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Rate of Convergence in Sobolev Space

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Abstract. In this paper, a new theorem on degree of approximation in $L_p^2(\Omega)$ Sobolev space of integrable functions of two variables by Bernstein-Chlodowsky polnomials on an unbounded triangular domain is studied. Also by using the K- functional of Peetre the order of approximation are established.
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1. Introduction

The aim of this paper is to study the problem on degree of the approximation of function of two variables of $f \in L_p^2(\Omega)$ by means of Bernstein - Chlodowsky polynomials in a triangular domain extending infinity, where $\Omega = \lim_{n\to\infty} \Delta_{b_n}, \Delta_{b_n} = \{(x,y) : x \leqslant 0, y \geqslant 0, x+y \leqslant b_n\}$ and (b_n) is a sequence of increasing positive number, such that:

$$\lim_{n\to\infty} b_n = \infty, \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{1}$$

Some properties of approximation of functions of two variable by Bernstein -Chlodowsky polynomials was proven in [1]-[5] and [7]. In addition, convergence of Bernstein-Chlodowsky polynomials of two variables were investigated on a triangular domain in [6] and [7]. In this paper we will use Bernstein-Chlodowsky polynomials on Ω which is introduced in [7]. Let,$f \in L_p^2(\Omega)$,

$$B_n(f;x,y) = \sum_{k=0}^{n} C_n^k (1 - \frac{x+y}{b_n})^{n-k} \sum_{i=0}^{k} f(\frac{k-i}{n} b_n, \frac{i}{n} b_n) C_k^i (\frac{x}{b_n})^{k-i} (\frac{y}{b_n})^i$$

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for $(x,y) \in \Delta_{b_n}$. We note that formula (1) is the sequence of linear positive operators in the space of integrable functions L_p of two variables, that is these linear positive operators translate a positive function to an another positive one. But, in general, the function is not necessarily a continuous one in L_p space. We can not use Korovkin's Theorem. First, We give certain results which are necessary to prove the main results.

Lemma 1. *Suppose that $e_{k,m}(t,) = t^{km}$ then*

$$\begin{aligned} B_n(e_{0,0};x,y) &= 1 \\ B_n(e_{1,0};x,y) &= x \\ B_n(e_{0,1};x,y) &= y \\ B_n(e_{2,0};x,y) &= x^2 + \frac{x(b_n - x)}{n} \\ B_n(e_{0,2};x,y) &= y^2 + \frac{y(b_n - y)}{n} \end{aligned}$$

Simple calculations can be calculated above Lemma.

Theorem 1 ([7])**.** *Let $f \in L_p(\Omega)$ and a be a fixed point in $(0,b_n)$. If, for every $(x,y) \in \Delta_a$ and $(t,s) \in \Delta_{b_n}$*

$$\frac{|f(t,s) - f(x,y)|}{|(t,s)-(x,y)|} \leqslant M \tag{2}$$

hold with the constant M,then

$$\|B_n(f) - f\|_{L_p(\Delta_a)} \to 0, n \to \infty.$$

Where $a > 0$.

2. Main Theorems

To simplify notation, we need the following.

$$L_p^2(\Omega_1) = \{f \in L_p(\Omega_1) : \Delta^{|\alpha|} f \in L_p(\Delta_a), |\alpha| = 2\}, \Omega_1 \subset [0,b_n) \times [0,b_n).$$

We consider also the following K-functional of Peetre;

$$K_p(f;\delta) = \inf_{g \in L_p^2(\Delta_a)} [\|f - g\|_{L_p(\Delta_a)} + \delta(\|g\|_{L_p^2(\Delta_a)})], \delta \geqslant 0.$$

for $f \in L_p(\Delta_a)$, we have $\lim_{\delta \to 0} K(f;\delta) = 0$. Therefore the K-functional gives the degree of approximation of a function $f \in L_p(\Delta_a)$ by smoother functions $g \in L_p^2(\Delta_a)$. Remember that the second order integral modulus of smoothness is given by

$$\omega_{2,p}(f;\delta) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) - 2f(x) + f(x-h)\|_{L_p(\Delta_a)}(I_h)$$

for an $f \in L_p(\Delta_a)$, where I_h indicates that the L_p-norm is taken over the interval $[h, b_n - h]$. It is also know that there are constants $a_1 > 0, a_2 > 0$, independent of f and p such that

$$a_1 \omega_{2,p}(f;\delta^{1/2}) \leqslant K_p(f;\delta) \leqslant \min(1,\delta)\|f\|_{L_p(\Delta_a)} + 2a_2\omega_{2,p}(f;\delta^{1/2}) \quad (3)$$

