



## On the Pairs of Orthogonal Ruled Surfaces

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**Abstract.** In this work, in three dimensional Euclidean space  $E^3$ , by using the ruled Bonnet surfaces which have been known up to now, the problem of finding some pairs of orthogonal ruled surfaces is examined and only one pair of orthogonal ruled surfaces can be obtained.

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### 1. Introduction

The orthogonal surfaces are related to the problem of isometric representation. These surfaces also play an important role in all questions connected with the study of the infinitesimal bending problem. Because the infinitesimal bending problem is reduced the problem of finding the orthogonal surfaces [5, 7, 8]. The orthogonal ruled surfaces are significant to investigate the isometric representation of the ruled surfaces or the infinitesimal bending of ruled surfaces. For this aim, we can use the Bonnet ruled surfaces to examine pairs of orthogonal ruled surfaces. Generally, Bonnet surfaces have been classified into three categories:

- The surfaces of constant mean curvature other than the planes and spheres.
- The isometric Weingarten surfaces of non-constant mean curvature which are isometric to a surface of revolution.
- The surfaces of non-constant mean curvature that admit a single non-trivial isometry.

In [2], by using the method given in [4], some special Bonnet ruled surfaces of non-constant mean curvature are given. It is shown that the ruled surfaces which are formed by the binormals two curves whose curvatures and absolute value of torsions are the same are the Bonnet pairs. Moreover the ruled minimal Bonnet surfaces and the ruled Weingarten surfaces are

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indicated (the developable surface are not investigated). The tangential Bonnet surfaces as the developable Bonnet surfaces are examined in [3].

In this work, the main goal is to find some ruled orthogonal surfaces by using the ruled Bonnet surfaces. Using this method, it is shown that the pairs of orthogonal surfaces except one pair can not be obtained.

## 2. Ruled Bonnet Surfaces and Some Orthogonal Pairs

### 2.1. Some Ruled Bonnet Surfaces

A ruled surface in three dimensional Euclidean space  $E^3$  can be given by the vectorial equation

$$\mathbf{X}(u, v) = \mathbf{r}(v) + u\mathbf{T}(v) \tag{1}$$

where  $\mathbf{r} = \mathbf{r}(v)$  is a directrix curve,  $\mathbf{T} = \mathbf{T}(v)$  is a unit vector with

$$\begin{aligned} \mathbf{T}^2(v) &= a^2(v) \neq 0, & \mathbf{T}(v) \cdot \mathbf{r}'(v) &= \cos \theta(v), \\ \mathbf{T}'(v) \cdot \mathbf{r}'(v) &= b(v), & (0 \leq \theta \leq \pi), & \text{ where } f' = \frac{df}{dv} \end{aligned} \tag{2}$$

The first fundamental form of the ruled surface (1) is

$$ds^2 = du^2 + 2 \cos \theta dudv + (a^2u^2 + 2bu + 1)dv^2 \tag{3}$$

The value of  $u$  for the central point; the parameter of the distribution  $\beta$  are expressed as

$$u = \alpha(v) = -\frac{b(v)}{a^2(v)}, \quad \beta = \frac{1}{a^2}(\mathbf{t}, \mathbf{T}, \mathbf{T}') \tag{4}$$

Here the central point is the limit of the point in which a generator  $g_1$  is met by the common perpendicular of  $g_1$  and a neighboring generator  $g_2$  as  $g_2$  approaches  $g_1$  over a ruled surface; the parameter of distribution is the limit of the ratio of the shortest distance between two generators and their included angle.

Now let an orthogonal trajectory to the generators be taken as a directrix and  $\mathbf{T}^2(v) = 1$ ,  $\mathbf{T}'^2(v) = 1$ . Such a ruled surface can be obtained from (1) under the conditions [6, 1]:

$$\begin{aligned} \mathbf{T} &= \frac{1}{\beta}(\mathbf{T}' \wedge \mathbf{r}'), \mathbf{T}' = -\frac{\alpha}{\beta^2 + \alpha^2}\mathbf{r}' + \frac{\beta}{\beta^2 + \alpha^2}\mathbf{r}' \wedge \mathbf{T} \\ \mathbf{r}' &= -\alpha\mathbf{T}' + \beta\mathbf{T} \wedge \mathbf{T}' \\ \mathbf{N} &= \frac{-1}{\beta w}[(\beta^2 + \alpha^2 - u\alpha)\mathbf{T}' + (\alpha - u)\mathbf{r}'], w = \sqrt{(u - \alpha)^2 + \beta^2}, (\beta \neq 0) \end{aligned} \tag{5}$$

We recall that the director-cone is the cone formed by drawing through a point lines parallel to the generators and it is determined by the function

$$D(v) = \frac{\mathbf{r}' \cdot \mathbf{T}''}{\beta} = (\mathbf{T}, \mathbf{T}', \mathbf{T}'') \tag{6}$$

Because  $\kappa^2 = 1 + D^2$  and  $\tau = \frac{D'}{1 + D^2}$  are the curvature and the torsion of the unit spherical curve  $\mathbf{x} = \mathbf{T}(v)$  which determines the director-cone [2]. The minimal surfaces and the isothermic Weingarten ruled surfaces are given by the equations  $\alpha' = \beta' = D = 0$  and  $\alpha'' = \beta' = D' = 0$  respectively [2, 6].

