EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 3, 2011, 221-229
ISSN 1307-5543 – www.ejpam.com

Transparent Ore Extensions over $\sigma(*)$-rings

Vijay Kumar Bhat

School of Mathematics, SMVD University, Katra, J and K, India-182320

Abstract. In this paper we introduce a stronger type of primary decomposition of a Noetherian ring. We call such a ring a *Transparent ring* and show that if R is a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$ (the field of rational numbers); σ an automorphism of R and δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$. Further more if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$, ($P(R)$ the prime radical of R), then $R[x;\sigma,\delta]$ is a *Transparent ring*.

2000 Mathematics Subject Classifications: 16S36, 16P40, 16P50, 16U20, 16W20

Key Words and Phrases: Automorphism, σ-derivation, $\sigma(*)$-ring, quotient ring, *Transparent ring*

1. Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The field of rational numbers and set of positive integers are denoted by $\mathbb{Q}$ and $\mathbb{N}$ respectively unless otherwise stated. The set of prime ideals of R is denoted by $Spec(R)$. The sets of minimal prime ideals of R is denoted by $MinSpec(R)$. Prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The set of associated prime ideals of R (viewed as a right R-module over itself) is denoted by $Ass(R_R)$. The notion of the quotient ring of a ring, the contractions and extensions of ideals arising thereby appear in chapter (9) of Goodearl and Warfield [10].

This article concerns the study of Ore extensions. Now let R be a ring and σ be an automorphism of R and δ is a σ-derivation of R. Recall that δ is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$.

Example 1. *Let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ defined by*

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix},$$

for all $r \in R$ be a homomorphism. Then δ is a σ-derivation of R.

Email address: vijaykumarbhat2000@yahoo.com

Recall that the Ore extension $R[x;\sigma,\delta]$ is the usual polynomial ring with coefficients in R, in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that $\sigma(I) = I$ and $\delta(I) \subseteq I$, then $O(I)$ denotes $I[x;\sigma,\delta]$, which is an ideal of $O(R)$. In case $\delta = 0$, we denote $R[x;\sigma]$ by $S(R)$. If J is an ideal of R such that $\sigma(J) = J$, then $S(J)$ denotes $J[x;\sigma]$, which is an ideal of $S(R)$. In case σ is the identity map, δ is just called a derivation of R, and we denote $R[x;\delta]$ by $D(R)$. If K is an ideal of R such that $\delta(K) \subseteq K$, then $D(K)$ denotes $K[x;\delta]$, which is an ideal of $D(R)$. Ore-extensions including skew-polynomial rings $S(R)$ and differential operator rings $D(R)$ have been of interest to many authors. For example [1, 2, 3, 5, 6, 7, 8, 10, 13, 14, 15, 16].

The classical study of any commutative Noetherian ring is done by studying its primary decomposition. Further there are other structural properties of rings, for example the existence of quotient rings or more particularly the existence of Artinian quotient rings etc. which can be nicely tied to primary decomposition of a Noetherian ring.

The first important result in the theory of non commutative Noetherian rings was proved in 1958 (GoldieŠs Theorem) which gives an analog of field of fractions for factor rings R/P, where R is a Noetherian ring and P is a prime ideal of R. In 1959 the one sided version was proved by Lesieur and Croisot (Theorem (5.12) of Goodearl and Warfield [10])and in 1960 Goldie generalized the result for semiprime rings (Theorem (5.10) of Goodearl and Warfield [10]).

In Blair and Small [8], it is shown that if R is embeddable in a right Artinian ring and if characteristic of R is zero, then the differential operator ring $R[x;\delta]$ embeds in a right Artinian ring, where δ is a derivation of R. It is also shown in [8] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew-polynomial ring $R[x;\sigma]$ embeds in an Artinian ring.

In this paper the above mentioned properties have been studied with emphasis on primary decomposition of the Ore extension $O(R)$, where R is a commutative Noetherian $\mathbb{Q}$-algebra.

A noncommutative analogue of associated prime ideals of a Noetherian ring has also been also discussed. We would like to note that a considerable work has been done in the investigation of prime ideals (in particular minimal prime ideals and associated prime ideals) of skew polynomial rings (K. R. Goodearl and E. S. Letzter [11], C. Faith [9], S. Annin [1], Leroy and Matczuk [15], Nordstrom [16] and Bhat [5]).

In section (4) of [11] Goodearl and Letzter have proved that if R is a Noetherian ring, then for each prime ideal P of $O(R)$, the prime ideals of R minimal over $P \cap R$ are contained within a single σ-orbit of $Spec(R)$.

