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Generalized Closed Sets with Respect to an Ideal

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Abstract. An ideal on a set X is a non empty collection of subsets of X with heredity property which is also closed under finite unions. The concept of generalized closed sets was introduced by Levine. In this paper, we introduce and investigate the concept of generalized closed sets with respect to an ideal.

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1. Introduction

Indeed ideals are very important tools in General Topology. It was the works of Newcomb [8], Rancin [9], Samuels [10] and Hamlet and Jankovic (see [1, 2, 3, 4, 5]) which motivated the research in applying topological ideals to generalize the most basic properties in General Topology. A nonempty collection I of subsets on a topological space (X, τ) is called a topological ideal [6] if it satisfies the following two conditions:

1. If A ∈ I and B ⊂ A implies B ∈ I (heredity)

2. If A ∈ I and B ∈ I, then A ∪ B ∈ I (finite additivity)

If A is a subset of a topological space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A, respectively. Let A⊂B⊂X. Then cl_B(A) (resp. int_B(A)) denotes closure of A (resp. interior of A) with respect to B. In 1963, Levine [7] introduced the concept of generalized closed sets. This notion has been studied extensively in recent years by many topologists. A subset A of a topological space (X, τ) is said to be generalized closed (briefly g-closed) if cl(A) ⊂ U whenever A ⊂ U and U is open in (X, τ).

In this paper, we introduce and study the concept of g-closed sets with respect to an ideal, which is the extension of the concept of g-closed sets.

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2. Generalized Closed Sets with Respect to an Ideal

Definition 1. *Let (X,τ) be a topological space and I be an ideal on X. A subset A of X is said to be generalized closed with respect to an ideal (briefly Ig-closed) if and only if cl(A) – B $\in$ I, whenever $A \subset B$ and B is open.*

Remark 1. *Every g-closed set is Ig-closed, but the converse need not be true, as this may be seen from the following example.*

Example 1. *Let X=$\{a,b,c\}$ with topology τ=$\{\varnothing$, $\{a\}$, $\{a,c\}$, X$\}$ and I=$\{\emptyset,\{b\},\{c\},\{b,c\}\}$. Clearly, the set $\{c\}$ is Ig-closed but not g-closed in (X,τ).*

The following theorem gives a characterization of Ig-closed sets.

Theorem 1. *A set A is Ig-closed in (X,τ) if and only if F $\subset$ cl(A) – A and F is closed in X implies F $\in$ I.*

Proof. Assume that A is Ig-closed. Let F $\subset$ cl(A) – A. Suppose F is closed. Then A $\subset$ X – F. By our assumption, cl(A) – (X – F) $\in$ I. But F $\subset$ cl(A) – $(X-F)$ and hence F $\in$ I.

Conversely, assume that F $\subset$ cl(A) – A and F is closed in X implies that F $\in$ I. Suppose A $\subset$ U and U is open. Then cl(A) – U = cl(A) $\cap$ (X–U) is a closed set in X, that is contained in cl(A) – A. By assumption, cl(A) – U $\in$ I. This implies that A is Ig-closed.

Theorem 2. *If A and B are Ig-closed sets of (X,τ), then their union $A \cup B$ is also Ig-closed.*

Proof. Suppose A and B are Ig-closed sets in (X,τ). If A $\cup$ B $\subset$ U and U is open, then A $\subset$ U and B $\subset$ U. By assumption, cl(A) – U $\in$ I and cl(B) – U $\in$I and hence cl(A $\cup$ B) – U = (cl(A)–U) $\cup$ (cl(B)–U) $\in$ I. That is A $\cup$ B is Ig-closed.

