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Stability Results for Set Differential Equations Involving Causal Operators with Memory

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Abstract. In this paper we study the stability concepts for set differential equations involving causal operators with memory by considering initial functions as a Hukuhara difference of two functions. This will enable to obtain results parallel to ordinary differential equations with delay.

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1. Introduction

It is well recognized and accepted that set differential equations are a generalization of ordinary differential equations and vector differential equations in a semilinear metric space and that they are useful in studying multivalued differential inclusions or multivalued differential equations.

Also causal operators or Volterra operators or non anticipative operators encompass a wide range of equations such as ordinary differential equations, integral equations, integro differential equations, to name a few.

Thus set differential equations involving causal operators with memory include the above said special cases for various types of equations and such a generalization is interesting as it gives a comprehensive view of different types of differential equations.

Also it is observed in [1] that solutions for set differential equations contain a lot of undesirable information that needs to be seperated, so that the equations in this setup satisfy the stability behaviour similar to that of scalar or vector equations. In order to take care of this situation, the Hukuhara difference in initial values is introduced.

In this paper we extend results in [1] to set differential equations involving causal operators with memory by considering the Hukuhara difference of initial functions. We obtain stability results using Lyapunov-like functions and the concept of minimal classes.

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2. Preliminaries

We begin with the definitions of $K_c(\mathbb{R}^n)$, the semilinear space in which we work. We next define the Hausdorff Metric, the Hukuhara difference, the Hukuhara derivative and the Hukuhara Integral. We also state all the important properties that are useful in this paper. We further define a partial order in $K_c(\mathbb{R}^n)$. We also state all the required results developed in [4] that will be used in this paper.

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Define the Hausdorff metric by

$$D[A,B] = \max[\sup_{x\in B} d(x,A), \quad \sup_{y\in A} d(y,B)], \tag{1}$$

where $d(x,A) = \inf[d(x,y) : y \in A]$, A,B are bounded sets in $\mathbb{R}^n$. We note that $K_c(\mathbb{R}^n)$ with this metric is a complete metric space.

It is known that if the space $K_c(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and non-negative scalar multiplication, then $K_c(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space.

The Hausdorff metric (1) satisfies the following properties:

$$D[A+C,B+C] = D[A,B] \text{ and } D[A,B] = D[B,A], \tag{2}$$

$$D[\lambda A, \lambda B] = \lambda D[A,B], \tag{3}$$

$$D[A,B] \le D[A,C] + D[C,B], \tag{4}$$

for all $A,B,C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$.

Let $A,B \in K_c(\mathbb{R}^n)$. The set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F : I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$\lim_{h\to 0+} \frac{F(t_0+h) - F(t_0)}{h} \text{ and } \lim_{h\to 0+} \frac{F(t_0) - F(t_0-h)}{h}$$

exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in $\mathbb{R}$.

With these preliminaries, we consider the set differential equation

$$D_H U = F(t,U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n), \quad t_0 \ge 0, \tag{5}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$.

The mapping $U \in C^1[J, K_c(\mathbb{R}^n)], J = [t_0, t_0 + a]$ is said to be a solution of (5) on J if it satisfies (5) on J.

Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s) ds, \quad t \in J. \tag{6}$$

Hence, we can associate with the IVP (5) the Hukuhara integral

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s))ds, \quad t \in J. \tag{7}$$

where the integral is the Hukuhara integral which is defined as,

$$\int F(s)ds = \{\int f(s)ds : f \ is \ any \ continuous \ selector \ of \ F\}$$

Observe also that U(t) is a solution of (5) on J iff it satisfies (7) on J.

We now proceed to define a partial order in the metric space $(K_c(\mathbb{R}^n), D)$. We begin with the definition of a cone in this set up.

Let $K(K^o)$ be the subfamily of $K_c(\mathbb{R}^n)$ consisting of set $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a non-negative(positive) vector of n components satisfying $u_i \geq 0$ $(u_i > 0)$ for $i = 1 \ldots n$. Then K is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K.

Definition 1. *For any U and $V \in K_c(\mathbb{R}^n)$, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and $U = V + Z$ then we say that $U \geq V$ $(U > V)$. Similarly we can define $U \leq V$ $(U < V)$.*

To define the causal operator we introduce the following notation.
Let $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ and $E_0 = C[[t_0 - h_1, T], K_c(\mathbb{R}^n)]$, where $U \in E_0$ implies $U(t) = \Phi_0(t),\ t_0 - h_1 \leq t \leq t_0$ and U(t) is any arbitrarily continuous function on $[t_0, T]$.