If a isometric representation between two surfaces preserves the principal curvatures of these surface, these surfaces are said to be Bonnet surfaces. In [4], in order to find a Bonnet surface a method is given. In according to this, A-net on a surface such that, when this net is parametrized, the conditions  $E = G, F = 0, M = c = \text{const.} \neq 0$  are satisfied, is called an A-net, where  $E, F, G$  are the coefficients of the first fundamental form of the surface and  $L, M, N$  are the coefficients of the second fundamental form. And necessary and sufficient condition for a surface to be a Bonnet surface is that the surface can have an A-net. In [2], on the ruled surface, the generators and orthogonal trajectories form an A-net if and only if the parameter of the distribution  $\beta$  and the abscissa of the central point  $\alpha$  are constants. And the parameter of distribution and the abscissa of the central point of a surface are constant, then the surface is a Bonnet surface. The torsion of the striction line of such a Bonnet surface is constant and is equal to the reciprocal of the parameter of distribution. And the ruled surfaces which are formed by the binormals two curves whose curvatures and absolute value of torsions are the same are the Bonnet pairs. This means that the ruled surfaces given by the equations

$$\begin{aligned} \mathbf{X}(u, v) &= \mathbf{r}(v) + |\beta| \sinh u \mathbf{T}(v) \\ \mathbf{Y}(u, v) &= -\mathbf{r}(v) + |\beta| \sinh u \mathbf{T}(v) \end{aligned} \tag{7}$$

respectively are Bonnet pairs. Here  $\beta \neq 0$  and  $D(v)$  is an arbitrary function. In [4], the cases  $D(v) = \text{const.}$  and  $D(v) = 0$  are indicated. In the case  $D(v) = \text{const.}$  the Bonnet pairs are obtained that

$$\begin{aligned} \mathbf{X}(u, v) = & \left( a \cos \frac{v}{\sqrt{|b|}} + \varepsilon \frac{a^2 + b^2}{b} \sqrt{|b|} \sinh u \sin \frac{v}{\sqrt{|b|}}, \right. \\ & a \sin \frac{v}{\sqrt{|b|}} - \varepsilon \frac{a^2 + b^2}{b} \sqrt{|b|} \sinh u \cos \frac{v}{\sqrt{|b|}}, \\ & \left. \varepsilon \sqrt{|b|} v + \frac{a(a^2 + b^2)}{b \sqrt{|b|}} \sinh u \right) \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbf{Y}(u, v) = & \left( -a \cos \frac{v}{\sqrt{|b|}} + \varepsilon \frac{a^2 + b^2}{b} \sqrt{|b|} \sinh u \sin \frac{v}{\sqrt{|b|}}, \right. \\ & -a \sin \frac{v}{\sqrt{|b|}} - \varepsilon \frac{a^2 + b^2}{b} \sqrt{|b|} \sinh u \cos \frac{v}{\sqrt{|b|}}, \\ & \left. -\varepsilon \sqrt{|b|} v + \frac{a(a^2 + b^2)}{b \sqrt{|b|}} \sinh u \right) \end{aligned}$$

where  $\varepsilon = \text{sgn}(b) = \text{sgn}(\beta)$ . In the case  $D(v) = 0$ ,

$$\begin{aligned} \mathbf{X}(u, v) &= (|\beta| \sinh u \cos v, |\beta| \sinh u \sin v, \beta v) \\ \mathbf{Y}(u, v) &= (|\beta| \sinh u \cos v, |\beta| \sinh u \sin v, -\beta v) \end{aligned} \tag{9}$$

are found. These solutions are valid for the non-developable surfaces ( $\beta \neq 0$ ) [2]. In [3], the tangential developable surfaces of the circular helices ( $\beta = 0$ ) are given as an example of the ruled Bonnet pairs. These surfaces are calculated as

$$\begin{aligned} \mathbf{X}(u, v) &= \left( a \cos \frac{v}{\sqrt{a^2 + b^2}} - \frac{au}{\sqrt{a^2 + b^2}} \sin \frac{v}{\sqrt{a^2 + b^2}}, \right. \\ & a \sin \frac{v}{\sqrt{a^2 + b^2}} + \frac{au}{\sqrt{a^2 + b^2}} \cos \frac{v}{\sqrt{a^2 + b^2}}, \\ & \left. \frac{bv}{\sqrt{a^2 + b^2}} + \frac{bu}{\sqrt{a^2 + b^2}} \right) \end{aligned} \tag{10}$$

$$\begin{aligned} \mathbf{Y}(u, v) &= \left( a \cos \left( \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) + \frac{au}{a^2 + b^2} \sin \left( \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right), \right. \\ & -a \sin \left( \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) + \frac{au}{a^2 + b^2} \cos \left( \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right), \\ & \left. \frac{-b(v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2}))}{\sqrt{a^2 + b^2}} + \frac{bu}{\sqrt{a^2 + b^2}} \right) \end{aligned}$$

All ruled Bonnet pairs which have been known up to now, can be taken the equations (7), (8), (9), (10).