Remark 2. *The intersection of two Ig-closed sets need not be an Ig-closed as shown by the following example.*

Example 2. *Let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing$, $\{b\}$, $X\}$. If $A=\{a,b\}$, $B=\{b,c\}$ and $I=\{\varnothing\}$, then A and B are Ig-closed but their intersection $A\cap B=\{b\}$ is not Ig-closed.*

Theorem 3. *If A is Ig-closed and $A \subset B \subset$ cl(A) in (X,τ), then B is Ig-closed in (X,τ).*

Proof. Suppose A is Ig-closed and A $\subset$ B $\subset$ cl(A) in (X,τ). Suppose B$\subset$ U and U is open. Then A $\subset$ U. Since A is Ig-closed, we have cl(A)–U $\in$ I. Now B $\subset$ cl(A). This implies that cl(B)–U $\subset$ cl(A)–U $\in$ I. Hence B is Ig-closed in (X,τ).

Theorem 4. *Let $A \subset Y \subset X$ and suppose that A is Ig-closed in (X,τ). Then A is Ig-closed relative to the subspace Y of X, with respect to the ideal $I_Y=\{F \subset Y\colon F \in I\}$.*

Proof. Suppose A $\subset$ U $\cap$ Y and U is open in (X,τ), then A $\subset$U. Since A is Ig-closed in (X,τ), we have cl(A)–U $\in$ I. Now (cl(A)$\cap$Y) – (U$\cap$Y) = (cl(A)–U)$\cap$Y $\in$ I, whenever A $\subset$ U $\cap$ Y and U is open. Hence A is Ig-closed relative to the subspace Y.

Theorem 5. *Let A be an Ig-closed set and F be a closed set in (X,τ), then $A \cap F$ is an Ig-closed set in (X,τ)*

Proof. Let A ∩ F ⊂ U and U is open. Then A ⊂ U ∪ (X–F). Since A is Ig-closed, we have cl(A)–(U ∪ (X–F)) ∈ I. Now, cl(A ∩ F) ⊂ cl(A) ∩ F = (cl(A)∩F) – (X–F). Therefore,

$$\begin{aligned} cl(A\cap F)-U &\subset (cl(A)\cap F)-(U\cap(X-F)) \\ &\subset cl(A)-(U\cup(X-F)) \\ &\in I \end{aligned}$$

Hence A ∩ F is Ig-closed in (X,τ).

Definition 2. *Let (X,τ) be a topological space and A be an ideal on X. A subset A ⊂ X is said to be generalized open with respect to an ideal (briefly Ig-open) if and only if X–A is Ig-closed.*

Theorem 6. *A set A is Ig-open in (X,τ) if and only if F–U ⊂ int(A), for some U ∈ I, whenever F ⊂ A and F is closed.*

Proof. Suppose A is Ig-open. Suppose F ⊂ A and F is closed. We have X – A ⊂ X – F. By assumption, cl(X–A) ⊂ (X–F) ∪ U, for some U ∈ I. This implies X–((X–F)∪U) ⊂ X–(cl(X–A)) and hence F – U ⊂ int(A).

Conversely, assume that F⊂A and F is closed imply F–U⊂int(A), for some U∈I. Consider an open set G such that X–A⊂G. Then X–G⊂A. By assumption,
(X–G)–U⊂int(A) = X–cl(X–A). This gives that X–(G∪U)⊂X–cl(X–A). Then, cl(X–A)⊂G∪U, for some U∈I. This shows that cl(X–A)–G∈I. Hence X–A is Ig-closed.

Recall that the sets A and B are said to be separated if cl(A)∩B = ∅ and A ∩ cl(B) = ∅.

Theorem 7. *If A and B are separated Ig-open sets in (X,τ), then A ∪ B is Ig-open.*

Proof. Suppose A and B are separated Ig-open sets in (X,τ) and F be a closed subset of A∪B. Then F∩cl(A)⊂A and F∩cl(B)⊂B. By assumption, (F∩cl(A))–U_1 ⊂ int(A) and (F∩cl(B))–U_2 ⊂ int(B), for some U_1, U_2 ∈ I. This mean that ((F∩cl(A))–int(A)) ∈ I and (F∩cl(B))–int(B)∈I. Then ((F∩cl(A))–int(A))∪((F∩cl(B))–int(B))∈I. Hence (F∩(cl(A)∪cl(B))–(int(A)∪int(B))) ∈ I. But F=F∩(A∪B) ⊂ F∩cl(A∪B), and we have

$$\begin{aligned} F-int(A\cup B) &\subset (F\cap cl(A\cup B))-int(A\cup B) \\ &\subset (F\cap cl(A\cup B))-(int(A)\cup int(B))\in I \end{aligned}$$

Hence, F–U⊂int(A∪B), for some U∈I. This proves that A∪B is Ig-open.

