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Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution

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Abstract. In this article, the two parameter Weibull probability distribution is embedded in a larger family obtained by introducing an additional parameter. We generalize the two parameter Weibull distribution using the quadratic rank transmutation map studied by Shaw *et al.* [9] to develop a transmuted Weibull distribution. We provide a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. The usefulness of the transmuted Weibull distribution for modeling reliability data is illustrated using real data.

2000 Mathematics Subject Classifications: 62N05; 90B25

Key Words and Phrases: Weibull distribution, hazard rate function, reliability function, parameter estimation

1. Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with revelent statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

The Weibull distribution is a very popular distribution named after Waladdi Weibull, a Swedish physicist. He applied this distribution in 1939 to analyze the breaking strength of materials. Since then, it has been widely used for analyzing lifetime data in reliability engineering. It is a versatile distribution that can take on the characteristics of other types of distributions, based on the value of the shape parameter. The Weibull distribution is a widely used statistical model for studying fatigue and endurance life in engineering devices and materials.

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Many examples can be found among the aerospace, electronics, materials, and automotive industries. Recent advances in Weibull theory have also created numerous specialized Weibull applications. Modern computing technology has made many of these techniques accessible across the engineering spectrum. Despite its popularity, and wide applicability the traditional 2-parameter and 3-parameter Weibull distribution is unable to capture all the lifetime phenomenon for instance the data set which has a non-monotonic failure rate function. Recently several generalization of Weibull distribution has been studied. An approach to the construction of flexible parametric models is to embed appropriate competing models into a larger model by adding shape parameter. Some recent generalizations of Weibull distribution including the exponentiated Weibull, extended Weibull, modified Weibull are discussed in Pham *et al.* [7] and references therein, along with their reliability functions. In this article we present a new generalization of Weibull distribution called the transmuted Weibull distribution. We will derive the subject distribution using the quadratic rank transmutation map studied by Shaw *et al.* [9].

A random variable X is said to have transmuted distribution if its cumulative distribution function(cdf) is given by

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \qquad |\lambda| \leq 1 \tag{1}$$

where $G(x)$ is the cdf of the base distribution. Observe that at $\lambda = 0$ we have the distribution of the base random variable. Aryal *et al.* [1] studied the the transmuted Gumbel distribution and it has been observed that transmuted Gumbel distribution can be used to model climate data. In the present study we will provide mathematical formulation of the transmuted Weibull distribution and some of its properties. We will also provide possible area of applications.

2. Transmuted Weibull Distribution

A random variable X is said to have a Weibull distribution with parameters $\eta > 0$ and $\sigma > 0$ if its probability density function (pdf) is given by

$$g(x) = \frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right) \qquad x > 0 \tag{2}$$

The cdf of X is given by

$$G(x) = 1 - \exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right). \tag{3}$$

Now using (1) and (3) we have the cdf of a transmuted Weibull distribution

$$F(x) = \left[1 - \exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right]\left[1 + \lambda\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right]. \tag{4}$$

Hence, the pdf of transmuted Weibull distribution with parameters η, σ and λ is

$$f(x) = \frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\left[1 - \lambda + 2\lambda\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right]. \tag{5}$$

Note that the transmuted Weibull distribution is an extended model to analyze more complex data and it generalizes some of the widely used distributions. In particular for $\eta = 1$ we have the transmuted exponential distribution as discussed in Shaw *et al.*[9]. The Weibull distribution is clearly a special case for $\lambda = 0$. When $\eta = \lambda = 1$ then the resulting distribution is an exponential distribution with parameter $\frac{\sigma}{2}$. Figure 1 illustrates some of the possible shapes of the pdf of a transmuted Weibull distribution for selected values of the parameters λ and η and for $\sigma = 1$.