We define a norm on E as follows: for $U, V \in E$

$$D_0[U, V] = Sup_{t_0 \leq t \leq T} D[U(t), V(t)]$$

where D denotes the Hausdorff Metric.

Definition 2. *By a causal operator or a Volterra operator or a nonanticipative operator we mean a mappling Q: $E \to E$ satisfying the property that if $U(s) = V(s),\ t_0 \leq s \leq t < T$ then $(QU)(s) = (QV)(s),\ t_0 \leq s \leq t < T$.*
By a causal operator with memory we mean a mapping Q:$E_0 \to E$ such that for $U(s) = V(s)$, $t_0 \leq s \leq t < T$, then $Q(U, \Phi_0)(s) = Q(V, \Phi_0)(s),\ t_0 \leq s \leq t < T$ and $\Phi_0 \in C_1 = C[[t_0 - h_1, t_0],\ K_c(\mathbb{R}^n)]$

3. Comparison Theorems

In this section we give the necessary notations and the comparison theorems required to prove the stability theorems.

Consider the IVP for set differential equation involving causal operators with memory given by

$$D_H U(t) = (QU)(t), \quad U_{t_0} = \Phi_0 \in C_1 \tag{8}$$

where $C_1 = C[[t_0 - h_1, t_0], K_c(\mathbb{R}^n)]$ with metric

$$D_1[\phi_0, \psi_0] = \sup_{t_0 - h_1 \le s \le t_0} D[\phi_0(s), \psi_0(s)], \quad \phi_0, \psi_0 \in C_1$$

Let $\tilde{E} = C[[t_0 - h_1, \infty), K_c(\mathbb{R}^n)]$ with norm

$$\tilde{D}[U, \theta] = \sup_{t_0 - h_1 \le t < \infty} \frac{D[U(t), \theta]}{h(t)}$$

where θ is the zero element in $\mathbb{R}^n$, which is regarded as a point set and $h : [t_0, \infty) \to \mathbb{R}_+$ is a continuous map and $(\tilde{E}, \tilde{D})$ is a Banach space. In this paper we consider the causal operator with memory as $Q \in C[\tilde{E}, \tilde{E}]$ such that $Q : \tilde{E} \to \tilde{E}$ and $U(s) = V(s)$ for $t_0 - h_1 \le s \le t$ implies $(QU)(s) = (QV)(s)$ for $t_0 - h_1 \le s \le t$.

In order to develop the comparison theorems using Lyapunov-like functions it is useful to select some class of functions in $K_c(\mathbb{R}^n)$ or elements in $\tilde{E}$ such that the generalized derivative of the Lyapunov function satisfies certain conditions on these classes. We begin by defining the following sets.

$$E_0 = \{U \in \tilde{E} : L(s, U(s)) \le f(L(t, U(t));\ t_1 \le s \le t, t_1 \ge t_0\};$$
$$E_1 = \{U \in \tilde{E} : L(s, U(s)) \le L(t, U(t));\ t_0 \le s \le t\};$$
$$E_\alpha = \{U \in \tilde{E} : L(s, U(s))\alpha(s) \le L(t, U(t))\alpha(t);\ t_0 \le s \le t\};$$

where

(i) $\alpha(t) \ge 0$ is a continuous function on $\mathbb{R}_+$,

(ii) $f(r)$ is continuous on $\mathbb{R}_+$, non decreasing in r and $f(r) \ge r$ for $r \ge 0$.

Now we proceed to state the comparison theorems using the Lyapunov-like functions. As the proofs are similar to that in [1] we omit them.

Theorem 1. *Let* $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$, $B = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \le \rho\} = B(\theta, \rho)$ *and let L(t,U) be locally Lipschitzian in U (i.e) for* $U, V \in B, t \in \mathbb{R}_+$ *and* $K > 0$,

$$|L(t, U) - L(t, V)| \le K\tilde{D}[U, V] \tag{9}$$

(i) Assume that for $t \ge t_0$ *and* $U \in E_1$

$$D_-L(t, U(t)) \le g(t, L(t, U(t))) \tag{10}$$

where

$$D_-L(t, U(t)) = \liminf_{h \to 0^-} \frac{1}{h}[L(t + h, U(t) + h(QU)(t)) - L(t, U(t)]$$

and $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$

(ii) Let $r(t) = r(t, t_0, W_0)$ be the maximal solution of scalar ordinary differential equation

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0 \tag{11}$$

existing on $t_0 \leq t < \infty$.

