. Further, since $g_j(z)\in \sum \mathscr{M}_\lambda^0(c_{n+p-1},\delta)$, we have

$$\sum_{n=1}^{\infty} c_{n+p-1}b_{n+p-1,j}\leq \delta b_{p-1,j} \tag{17}$$

for every $j=1,2,\dots,q$. Hence we obtain

$$b_{n+p-1,j}\leq \left(\frac{n+p-1}{p}\right)^{-1} b_{p-1,j} \tag{18}$$

Using (16) for $i = 1, 2, \ldots, m$, and (18) for $j = 1, 2, \ldots, q-1$, and (17) for $j = q$, we get

$$\sum_{n=1}^{\infty}\left[\left(\frac{n+p-1}{p}\right)^{m(k+1)+q-1} c_{n+p-1}\left\{\prod_{i=1}^{m} a_{n+p-1,i}\prod_{j=1}^{q} b_{n+p-1,j}\right\}\right]$$

$$\leq \sum_{n=1}^{\infty}\left[\left(\frac{n+p-1}{p}\right)^{m(k+1)+q-1}\left(\frac{n+p-1}{p}\right)^{-m(k+1)}\left(\frac{n+p-1}{p}\right)^{-(q-1)}\right.$$

$$\left.\left\{\prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q-1} b_{n+p-1,j}\right\} c_{n+p-1} b_{n+p-1,q}\right]$$

$$= \left(\prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q-1} b_{p-1,j}\right)\left(\sum_{n=1}^{\infty} c_{n+p-1} b_{n+p-1,q}\right)$$

$$\leq \delta \prod_{i=1}^{m} a_{p-1,i}\prod_{j=1}^{q} b_{p-1,j} \text{ (by (17))}$$

Hence $h(z) \in \sum \mathscr{B}_{\lambda}^{(k+1)m+q-1}(c_{n+p-1}, \delta)$. This completes the proof of the Theorem 1.

Taking $k = 0$ in the proof of the above theorem, we obtain

Corollary 1. *Let the functions $f_i(z)$ defined by (2) and the functions $g_j(z)$ defined by (4) belong to the class $\sum \mathscr{M}_{\lambda}^{0}(c_{n+p-1}, \delta)$ for every $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, q$. If $c_{n+p-1} \geq \left(\frac{n+p-1}{p}\right)\delta$, then the quasi-Hadamard product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_q(z)$ belongs to the class $\sum \mathscr{B}_{\lambda}^{m+q-1}(c_{n+p-1}, \delta)$.*

Now taking into account the quasi-Hadamard product functions $g_1(z) * g_2(z) * \ldots * g_q(z)$ only, in the proof of the above theorem, and using (18) for $j = 1, 2, 3 \ldots, q-1$, and (17) for $j = m$, we obtain

Corollary 2. *Let the functions $g_j(z)$ defined by (4) belong to the class $\sum \mathscr{M}_{\lambda}^{0}(c_{n+p-1}, \delta)$ for $j = 1, 2, \ldots, q$. If $c_{n+p-1} \geq \left(\frac{n+p-1}{p}\right)\delta$, then Hadamard product $g_1 * g_2 * \ldots * g_q(z)$ belongs to the class $\sum \mathscr{B}_{\lambda}^{q-1}(c_{n+p-1}, \delta)$.*

Remark 1.

(i) Putting $c_{n+p-1} = (n+2p-1) + \beta(n+2\alpha-1)$ and $\delta = 2\beta(p-\alpha)$ in the above theorem, we obtain the results obtained by Aouf [1].

(ii) Putting $p = 1$, $c_n = n((n+1) + \beta(n+2\alpha-1))$ and $\delta = 2\beta(1-\alpha)$ in the above theorem, we obtain the results obtained by Mogra [11].

(iii) Putting $p = 1$, in Corollary 2, we obtain the results obtained by El-Ashwah and Aouf [5].

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