The author has proved in Theorem (2.4) of [5] that if σ is an automorphism of a Noetherian ring R , then $P \in Ass(S(R)_{S(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $S(P \cap R) = P$ and $P \cap R = \cap_{i=0}^{m} \sigma^i(U)$, where $m \geq 1$ is an integer such that $\sigma^m(V) = V$ for all $V \in Ass(R_R)$. (The same result has been proved for minimal prime ideal case).

Carl Faith has proved in [9] that if R is a commutative ring, then the associated prime ideals of the usual polynomial ring $R[x]$ (viewed as a module over itself) are precisely the ideals of the form $P[x]$, where P is an associated prime ideal of R.

S. Annin has proved in Theorem (2.2) of [1] that if R is a ring and M be a right R-module. If σ is an endomorphism of R and $S = R[x;\sigma]$ and M_R is σ-compatible, then $Ass(M[x]_S) = \{P[x]$ such that $P \in Ass(M_R)\}$.

In [15], Leroy and Matczuk have investigated the relationship between the associated prime ideals of an R-module M_R and that of the induced $S(R)$-module $M_{S(R)}$, where as usual $S(R) = R[x;\sigma]$ (σ an automorphism of a ring R). They have proved the following:

Theorem (Theorem 5.7 of [15])**.** *Suppose M_R contains enough prime submodules and let for $Q \in Ass(M_{S(R)})$. If for every $P \in Ass(M_R)$, $\sigma(P) = P$, then $Q = P(S(R))$ for some $P \in Ass(M_R)$.*

In Theorem (1.2) of [16] Nordstrom has proved that if R is a ring with identity and σ is a surjective endomorphism of R, then for any right R-module M, $Ass(M[x;\sigma]) = \{I[x;\sigma]$, $I \in \sigma - Ass(M)\}$. In Corollary (1.5) of [16] it has been proved that if R is Noetherian and σ an automorphism of R, then $Ass(M[x;\sigma]_{S(R)}) = \{P_\sigma[x;\sigma], P \in Ass(M)\}$, where $P_\sigma = \cap_{i \in N} \sigma^{-i}(P)$ and $S(R) = R[x;\sigma]$.

The above discussion leads to a stronger type of primary decomposition of a Noetherian ring. We call a Noetherian ring with such a decomposition a *Transparent ring*.

Before we give the definition of a *Transparent ring*, we need the following:

Definition 1. *A ring R is said to be an irreducible ring if the intersection of any two non-zero ideals of R is non-zero. An ideal I of R is called irreducible if $I = J \cap K$ implies that either $J = I$ or $K = I$. Note that if I is an irreducible ideal of R, then R/I is an irreducible ring.*

Proposition 1. *Let R be a Noetherian ring. Then there exist irreducible ideals I_j, $1 \leq j \leq n$ of R such that $\cap_{j=1}^n I_j = 0$.*

Proof. The proof is obvious and we leave the details to the reader.

Definition 2 (A)**.** *A Noetherian ring R is said to be a Transparent ring if there exist irreducible ideals I_j, $1 \leq j \leq n$ such that $\cap_{j=1}^n I_j = 0$ and each R/I_j has a right Artinian quotient ring.*

It can be easily seen that an integral domain is a *Transparent ring*, a commutative Noetherian ring is a *Transparent ring* and so is a Noetherian ring having an Artinian quotient ring. A fully bounded Noetherian ring is also a *Transparent ring*.

This type of decomposition was actually introduces by the author in [2]. Such a ring was called a *decomposable ring*, but in order to distinguish between one more definition of a *decomposable ring* given below and pointed out by the referee of one of the papers of the author, we now call such a ring a *Transparent ring*.

Definition 3 (Decomposable ring, Hazewinkel, Gubareni and Kirichenko[12])**.** *Let R be a ring. An R-module M is said to be decomposable if $M \simeq M_1 \oplus M_2$ of non zero R-modules M_1 and M_2. A ring R is called a decomposable ring if it is a direct sum of two rings.*

In this paper we investigate the *Transparent ring* property for $O(R) = R[x;\sigma,\delta]$. In this direction we get a motivation from the following:

Recall that in [13], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [14], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 2. *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$*, where* F *is a field. Then* $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$*. Let* $\sigma : R \to R$ *be defined by* $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$*. Then it can be seen that* σ *is an endomorphism of* R *and that* R *is a* $\sigma(*)$*-ring.*