Let $U(t, t_0, \phi_0(t_0))$ be any solution of the system (8) such that $U(t, t_0, \phi_0(t_0)) \in B$ for $t \in [t_0, t_1]$ and let $L(t_0, \phi_0(t_0)) \leq w_0$ then

$$L(t, U(t, t_0, \phi_0(t_0)) \leq r(t), \quad \forall t \in [t_0, t_1], t \geq t_0.$$

Theorem 2. *Assume that the hypothesis of Theorem 1 holds, except for inequality (10) which is replaced by*

$$\alpha(t)D_-L(t, U(t)) + L(t, U(t))D_-\alpha(t) \leq w(t, L(t, U(t))\alpha(t)), \tag{12}$$

for $t > t_0, U \in E_\alpha$, where $\alpha(t) > 0$ is continuous on $\mathbb{R}_+$ and

$$D_-\alpha(t) = \liminf_{h \to o^-} \frac{\alpha(t+h) - \alpha(t)}{h}$$

then $\alpha(t_0)L(t_0, \phi_0) \leq w_0$ implies $\alpha(t)L(t, U(t)) \leq r(t)$, $t \geq t_0$.

Theorem 3. *Assume that*

(i) $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ and $L(t, U)$ be locally Lipschitzian in U.

(ii) $g_0, g \in C[\mathbb{R}_+^2, \mathbb{R}]$ such that $g_0(t, w) \leq g(t, w), (t, w) \in {\mathbb{R}_+}^2$, and $\eta(t, T_0, v_0)$ is the left maximal solution of

$$v' = g_0(t, v), \quad v(T_0) = v_0 \tag{13}$$

existing on $t_0 \leq t \leq T_0$ and $r(t, t_0, w_0)$ is the right maximal solution of

$$w' = g(t, w), \quad w(t_0) = w_0 \tag{14}$$

existing on $[t_0, \infty)$

(iii)

$$D_-L(t, U(t)) \leq g(t, L(t, U(t))) \tag{15}$$

on Ω where $\Omega = \{U \in \tilde{E} : L(s, U(s)) \leq \eta(s, t, L(t, U(t))),\ t_0 \leq s \leq t\}$ then we have $L(t, U(t, t_0, \phi_0(t_0)) \leq r(t, t_0, w_0)$, $t \geq t_0$ whenever $L(t_0, \phi_0(t_0)) \leq w_0$.

In the proof of Theorem 3 we use the Lemma 1.4.1 in [2].

4. Stability Results

In this section we study the stability properties of the solution of (8) in order to do so, we assume that the solutions of (8) exist for $t \geq t_0$ and are unique.

It is to be noted that if we have to study the stability properties of any solution $U(t, t_0, \phi_0)$ of (8). Then we have to find two functions Z and V such that the Hukuhara difference $Z = U - V$ exists and $D_H Z = D_H U - D_H V$ exists and further $Q(0)(t) \equiv 0$ for all t. Thus if all the above conditions are satisfied then studying the stability properties of any solution $U(t, t_0, \phi_0)$ of (8) reduces to the study of zero solution of (8).

We observe that in the generation of set differential equation involving causal operators with memory from ordinary differential equations involving causal operators with memory, certain undesirable elements may enter the solution $U(t)$ of (8). In order that the solutions of (8) project the behaviour of ordinary differential equations involving causal operator with memory, from which they can be generated, we introduce the concept of Hukuhara difference in initial functions. Before we introduce the theory, we consider the following example.