Corollary 1. *Let A and B are Ig-closed sets and suppose X–A and X–B are separated in (X,τ). Then A∩B is Ig-closed.*

Corollary 2. *If A and B are Ig-open sets in (X,τ), then A∩B is Ig-open.*

Proof. If A and B are Ig-open, then X–A and X–B are Ig-closed. By Theorem 2, X–(A∩B) is Ig-closed, which implies A∩B is Ig-open.

Theorem 8. *If A⊂B⊂X, A is Ig-open relative to B and B is Ig-open relative to X, then A is Ig-open relative to X.*

Proof. Suppose A⊂B⊂X, A is Ig-open relative to B and B is Ig-open relative to X. Suppose F⊂A and F is closed. Since A is Ig-open relative to B, by theorem 6, F–U_1 ⊂ int_B(A), for some U_1 ∈ I. This implies there exists an open set G_1 such that F–U_1 ⊂ G_1 ∩ B⊂A, for some U_1 ∈ I. Since B is Ig-open, F ⊂ B and F is closed; we have F–U_2 ⊂ int(B), for some U_2 ∈ I. This implies there exists an open set G_2 such that F–U_2 ⊂ G_2 ⊂ B, for some U_2 ∈ I. Now
F – (U_1 ∪ U_2) ⊂ (F–U_1) ∩ (F–U_2) ⊂ G_1 ∩ G_2 ⊂ G_1 ∩ B ⊂ A. This implies that
F – (U_1 ∪ U_2) ⊂ int(A), for some U_1 ∪ U_2 ∈ I and hence A is Ig-open relative to X.

Theorem 9. *If int(A) ⊂ B ⊂ A and if A is Ig-open in (X,τ), then B is Ig-open in X.*

Proof. Suppose int(A) ⊂ B ⊂ A and A is Ig-open. Then X–A ⊂ X–B ⊂ cl(X–A) and X–A is Ig-closed. By Theorem 3, X–B is Ig-closed and hence B is Ig-open.

Theorem 10. *A set A is Ig-closed in (X,τ) if and only if cl(A)–A is Ig-open.*

Proof.
Necessity: Suppose F ⊂ cl(A)–A and F be closed. Then F ∈ I. This implies that F–U = ∅, for some U ∈ I. Clearly, F–U ⊂ int(cl(A)–A). By Theorem 6 cl(A)–A is Ig-open.
Sufficiency: Suppose A ⊂ G and G is open in (X,τ) . Then
cl(A)∩(X–G) ⊂ cl(A) ∩ (X–A) = cl(A)–A. By hypothesis,
(cl(A)∩(X–G)) – U ⊂ int(cl(A)–A) = ∅, for some U ∈ I. This implies that
cl(A) ∩ (X–G) ⊂ U ∈ I and hence cl(A)–G ∈I. Thus, A is Ig-closed.

Theorem 11. *Let f: $(X,\tau) \rightarrow (Y,\sigma)$ be continuous and closed. If A ⊂ X is Ig-closed in X, then f(A) is f(I)-g-closed in (Y,σ), where f(I) = {f(U): U ∈ I}.*

Proof. Suppose A ⊂ X and A is Ig-closed. Suppose f(A) ⊂ G and G is open. Then
A ⊂ f^{-1}(G). By definition, cl(A) – f^{-1}(G) ∈I and hence f(cl(A)) – G ∈ f(I). Since f is closed, cl(f(A)) ⊂ cl(f(cl(A))) = f(cl(A)). Then cl(f(A)) – G ⊂ f(cl(A)) – G ∈ f(I) and hence f(A) is f(I)-g-closed.

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