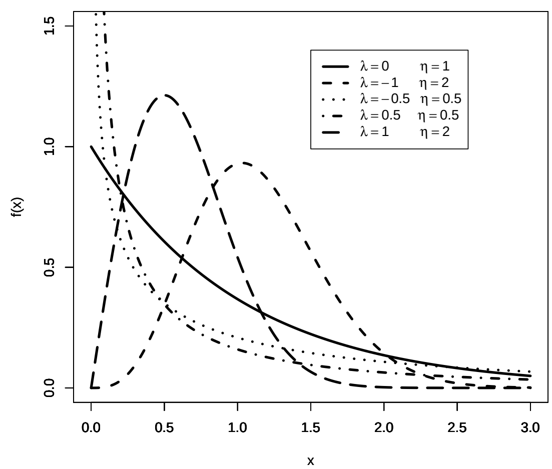

Figure 1: pdf of transmuted Weibull distribution for $\sigma = 1$ and different values of λ and η

3. Moments and Quantiles

In this section we shall present the moments and qunatiles for the transmuted Weibull distribution. The k^{th} order moments of a transmuted Weibull random variable X, in terms of gamma function $\Gamma(.)$, is given by

$$E(X^k) = \sigma^k \Gamma\left(1 + \frac{k}{\eta}\right)\left\{1 - \lambda + \lambda 2^{-\frac{k}{\eta}}\right\} \tag{6}$$

Moreover, if $k/\eta = r$ is a positive integer, then

$$E(X^k) = \sigma^k r! \left[1 - \lambda + \lambda 2^{-r}\right]$$

Therefore, the expected value $E(X)$ and variance $Var(X)$ of a transmuted Weibull random variable X are, respectively, given by

$$E(X) = \sigma\Gamma\left(1 + \frac{1}{\eta}\right)\left(1 - \lambda + \lambda 2^{-\frac{1}{\eta}}\right),$$

$$Var(X) = \sigma^2\left\{\Gamma\left(1+\frac{2}{\eta}\right)\left[1-\lambda+\lambda 2^{-\frac{2}{\eta}}\right]-\Gamma^2\left(1+\frac{1}{\eta}\right)\left(1-\lambda+\lambda 2^{-\frac{1}{\eta}}\right)^2\right\}.$$

Note that when $\eta = k$,

$$E(X^k) = \sigma^k\left[\frac{2-\lambda}{2}\right].$$

The q^{th} quantile x_q of the transmuted Weibull distribution can be obtained from (4) as

$$x_q = \sigma\left[-\ln\left\{1-\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right)\right\}\right]^{1/\eta} \tag{7}$$

In particular, the distribution median is

$$x_{0.5} = \sigma\left[-\ln\left(\frac{\lambda-1+\sqrt{1+\lambda^2}}{2\lambda}\right)\right]^{1/\eta}.$$

3.1. Random Number Generation and Parameter Estimation

Using the method of inversion we can generate random numbers from the transmuted Weibull distribution as

$$1-\exp\left(-\left(\frac{x}{\sigma}\right)^\eta\right)\left[(1-\lambda)+\lambda\exp\left(-\left(\frac{x}{\sigma}\right)^\eta\right)\right] = u$$

where $u \sim U(0,1)$. After simple calculation this yields

$$x = \sigma\left[-\ln\left\{1-\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}\right)\right\}\right]^{1/\eta}. \tag{8}$$

One can use equation (8) to generate random numbers when the parameters η, σ and λ are known. The maximum likelihood estimates, MLEs, of the parameters that are inherent within the transmuted Weibull probability distribution function is given by the following: Let $X_1, X_2, \cdots, X_n$ be a sample of size n from a transmuted Weibull distribution. Then the likelihood function is given by

$$L = \left(\frac{\eta}{\sigma}\right)^n \exp\left[-\sum_{i=1}^n\left(\frac{x_i}{\sigma}\right)^\eta\right]\prod_{i=1}^n\left\{\left(\frac{x_i}{\sigma}\right)^{\eta-1}\times\left[1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)^\eta\right]\right\}.$$