We prove the following theorems:

Theorem 2. *Let $f \in L_p^2(\Omega_1), 1 \leqslant p < \infty$ and a, M are constants, If the condition,*

$$\frac{|f(t,s) - f(x,y)|}{|(t,s) - (x,y)|} \leqslant M, t \in (a, b_n], s \in (a, b_n], (x,y) \in \Delta_a$$

is satisfied, then

$$\begin{aligned} \|B_n(f) - f\|_{L_p^2(\Delta_a)} \leqslant C_p(\|f\|_{L_p^2(\Delta_a)})\delta_n, \delta_n &= \frac{a(b_n + a)}{n} \\ C_p &= \begin{cases} p > 1, 2(\dfrac{p}{p+1})^p \\ p = 1, a^2 \end{cases} \end{aligned}$$

Proof. For $f \in L_p^2(\Omega_1)$ we can write that,

$$\begin{aligned} B_n(f(t,s) - f(x,y); x, y) &= f_x(x,y)B_n((t-x); x, y) + f_y(x,y)B_n((s-y); x, y) \\ &+ B_n(\int_x^t f_{uu}(u,y)(u-t)du; x, y) \\ &+ B_n(\int_y^s f_{kk}(x,k)(k-s)dk; x, y) \\ &+ B_n(\int_x^t \int_y^s f_{ts}(t,s)dsdt; x, y) \end{aligned}$$

Now, we need the Hardy-Littlewood majorante of f_{xx} at x , Which is defined as following:

$$\varphi_{f_{xx}(x,y)} = \sup_{0 \leqslant t \leqslant x, t \neq x} (\frac{1}{t-x}) \int_x^t f_{uu}(u,y)du$$

and using following inequality

$$\int_\Omega |\varphi_{f_{xy}(x,y)}|^p dxdy \leqslant 2(\frac{p}{p+1})^p \int_0^a \int_0^a |f_{ts}(t,s)|^p dsdt$$

using L_p-norm, we get

$$|B_1(x,y)| + |B_2(x,y)| + |B_3(x,y)| \leqslant \varphi_{f_{xx}(x,y)}\delta_n + \varphi_{f_{yy}(x,y)}\delta_n + \varphi_{fxy(x,y)}\sqrt{\delta_n}$$

$$\begin{aligned}|B_1|_{L_p(\Delta_a)}+|B_2|_{L_p(\Delta_a)}+|B_3|_{L_p(a)} &\leqslant C_p(\|f_{xx}\|_{L_p(\Delta_a)}+\|f_{yy}\|_{L_p(\Delta_a)}+\|f_{xy}\|_{L_p(\Delta_a)})\delta_n\\ &< C_p(\|f\|_{L_p^2(\Delta_a)})\delta_n\\ \|B_nf-f\|_{L_p(\Delta_a)} &\leqslant C_p(\|f\|_{L_p^2(\Delta_a)})\delta_n\end{aligned}$$

where

$$C_p=2^{1/p}(\frac{p}{p-1}),(1<p<\infty)$$

If $p=1$,

$$\begin{aligned}\int_{(\Delta_a)}|B_1(x,y)|dxdy &\leqslant \int_0^a\int_0^a|B_n(\int_x^t f_{uu}(u,y)(u-t)du;x,y)|dxdy\\ &\leqslant \int_0^a\int_0^a B_n(|t-x|\int_x^t f_{uu}(u,y)du;x,y)dxdy\\ &= \|f_{xx}\|_{L_1(\Delta_a)}a^2\delta_n.\\ \int_{(\Delta_a)}|B_2(x,y)|dxdy &\leqslant \int_0^a\int_0^a B_n(|s-y|\int_x^t f_{kk}(x,k)dk;x,y)dxdy\\ &= \|f_{yy}\|_{L_1(\Delta_a)}a^2\delta_n.\\ \int_{(\Delta_a)}|B_3(x,y)|dxdy &\leqslant \int_0^a\int_0^a|B_n(\int_x^t\int_y^s f_{ts}(t,s)dsdt;x,y)|dxdy\\ &= \|f_{xy}\|_{L_1(\Delta_a)}a^2\delta_n.\end{aligned}$$