We note that the above ring is not σ-rigid. For let $0 \neq a \in F$. Then

$$\begin{pmatrix} 0 & a \\ 0 & o \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In [14], Kwak also establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, or equivalently if the prime radical is a completely semiprime ideal. Recall that an ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. Also an ideal J of R is called completely prime if $ab \in J$ implies $a \in J$ or $b \in J$ for $a, b \in R$. Completely prime ideals of skew polynomial rings have been recently characterized. In this direction the following has been proved:

Theorem (Theorem 2.4 of Bhat [7]). *Let* R *be a ring. Let* σ *be an automorphism of* R *and* δ *a* σ*-derivation of* R*. Then:*

1. *For any completely prime ideal* P *of* R *with* $\delta(P) \subseteq P$ *and* $\sigma(P) = P$*,* $O(P) = P[x; \sigma, \delta]$ *is a completely prime ideal of* $O(R)$*.*
2. *For any completely prime ideal* U *of* $O(R)$*,* $U \cap R$ *is a completely prime ideal of* R*.*

The 2-primal property has also been extended to the skew-polynomial ring $R[x; \sigma]$ in Kwak [14]. Clearly R is a $I(*)$-ring if and only if R is a 2-primal ring, where I is the identity map on R. The ring in Example (2) is 2-primal.

We now give an example of a ring R, and an endomorphism σ of R such that R is not a $\sigma(*)$-ring, however R is 2-primal.

Example 3. *Let* $R = F[x]$ *be the polynomial ring over a field* F*. Then* R *is 2-primal with* $P(R) = 0$*. Let* $\sigma : R \to R$ *be an endomorphism defined by* $\sigma(f(x)) = f(0)$*. Then* R *is not a* $\sigma(*)$*-ring. For example consider* $f(x) = xa$*,* $a \neq 0$*.*

In this paper, we prove the following Theorem:

Theorem 1. *Let* R *be a commutative Noetherian* $\sigma(*)$*-ring, which is also an algebra over* $\mathbb{Q}$ *(*σ *an automorphism of* R*). Let* δ *be a* σ*-derivation of* R *such that* $\delta(\sigma(a)) = \sigma(\delta(a))$*, for all* $a \in R$*. Then* $R[x; \sigma, \delta]$ *is a Transparent ring.*

This is proved in Theorem (3).

2. *Transparent Rings* and Polynomial Rings over $\sigma(*)$-rings

We begin with the following Proposition:

Proposition 2. *Let R be a ring. Let σ be an automorphism of R. Then R is a $\sigma(*)$-ring implies $P(R)$ is completely semiprime.*

Proof. Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.

Proposition 3. *Let R be a $\sigma(*)$-ring and $U \in MinSpec(R)$ be such that $\sigma(U) = U$. Then $U(S(R)) = U[x;\sigma]$ is a completely prime ideal of $S(R) = R[x;\sigma]$.*

Proof. Proposition (2) implies that $P(R)$ is completely semiprime ideal of R and further more U is completely prime by Proposition (1.11) of [17]. Now we note that σ can be extended to an automorphism $\overline{\sigma}$ of R/U. Now it is well known that $S/U(S(R)) \simeq (R/U)[x;\overline{\sigma}]$ and hence $U(S(R))$ is a completely prime ideal of $S(R)$.

A necessary and sufficient condition for a Noetherian ring R to be a $\sigma(*)$-ring has been given in the following Theorem:

Theorem 2 (Theorem 2.4 of Bhat and Kumari [4])**.** *Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. Now for each i, $a \in U_i$ or $\sigma(a) \in U_i$ and U_i is completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring and let $U = U_1$ be a minimal prime ideal of R. Now by Proposition (2), $P(R)$ is completely semiprime. Let $U_2, U_3, \ldots, U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now suppose that $U = U_1$ is not completely prime. Then there exist $a, b \in R \setminus U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap \ldots \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. So $c \in P(R)$ and, thus $b(U_2 \cap U_3 \cap \ldots \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap \ldots \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime.

Corollary 1. *Let R be a Noetherian $\sigma(*)$-ring and $U \in MinSpec(R)$. Then $U(S(R)) = U[x;\sigma]$ is a completely prime ideal of $S(R) = R[x;\sigma]$.*

Proof. Let $U \in MinSpec(R)$. Then $\sigma(U) = U$ by Theorem (2). Now result follows from Proposition 2.

Proposition 4. *Let R be a Noetherian $\sigma(*)$-ring which is also an algebra over $\mathbb{Q}$ and δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$. Then $\delta(U) \subseteq U$ for all $U \in MinSpec(R)$.*

Proof. Let $U \in MinSpec(R)$. Then $\sigma(U) = U$ by Theorem (2). Consider the set

$$T = \{a \in U \mid \text{ such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}.$$

First of all, we will show that T is an ideal of R. Let $a, b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1$. Now $\delta^k(a-b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1\}$. Therefore $a - b \in T$. Therefore T is a δ-invariant ideal of R.