Hence, the log-likelihood function $\mathscr{L} = \ln L$ becomes

$$\mathscr{L} = n\ln\frac{\eta}{\sigma}-\sum_{i=1}^n\left(\frac{x_i}{\sigma}\right)^\eta+\sum_{i=1}^n\ln\left(\frac{x_i}{\sigma}\right)^{\eta-1}+\sum_{i=1}^n\ln\left[1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)^\eta\right]$$

$$= \quad n\ln\eta - n\eta\ln\sigma + (\eta-1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\eta} + \sum_{i=1}^{n}\ln\left[1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)^{\eta}\right] \tag{9}$$

Therefore, the MLEs of η, σ and λ which maximize (9) must satisfy the following normal equations

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \eta} &= \frac{n}{\eta} + \sum_{i=1}^{n}\left[1-\left(\frac{x_i}{\sigma}\right)^{\eta}\right]\ln\left(\frac{x_i}{\sigma}\right) - 2\lambda\sum_{i=1}^{n}\frac{\ln(x_i/\sigma)(x_i/\sigma)^{\eta}\exp(-(x_i/\sigma)^{\eta})}{[1-\lambda+2\lambda\exp(-(x_i/\sigma)^{\eta})]} = 0,\\
\frac{\partial \mathscr{L}}{\partial \sigma} &= -\frac{\eta}{\sigma}\sum_{i=1}^{n}\left[1-\left(\frac{x_i}{\sigma}\right)^{\eta}\right] + \frac{2\lambda\eta}{\sigma}\sum_{i=1}^{n}\frac{(x_i/\sigma)^{\eta}\exp(-(x_i/\sigma)^{\eta})}{[1-\lambda+2\lambda\exp(-(x_i/\sigma)^{\eta})]} = 0,\\
\frac{\partial \mathscr{L}}{\partial \lambda} &= \sum_{i=1}^{n}\frac{2\exp(-(x_i/\sigma)^{\eta})-1}{[1-\lambda+2\lambda\exp(-(x_i/\sigma)^{\eta})]} = 0.
\end{aligned}$$

The maximum likelihood estimator $\hat{\boldsymbol{\theta}} = \left(\hat{\eta}, \hat{\sigma}, \hat{\lambda}\right)'$ of $\boldsymbol{\theta} = (\eta, \sigma, \lambda)'$ is obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function given in (9). In order to compute the standard error and asymptotic confidence interval we use the usual large sample approximation in which the maximum likelihood estimators of $\boldsymbol{\theta}$ can be treated as being approximately trivariate normal. Hence as $n \to \infty$, the asymptotic distribution of the MLE $\left(\hat{\eta}, \hat{\sigma}, \hat{\lambda}\right)$ is given by, see Zaindin *et al.* [10],

$$\begin{pmatrix}\hat{\eta}\\ \hat{\sigma}\\ \hat{\lambda}\end{pmatrix} \sim N\left[\begin{pmatrix}\eta\\ \sigma\\ \lambda\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33}\end{pmatrix}\right] \tag{10}$$

where, $\hat{V}_{ij} = V_{ij}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$ and

$$\begin{pmatrix}V_{11} & V_{12} & V_{13}\\ V_{21} & V_{22} & V_{23}\\ V_{31} & V_{32} & V_{33}\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}^{-1}$$

is the approximate variance covariance matrix with its elements obtained from

$$\begin{aligned}
A_{11} &= -\frac{\partial^2 \mathscr{L}}{\partial \eta^2}, & A_{12} &= -\frac{\partial^2 \mathscr{L}}{\partial \eta \partial \sigma},\\
A_{22} &= -\frac{\partial^2 \mathscr{L}}{\partial \sigma^2}, & A_{23} &= -\frac{\partial^2 \mathscr{L}}{\partial \sigma \partial \lambda},\\
A_{33} &= -\frac{\partial^2 \mathscr{L}}{\partial \lambda^2}, & A_{13} &= -\frac{\partial^2 \mathscr{L}}{\partial \eta \partial \lambda}.
\end{aligned}$$

where $\mathscr{L}$ is the log-likelihood function given in (9). An Approximate $100(1-\alpha)\%$ two sided confidence intervals for η, σ and λ are, respectively, given by

$$\hat{\eta} \pm z_{\alpha/2}\sqrt{\hat{V}_{11}}, \quad \hat{\sigma} \pm z_{\alpha/2}\sqrt{\hat{V}_{22}} \text{ and } \hat{\lambda} \pm z_{\alpha/2}\sqrt{\hat{V}_{33}}.$$

where z_α is the upper $\alpha - th$ percentiles of the standard normal distribution. For details see [3, 10] etc.. Using R [2] we can easily compute the Hessian matrix and its inverse and hence the values of the standard error and asymptotic confidence intervals.