### 2.2. Some Orthogonal Ruled Surfaces

If two surfaces  $S(\mathbf{a})$  and  $S(\mathbf{b})$  given by the equations  $\mathbf{a} = \mathbf{a}(u, v)$  and  $\mathbf{b} = \mathbf{b}(u, v)$  map upon each other such that their linear elements are orthogonal at corresponding points,  $d\mathbf{a} \cdot d\mathbf{b} = 0$ , then, these surfaces are called to be orthogonal surfaces. Now, we recall the relationship between the orthogonal surfaces and the isometric surfaces: if two surfaces  $S(\mathbf{x})$  and  $S(\mathbf{y})$  given by the equations  $\mathbf{x} = \mathbf{x}(u, v)$  and  $\mathbf{y} = \mathbf{y}(u, v)$  are isometric ( $dx^2 = dy^2$ ), two surfaces written as  $S(\mathbf{x} + \mathbf{y})$  and  $S(\mathbf{x} - \mathbf{y})$  are orthogonal surfaces  $d(\mathbf{x} + \mathbf{y}) \cdot d(\mathbf{x} - \mathbf{y}) = 0$ . By using two surfaces  $S(\mathbf{x})$  and  $S(\mathbf{y})$  which can be map isometrically on the each other, the orthogonal surfaces  $S(\mathbf{a})$  and  $S(\mathbf{b})$  given by the equations

$$\mathbf{a} = \mathbf{x} + \mathbf{y}, \quad \mathbf{b} = \mathbf{x} - \mathbf{y} \tag{11}$$

can be written. By using this method, we can get the orthogonal surfaces easily. For this aim, we can use the ruled surfaces given by the equations (7), (8), (9), (10) to find orthogonal

ruled surfaces. But the process of using the ruled surfaces is not as easy as the general process. Since the equation of a ruled surface consists of two parts  $(\mathbf{X}(u, v) = \mathbf{r}(v) + u\mathbf{T}(v))$ , mostly, one of the first  $(\mathbf{r}(v))$  or second  $(\mathbf{T}(v))$  part can vanish and the surface becomes a curve. Because of this, only one pairs of orthogonal surfaces given by

$$\begin{aligned}
 \mathbf{x} &= a \left( \cos \frac{v}{\sqrt{a^2 + b^2}} + \cos \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) + \\
 &\quad + \frac{au}{\sqrt{a^2 + b^2}} \left( -\sin \frac{v}{\sqrt{a^2 + b^2}} + \sin \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) \\
 \mathbf{y} &= a \left( \sin \frac{v}{\sqrt{a^2 + b^2}} - \sin \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) + \\
 &\quad + \frac{au}{\sqrt{a^2 + b^2}} \left( \cos \frac{v}{\sqrt{a^2 + b^2}} + \cos \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) \\
 \mathbf{z} &= -\frac{\frac{2b(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} + \frac{2bu}{\sqrt{a^2 + b^2}} \tag{12}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{x} &= a \left( \cos \frac{v}{\sqrt{a^2 + b^2}} - \cos \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) - \\
 &\quad - \frac{au}{\sqrt{a^2 + b^2}} \left( \sin \frac{v}{\sqrt{a^2 + b^2}} + \sin \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) \\
 \mathbf{y} &= a \left( \sin \frac{v}{\sqrt{a^2 + b^2}} + \sin \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) + \\
 &\quad + \frac{au}{\sqrt{a^2 + b^2}} \left( \cos \frac{v}{\sqrt{a^2 + b^2}} - \cos \frac{v + \frac{2(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}} \right) \\
 \mathbf{z} &= \frac{2bv + \frac{2b(a^2+b^2)^{\frac{3}{2}}}{a} \arctan(\frac{au}{a^2+b^2})}{\sqrt{a^2 + b^2}}
 \end{aligned}$$

can be found.

**Theorem 1.** *By using the Bonnet Ruled surfaces; only one pair of orthogonal surfaces can be found. This surfaces are generated by the tangential developable surfaces of the circular helices.*

The infinitesimal bending problem is reduced the finding them [5, 7, 8]: the equation

$$\mathbf{x}_t(u, v, t) = \mathbf{x}(u, v) + t\mathbf{y}(u, v) \quad (13)$$

gives the infinitesimal bending surfaces of the surface  $S(\mathbf{x})$  under the condition

$$d\mathbf{x}_t^2(u, v, t) = (d\mathbf{x}(u, v) + t d\mathbf{y}(u, v))^2 \quad (14)$$

where  $t$  is a real infinitesimal parameter. Here the condition (14) is equivalent to the condition  $d\mathbf{x} \cdot d\mathbf{y} = 0$ . Consequently, we give the following corollary:

**Corollary 1.** *One of the surfaces given by the equation (12) can be infinitesimal deformation by using the other one.*

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