We will now show that $T \in Spec(R)$. Suppose $T \notin Spec(R)$. Let $a \notin T$, $b \notin T$ be such that $aRb \subseteq T$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq T$. This implies on simplification that

$$\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U,$$

where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $T \in Spec(R)$. Now $T \subseteq U$, so $T = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$.

Remark 1. *In above proposition the condition that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$ is necessary. For example if $s = t = 1$, then $a \in U$, $b \in U$ and therefore, $\sigma^i(a) \in U$, $\sigma^i(b) \in U$ for all integers $i \geq 1$ as $\sigma(U) = U$. Now $\delta^2(adb) \in U$ implies that*

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) + u \in U.$$

where u is sum of terms involving a or b, or $\sigma^i(b)$. Therefore by assumption $u \in U$. This implies that

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) \in U.$$

If $\delta(\sigma(a)) \neq \sigma(\delta(a))$, for all $a \in R$, then we get nothing out of it and if $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$, we get $\delta(a)\sigma(d)\sigma(\delta(b)) \in U$ which gives a contradiction.

Proposition 5. *Let R be a Noetherian ring having an Artinian quotient ring. Then R is a Transparent ring.*

Proof. Let $Q(R)$ be the quotient ring of R . Now for any ideal J of $Q(R)$, the contraction J^c of J is an ideal of R and the extension of J^c is J; i.e. $(J^c)^e = J$. For this see Proposition (9.19) of Goodearl and Warfield [10]. Now there exist ideals I_j, $1 \leq j \leq n$ of $Q(R)$ such that $0 = \cap_{j=1}^n I_j$, and each $Q(R)/I_j$ is an Artinian ring. Let $I_j^c = K_j$. Then it is not difficult to see that R/K_j has Artinian quotient ring $Q(R)/I_j$. Moreover $\cap_{j=1}^n K_j = 0$. Hence R is a *Transparent ring*.

Definition 4. *Let P be a prime ideal of a commutative ring R. Then the symbolic power of P for any $n \in \mathbb{N}$ is denoted by $P^{(n)}$ and is defined as*

$$P^{(n)} = \{a \in R \text{ such that there exists some } d \in R, d \notin P \text{ such that } da \in P^n\}.$$

Also if I is an ideal of R, define as usual $\sqrt{I} = \{a \in R$ such that $a^n \in I$ for some $n \in \mathbb{N}$.

Lemma 1. *Let R be a commutative Noetherian ring, and σ an automorphism of R. If P is a prime ideal of R such that $\sigma(P) = P$, then $\sigma(P^{(n)}) = P^{(n)}$ for all integers $n \geq 1$.*

Proof. See Lemma (2.10) of Bhat [6].

Lemma 2. *Let R be a commutative Noetherian ring; σ and δ as usual. Let P be a prime ideal of R such that $\sigma(P) = P$ and $\delta(P) \subseteq P$. Then $\delta(P^{(k)}) \subseteq P^{(k)}$ for all integers $k \geq 1$.*

Proof. See Lemma (2.11) of [6].

Theorem 3. *Let R be a commutative Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$, (σ an automorphism of R). Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$. Then $O(R) = R[x; \sigma, \delta]$ is a Transparent ring.*

Proof. R is a commutative Noetherian $\mathbb{Q}$-algebra, therefore, the ideal (0) has a reduced primary decomposition. Let I_j, $1 \leq j \leq n$ be irreducible ideals of R such that $(0) = \cap_{j=1}^n I_j$. For this see Theorem (4) of Zariski and Samuel [18]. Let $\sqrt{I_j} = P_j$, where P_j is a prime ideal belonging to I_j. Now $P_j \in Ass(R_R)$, $1 \leq j \leq n$ by first uniqueness Theorem. Now by Theorem (23) of Zariski and Samuel [18] there exists a positive integer k such that $P_j^{(k)} \subseteq I_j$, $1 \leq j \leq n$. Therefore we have $\cap_{j=1}^n P_j^{(k)} = 0$. Now each P_j contains a minimal prime ideal U_j by Proposition (2.3) of Goodearl and Warfield [10], therefore $\cap_{j=1}^n U_j^{(k)} = 0$. Now Theorem (2) implies that $\sigma(U_j) = U_j$, for all j, $1 \leq j \leq n$. Therefore Proposition (4) implies that $\delta(U_j) \subseteq U_j$, for all j, $1 \leq j \leq n$. Now Lemma (1) implies that $\sigma(U_j)^{(k)} = U_j^{(k)}$ and Lemma (2) implies that $\delta(U_j)^{(k)} \subseteq U_j^{(k)}$, for all j, $1 \leq j \leq n$ and for all $k \geq 1$. Therefore $O(U_j^{(k)})$ is an ideal of $O(R)$ and $\cap_{j=1}^n O(U_j^{(k)}) = 0$.