4. Reliability Analysis

The transmuted Weibull distribution can be a useful characterization of failure time of a given system because of the analytical structure. The reliability function $R(t)$, which is the probability of an item not failing prior to some time t, is defined by $R(t) = 1 - F(t)$. The reliability function of a transmuted Weibull distribution is given by

$$R(t) = \exp\left(-\left(\frac{t}{\sigma}\right)^\eta\right)\left[1-\lambda+\lambda\exp\left(-\left(\frac{t}{\sigma}\right)^\eta\right)\right]. \tag{11}$$

The other characteristic of interest of a random variable is the hazard rate function defined by

$$h(t) = \frac{f(t)}{1-F(t)}$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time t. The hazard rate function for a transmuted Weibull random variable is given by

$$h(t) = \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1}\left\{\frac{1-\lambda+2\lambda\exp\left(-\left(\frac{t}{\sigma}\right)^\eta\right)}{1-\lambda+\lambda\exp\left(-\left(\frac{t}{\sigma}\right)^\eta\right)}\right\}. \tag{12}$$

Theorem 1. *The hazard rate function of a transmuted Weibull distribution has the following properties:*

(i) If $\eta = \lambda = 1$, the failure rate is constant.

(ii) If $\lambda = 1$, then the failure rate is increasing for $\eta > 1$ and decreasing for $\eta < 1$

(iii) If $\eta = 1$ then the failure rate is increasing if $\lambda < 0$ and is decreasing if $\lambda > 0$.

(iv) If $\lambda = 0$ and $\eta = 1$ the the failure rate is a constant.

Proof.

(i) If $\lambda = \eta = 1$ then

$$h(t) = \frac{2}{\sigma}.$$

This is a constant.

(ii) If $\lambda = 1$, then

$$h(t) = \frac{2\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1}.$$

which is increasing for $\eta > 1$ and is decreasing for $\eta < 1$.
Note that for $\eta = 2$ we have linear hazard function as in the case of Rayleigh distribution.

(iii) If $\eta = 1$ then we have

$$h(t) = \frac{1}{\sigma} + \frac{\lambda}{\sigma}\frac{\exp(-t/\sigma)}{[1-\lambda+\lambda\exp(-t/\sigma)]}.$$

It can be easily shown that $h(t)$ is increasing for $\lambda < 0$ and is decreasing for $\lambda > 0$.
Note that

$$h(0) = \frac{1+\lambda}{\sigma} \text{ and } h(\infty) = \frac{1}{\sigma}.$$

It is clear that that the hazard rate function increases from 0 to $\frac{1}{\sigma}$ for $\lambda < 0$ and it decreases from $\frac{2}{\sigma}$ to $\frac{1}{\sigma}$ for $\lambda > 0$.

(iv) If $\lambda = 0$ and $\eta = 1$ the the failure rate is a constant as the resulting distribution is exponential distribution.

Figure 2 illustrates the reliability behavior of a transmuted Weibull distribution as the value of the parameter λ varies from -1 to 1. Note that the figure on the left shows how