then

$$\begin{aligned}\|B_1\|_{L_p(\Delta_a)}+\|B_2\|_{L_p(\Delta_a)}+\|B_3\|_{L_p(a)} &\leqslant a^2(\|f_{xx}\|_{L_p(\Delta_a)}+\|f_{yy}\|_{L_p(\Delta_a)}+\|f_{xy}\|_{L_p(\Delta_a)})\delta_n\\ \|B_nf-f\|_{L_p(\Delta_a)} &\leqslant a^2\|f\|_{L_p^2(\Delta_a)}\delta_n.\end{aligned}$$

Thus, the proof is completed.

Theorem 3. *Let $f\in L_p^2(\Omega^1)$, $1\le p<\infty$ and f satisfies the condition (2) then the following inequality*

$$\|B_nf-f\|_{L_p(\Delta_a)}\leqslant M_p[\|f\|_{L_p^2(\Delta_a)}\delta_n+\omega_{2,p}(f;\delta^{(1/2)})] \quad (4)$$

holds. Where a, M are constants.

Proof. For all sufficiently large n, from Theorem 2 we can write

$$\|B_nh-h\|_{L_p\Delta_a} \leqslant \begin{cases}(\varepsilon+M\delta_na)\|h\|_{L_p(\Delta_a)} & ,h\in L_p(\Delta_a)\\ C_p\|f\|_{L_p(\Delta_a)}\delta_n & ,h\in L_P^2(\Delta_a)\end{cases}$$

where C_p is positive constant which independent of h,n and where h satisfies (2). When $f\,L_p^2(\Omega_1)$ and $g\in L_p^2(\Delta_a)$ the condition (2) is satisfied then

$$\|B_nf-f\|_{L_p(\Delta_a)} \leqslant \|B_n(f-g)-(f-g)\|_{L_p(\Delta_a)}+\|B_ng-g\|_{L_p(\Delta_a)}$$

$$\begin{aligned} &\leqslant (\varepsilon + M\delta_n a)\|f-g\|_{L_p(\Delta_a} + C_p\|g\|_{L_p^2(\Delta_a}\delta_n \\ &\leqslant \widetilde{M}[\|f-g\|_{L_p(\Delta_a} + \|g\|_{L_p^2(\Delta_a}\delta_n] \end{aligned}$$

where $\widetilde{M} = \max\{\varepsilon + M\delta_n a, C_p\}$.
Using the K-functional we get,

$$\|B_n f - f\|_{L_p(\Delta_a)} \leqslant \widetilde{M} \sup_{g\in L_p^2(\Delta_a)} [\|f-g\|_{L_p(\Delta_a} + \|g\|_{L_p^2(\Delta_a)}\delta_n]$$

since, for a sufficiently large n,δ_n and from (3),

$$\begin{aligned} K_p(f;\delta) &\leqslant \delta_n\|f\|_{L_p(\Delta_a)} + 2a_1\omega_{2,p}(f;\delta^{(1/2)}) \\ \widetilde{M}K_p(f;\delta) &\leqslant \widetilde{M}[\delta_n\|f\|_{L_p(\Delta_a)} + 2a_1\omega_{2,p}(f;\delta^{(1/2)})] \end{aligned}$$

we obtain (4),

$$\|B_n f - f\|_{L_p(\Delta_a)} \leqslant M_p[\|f\|_{L_p^2(\Delta_a)}\delta_n + \omega_{2,p}(f;\delta^{(1/2)})].$$

Thus, the proof is completed.

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