Now $R/U_j^{(k)}$ has an Artinian quotient ring, as it has no embedded primes, therefore $O(R)/O(U_j^{(k)})$ has also an Artinian quotient ring by Theorem (2.11) of Bhat [3]. Hence $O(R) = R[x; \sigma, \delta]$ is *Transparent ring*.

Corollary 2. *Let R be a commutative Noetherian $\sigma(*)$-ring (σ an automorphism of R). Then $S(R) = R[x; \sigma]$ is a Transparent ring.*

Corollary 3. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Then $D(R) = R[x; \delta]$ is a Transparent ring.*

Question 1. *Let R be a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$, (σ an automorphism of R) and δ a σ-derivation of R. Is $O(R) = R[x; \sigma, \delta]$ is a Transparent ring?.*

References

[1] S. Annin, Associated primes over skew polynomial rings, Comm. Algebra, Vol. 30, 2511-2528. MR1940490 (2003k:16037). 2002.

[2] V. K. Bhat, Decomposability of iterated extensions, Int. J. Math. Game Theory Algebra, Vol. 15(1), 45-48. MR2257766. 2006.

[3] V. K. Bhat, Ring extensions and their quotient rings, East-West J. Math., Vol. 9(1), 25-30. MR2444121 (2009f:16049). 2007.

[4] V. K. Bhat and Neetu Kumari, Transparency of $\sigma(*)$-rings and their extensions, Int. J. Algebra , Vol. 2(19), 919-924. MR2481211 (2010d:16032). 2008.

[5] V. K. Bhat, Associated prime ideals of skew polynomial rings, Beitrage Algebra Geom., Vol. 49(1), 277-283. MR2410584 (2009e:16046). 2008.

[6] V. K. Bhat, Transparent rings and their extensions, New York J. Math., Vol. 15, 291-299. MR2530150 (2010j:16060). 2009.

[7] V. K. Bhat, A note on completely prime ideals of Ore extensions, Internat. J. Algebra Comput., Vol. 20(3), 457-463. MR2658421. 2010.

[8] W. D. Blair, L.W. Small, Embedding differential and skew polynomial rings into artinian rings, Proc. Amer. Math. Soc., Vol. 109(4), 881-886. MR1025276 (90k:16003). 1990.

[9] C. Faith, Associated primes in commutative polynomial rings, Comm. Algebra, Vol. 28, 3983-3986. MR1767601 (2001a:13038). 2000.

[10] K. R. Goodearl, R.B. Warfield, An introduction to Non-commutative Noetherian rings. Camb. Uni. Press. MR1020298 (91c:16001). 1989.

[11] K. R. Goodearl and E. S. Letzter, Prime ideals in skew and q-skew polynomial rings, Memoirs of the Amer. Math. Soc., No. 521. MR1197519 (94j:16051). 1994.

[12] M. Hazewinkel, N. Gubareni and V. V. Kirichenko, Algebras, rings and modules; Vol. 1, Mathematics and its applications, Kluwer Academic Press. MR2106764 (2006a:16001). 2004.

[13] J. Krempa, Some examples of reduced rings, Algebra Colloq., Vol. 3(4), 289-300. MR1422968 (98e:16027). 1996.

[14] T. K. Kwak, Prime radicals of skew-polynomial rings, Int. J. Math. Sci., Vol. 2(2), 219-227. MR2061508 (2006a:16035). 2003.

[15] A. Leroy and J. Matczuk, On induced modules over Ore extensions, Comm. Algebra, Vol. 32(7), 2743-2766. MR2099932 (2005g:16051). 2004.

[16] H. E. Nordstorm, Associated primes over Ore extensions, J. Algebra, Vol. 286(1), 69-75. MR2124809 (2006c:16049). 2005.

[17] G. Y. Shin, Prime ideals and sheaf representations of a pseudo symmetric ring, Trans. Amer. Math. Soc., Vol. 184, 43-60. MR0338058 (49:2825). 1973.

[18] O. Zariski and P. Samuel, Commutative Algebra, Vol. I, D. Van Nostrand Company, Inc. MR0384768 (52:5641). 1967.