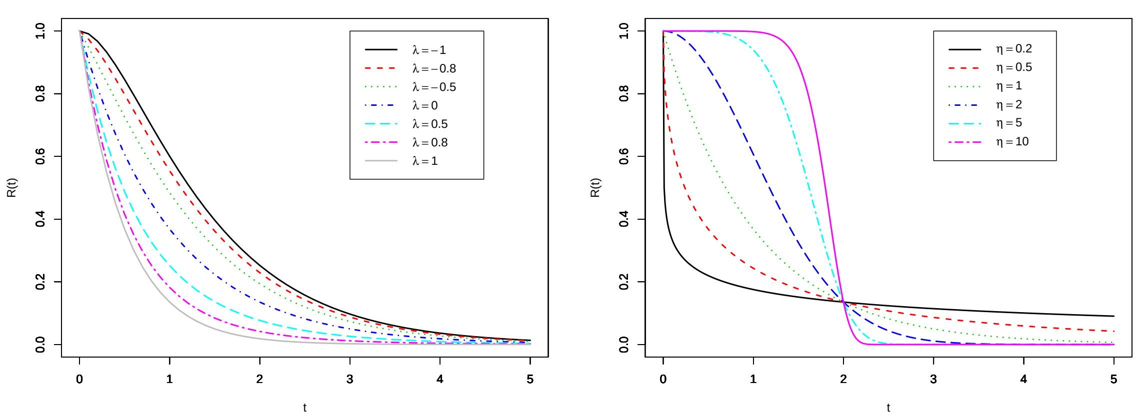

Figure 2: Reliability Function of Transmuted Weibull Distribution

the reliability function changes its shape when we vary the parameter λ keeping $\eta = 1$ and $\sigma = 1$ whereas the figure on the right exhibits the behavior as the value of η changes keeping $\lambda = 1$ and $\sigma = 2$. Also notice that for $\lambda = 1$ and $t = \sigma$ the reliability function takes the value

$\exp(-2)$ which is independent of η.

Figure 3 illustrates the behavior of the hazard rate function of a transmuted Weibull distribution. Note that the figure on the left shows how the hazard rate function changes its shape when we vary the parameter λ keeping $\eta = 1$ and $\sigma = 1$ whereas the figure on the right exhibits the behavior as the value of η changes keeping $\lambda = 1$ and $\sigma = 1$. Observing the be-

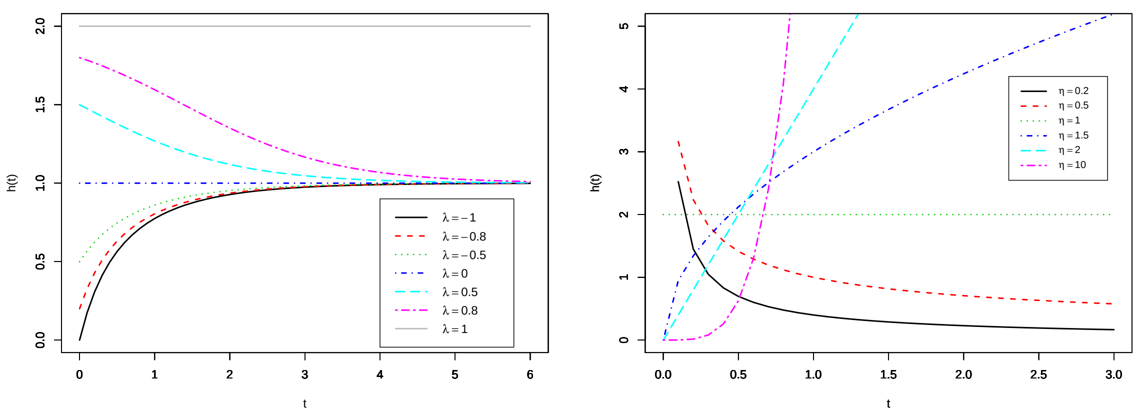

Figure 3: Reliability Function of Transmuted Weibull Distribution

havior of the hazard rate function it is worth noting that the transmuted Weibull distribution probably will have more applicability than other generalization of the Weibull distribution.

Many generalized Weibull models have been proposed in reliability literature through the fundamental relationship between the reliability function $R(t)$ and its cumulative hazard rate function $H(t)$ given by $H(t) = -\ln R(t)$. The cumulative hazard rate function of a transmuted Weibull random variable is given by

$$H(t) = \int_0^t h(x)dx = \left(\frac{t}{\sigma}\right)^\eta - \ln\left[1 - \lambda + \lambda \exp\left(-\left(\frac{t}{\sigma}\right)^\eta\right)\right].$$

Observe that:

(i) $H(t)$ is nondecreasing for all $t \geq 0$,

(ii) $H(0) = 0$,

(iii) $\lim_{t\to\infty} H(t) = \infty$.

Given that a unit is of age t, the remaining time after time t is random. The expected value of this random residual life is called the mean residual life(MRL) at time t. The mean residual life (MRL) at a given time t measures the expected remaining life time of an individual of age t. It is given by

$$m(t) = E(T - t | T \geq t)$$

$$= \frac{1}{R(t)}\int_t^\infty R(u)du$$

Note that $m(0)$ is the mean time to failure. The MRL can be expressed in terms of the cumulative hazard rate function as

$$m(t) = \int_0^\infty \exp\left[H(t) - H(t+x)\right]dx.$$

The mean residual life can also be related to the failure rate $h(t)$ of the random variable through $m^{'}(t) = m(t)h(t) - 1$.
The MRL function as well as the hazard rate or failure rate(FR) function is very important as each of them can be used to characterize a unique corresponding life time distribution. Life times can exhibit IMRL (increasing MRL) or DMRL (decreasing MRL). MRL functions that first decreases (increases) and then increases (decreases) are usually called bathtub (upside-down bathtub) shaped, BMRL (UMRL). The relationship between the behaviors of the two functions of a distribution was studied by many authors. The following theorem in [4] summarizes the results of the studies.

For a nonnegative random variable T with pdf $f(t)$, finite mean μ, and differentiable FR function $h(t)$, the MRL function is

(i) Constant $= \mu$ if T has an exponential distribution.

(ii) DMRL (IMRL) if $h(t)$) is IFR (DFR).

(iii) UMRL (BMRL) with a unique change point t_m if $h(t)$ is BFR (UFR) with a unique change point t_r, $0 < t_m < t_r < \infty$ and $f(0)\mu > 1(< 1)$.

In industrial reliability studies of repair, replacement and other maintenance strategies, the mean residual life function may be proven to be more relevant than the failure rate function. If the goal is to improve the average system lifetime then the mean residual life is the relevant measure. Note that the failure rate function relates only to the risk of immediate failure where as the mean residual life summaries the entire residual life distribution.
The MRL function $m(t)$for a transmuted Weibull random variable can be expressed in terms of incomplete Gamma function as given below:

$$m(t) = \frac{\sigma}{\eta}\frac{\exp\left((\frac{t}{\sigma})^\eta\right)}{\left[1-\lambda+\lambda\exp\left(-(\frac{t}{\sigma})^\eta\right)\right]}\left\{(1-\lambda)\Gamma\left(\frac{1}{\eta},\left[\frac{t}{\sigma}\right]^\eta\right) + \lambda 2^{-\frac{1}{\eta}}\Gamma\left(\frac{1}{\eta},2\left[\frac{t}{\sigma}\right]^\eta\right)\right\},$$

where $\Gamma(a,x) = \int_x^\infty e^{-z}z^{a-1}dz$ are the upper incomplete Gamma function.

5. Order Statistics

We know that if $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \cdots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}$$

for $j = 1, 2, \cdots, n$.

We have from (2) and (3) the pdf of the j^{th} order Weibull random variable $X_{(j)}$ given by

$$g_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left[-(n+1-j)\left(\frac{x}{\sigma}\right)^{\eta}\right]\left[1-\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right]^{j-1}$$

Therefore, the pdf of the n^{th} order Weibull statistic $X_{(n)}$ is given by

$$g_{X_{(n)}}(x) = \frac{n\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left[-\left(\frac{x}{\sigma}\right)^{\eta}\right]\left[1-\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right]^{n-1} \tag{13}$$

and the pdf of the 1^{st} order Weibull statistic $X_{(1)}$ is given by

$$g_{X_{(1)}}(x) = \frac{n\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left[-n\left(\frac{x}{\sigma}\right)^{\eta}\right] \tag{14}$$

Note that in a particular case of $n = 2$, (13) yields

$$g_{X_{(2)}}(x) = \frac{2\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left[-\left(\frac{x}{\sigma}\right)^{\eta}\right]\left[1-\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right] \tag{15}$$

and (14) yields

$$g_{X_{(1)}}(x) = \frac{2\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left[-2\left(\frac{x}{\sigma}\right)^{\eta}\right] \tag{16}$$

Observe that (15) and (16) are special cases of (5) for $\lambda = -1$ and $\lambda = 1$ respectively. Theorem below summarizes this relationship.

Theorem 2. *Suppose we have a system containing two components with each of them having independent and identical Weibull distribution. If the components are connected in series then the overall system will have transmuted Weibull distribution with* $\lambda = 1$ *whereas if the components are parallel then the overall system will have transmuted Weibull distribution with* $\lambda = -1$.

We know that a series with n components work if and only if all the components work. Examples of systems with components in series include chains, high-voltage multi-cell batteries, inexpensive computer systems, and inexpensive decorative tree lights using low voltage bulbs etc.. A parallel structure with n components works if at least one of the components works.

Examples of systems with components in parallel include automobile headlights, RAID computer disk array system, stairwells with emergency lightings, overhead projectors with backup bulbs etc.

It has been observed that a transmuted Weibull distribution with $\lambda = 1$ is the distribution of $min(X_1, X_2)$ and a transmuted Weibull distribution with $\lambda = -1$ is the distribution of the $max(X_1, X_2)$ where X_1 and X_2 are independent and identically distributed 2-parameter Weibull random variables.

Now we provide the distribution of the order statistics for transmuted Weibull random variable. The pdf of the j^{th} order statistic for transmuted Weibull distribution is given by

$$\begin{aligned} f_{X_{(j)}}(x) &= \frac{n!}{(j-1)!(n-j)!}\frac{\eta}{\sigma}y^{\frac{\eta-1}{\eta}}\exp(-y)\left[1-\lambda+2\lambda\exp\left(-y\right)\right]\left[1+\lambda\exp\left(-y\right)\right]^{j-1} \\ &\quad \times\left[1-\exp\left(-y\right)\right]^{j-1}\exp\left[-(n-j)y\right]\left[1-\lambda+\lambda\exp\left(-y\right)\right]^{n-j}, \end{aligned}$$

where,

$$y = \left(\frac{x}{\sigma}\right)^{\eta}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$\begin{aligned} f_{X_{(n)}}(x) &= \frac{n\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\left[1-\lambda+2\lambda\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right] \\ &\quad \times\left[1+\lambda\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right]^{n-1}\left[1-\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right]^{n-1} \end{aligned}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{n\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1}\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\left[1-\lambda+2\lambda\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\right] \\ &\quad \times\left[\exp\left(-\left(\frac{x}{\sigma}\right)^{\eta}\right)\left\{1-\lambda+\lambda\exp\left(-\left(\frac{x}{\sigma}\right)\right)\right\}\right]^{n-1}. \end{aligned}$$

Note that $\lambda = 0$ yields the order statistics of the two parameter Weibull distribution.

6. Applications of Transmuted Weibull Distribution

In this section we will study two real data sets to illustrate the usefulness of the transmuted Weibull distribution for modeling reliability data. We will make comparison of the results with the exponentiated Weibull distribution whose cdf is given by

$$F(x) = \left[1-\exp\left(-\left(\frac{x}{\sigma}\right)^{\gamma}\right)\right]^{\alpha}, \quad x > 0, \alpha, \sigma, \gamma > 0 \tag{17}$$

Note that Weibull distribution is a submodel of both the transmuted Weibull and exponentiated Weibull distribution. Our first data set is about testing the tensile fatigue characteristics of a polyster/viscose yarn to study the problem of warp breakage during weaving. The study consists of 100 centimeter yarn sample at 2.3 percent strain level. This data was studied by Quesenberry *et al.* [8]. The Weibull distribution (2) is fitted to the subject data and the parameter estimates computed are given in the table below. We fitted both the exponentiated Weibull and transmuted Weibull distribution to the subject data. The MLEs and the values of maximized log-likelihoods for Weibull, exponentiated Weibull and transmuted Weibull distribution are given in the table below. One can use the likelihood ratio test to show that the

Table 1: Estimated Parameters of Weibull, Exponentiated Weibull and Transmuted Weibull Distributions

Distribution	Parameter Estimates	Log-likelihood
Weibull	$\hat{\eta} = 1.60, \hat{\sigma} = 247.9$	-627.0500
Exp.Weibull	$\hat{\alpha} = 1.000001588, \hat{\sigma} = 243.541641, \hat{\gamma} = 1.500001745$	-625.5765
Tran. Weibull	$\hat{\eta} = 1.7187616, \hat{\sigma} = 330.2877498, \hat{\lambda} = 0.7502233$	-624.5224

transmuted Weibull distribution fits the subject data better than the 2-parameter Weibull and exponentiated Weibull distribution. Furthermore, the graphical comparison corresponding to these fits to conform our claim is illustrated in figure 4. The second data set is the breaking

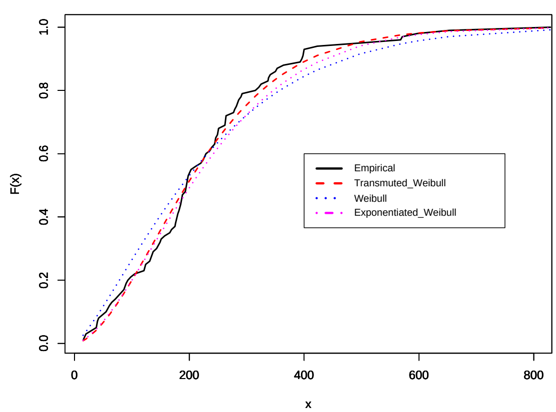

Figure 4: Empirical, Fitted Weibull, Exponentiated Weibull and Transmuted Weibull CDF of Warp Breakage Data

stress of carbon fibers. The data set contains 100 observations on breaking stress of carbon fibers (in Gba) studied by Nichols and Padgett [5] and discussed in [6] to show the usefulness of exponentiated Weibull distribution (17). It is shown (in [6]) that the subject data fits well the exponentiated Weibull distribution with its MLEs and the maximized log-likelihood given by

$$\hat{\alpha} = 1.17262, \hat{\sigma} = 2.79673, \hat{\gamma} = 2.57902 \text{ and } -\mathscr{L} = 141.369$$

A quasi Newton algorithm was implemented in R[2] in order to solve the system of normal equations inherent within the transmuted Weibull distribution. The parameter estimates along with asymptotic 95% confidence interval are given in the table below. Note that for

Table 2: Estimated Parameters of the Transmuted Weibull Distribution for Breaking Stress Data

			95% Confidence Interval	
Parameter	ML estimate	standard error	Lower	Upper
η	2.993459	0.2413740	2.520374	3.466543
σ	3.412541	0.3379128	2.750244	4.074838
λ	0.6788985	0.3798750	-0.065643	1.423439*

the transmuted Weibull distribution $-\mathscr{L} = 141.1349$.
This shows that the transmuted Weibull distribution fits equally well the subject data. Moreover, we can perform the graphical comparison such as probability plots (not shown here) corresponding to these fits to conform our claim. From these examples it can be observed that transmuted Weibull distribution can be a good competitor of other generalized Weibull distributions to model the reliability data.

7. Concluding Remarks

In the present study, we have introduced a new generalization of the Weibull distribution called the transmuted Weibull distribution. The subject distribution is generated by using the quadratic rank transmutation map and taking the 2-parameter Weibull distribution as the base distribution. Some mathematical properties along with estimation issues are addressed. The hazard rate function and reliability behavior of the transmuted Weibull distribution shows that the subject distribution can be used to model reliability data. We have studied two data sets published in the literature to show the usefulness of the transmuted Weibull distribution and to make comparison with exponentiated Weibull distribution. We expect that this study will serve as a reference and help to advance future research in this area.

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*Note that $-1 \leq \lambda \leq 1$ therefore the upper limit of λ can be simply replaced by 1.

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