Example 1. *Consider the set differential equation with delay on* $\mathbb{R}$.

$$D_H U = -U(t - \tau), \quad U_0 = [\phi_1, \phi_2] \tag{16}$$

where ϕ_1, ϕ_2 *are real valued functions and* $U(t) = [u_1(t), u_2(t)]$

$$\begin{aligned} U'(t) =& [u_1'(t), u_2'(t)] \\ =& -U(t - \tau) \\ =& -[u_1(t - \tau), u_2(t - \tau)] \\ =& [-u_2(t - \tau), -u_1(t - \tau)] \end{aligned}$$

therefore

$$u_1' = -u_2(t - \tau) \text{ and } u_2' = -u_1(t - \tau) \tag{17}$$

where $u_1(0) = \phi_1(0),\ u_2(0) = \phi_2(0)$

$$u_1'' = u_1(t - 2\tau) \text{ and } u_2'' = u_2(t - 2\tau), -2\tau \leq t \leq 0. \tag{18}$$

Suppose $e^{\lambda_1 t}$ *and* $e^{-\lambda_2 t}$ *are solutions of the system (18). Let* $u_1(t) = e^{\lambda_1 t}$ *and* $u_2(t) = e^{-\lambda_2 t}$

$$\begin{aligned} u_1'(t) &= \lambda_1 e^{\lambda_1 t} & u_2'(t) &= -\lambda_2 e^{-\lambda_2 t} \\ u_1''(t) &= \lambda_1^2 e^{\lambda_1 t} & u_2''(t) &= \lambda_2^2 e^{-\lambda_2 t} \\ u_1(t - 2\tau) &= e^{\lambda_1 (t - 2\tau)} & u_2(t - 2\tau) &= e^{-\lambda_2 (t - 2\tau)} \end{aligned}$$

since $u_1(t)$ *and* $u_2(t)$ *are solutions of the system (18) we have to choose* $\lambda_1, \lambda_2 > 0$ *such that* $\lambda_1^2 = e^{-2\lambda_1 \tau}, \lambda_2^2 = e^{2\lambda_2 \tau}$*, then* $u_1(t) = e^{\lambda_1 t}$ *and* $u_2(t) = e^{-\lambda_2 t}$ *are solutions of the system (18). As a result we have*

$$u_1(t) = C_1 e^{\lambda_1 t} + C_2 e^{-\lambda_2 t} \text{ and } u_2(t) = C_3 e^{\lambda_1 t} + C_4 e^{-\lambda_2 t}. \tag{19}$$

Suppose the initial functions are given by

$$\Phi_1(s) = \frac{1}{2}[u_{10} - u_{20}]e^{\lambda_1 s} + \frac{1}{2}[u_{10} + u_{20}]e^{-\lambda_2 s} \tag{20}$$

$$\Phi_2(s) = \frac{1}{2}[u_{20} - u_{10}]e^{\lambda_1 s} + \frac{1}{2}[u_{10} + u_{20}]e^{-\lambda_2 s} \tag{21}$$

for $-2\tau \leq s \leq 0$*. By using the relations (17), (19), and (20) we have to compute the values* C_1, C_2, C_3 *and* C_4*. Taking* $t = 0$ *and* $t = \tau$ *we get*

$$C_1 = \frac{1}{2}[u_{10} - u_{20}], \quad C_2 = \tfrac{1}{2}[u_{10} + u_{20}],$$
$$C_3 = \frac{1}{2}[u_{20} - u_{10}], \quad C_4 = \tfrac{1}{2}[u_{10} + u_{20}].$$

Therefore

$$U(t) = [-\frac{1}{2}[u_{20} - u_{10}], \frac{1}{2}[u_{20} - u_{10}]]e^{\lambda_1 t} + [\frac{1}{2}[u_{10} + u_{20}], \frac{1}{2}[u_{10} + u_{20}]]e^{-\lambda_2 t}$$

Choose

$$\psi_0 = [-\frac{1}{2}[u_{20} - u_{10}], \frac{1}{2}[u_{20} + u_{10}]]e^{\lambda_1 s}$$

and

$$\chi_0 = [\frac{1}{2}[u_{10} + u_{20}], \frac{1}{2}[u_{10} + u_{20}]]e^{-\lambda_2 s}$$

for $\Phi_0 = [\Phi_1, \Phi_2] = \psi_0 + \chi_0$*. Then* $U(t, t_0, \chi_0) = [\frac{1}{2}[u_{10} + u_{20}], \frac{1}{2}[u_{10} + u_{20}]]e^{-\lambda_2 s}$ *which implies the stability of the trivial solution of the equation (16).*

We suppose that given any solution $U(t, t_0, \Phi_0)$ of (8), we can find a function V such that the Hukuhara difference $U - V$ exists, satisfies the properties mentioned earlier in this context so that it is sufficient to study the stability properties of the trivial solution. Other notations of Lyapunov stability can be formulated in a similar way following the standard stability definitions in [3]. We now present the stability theorems in this context.

Theorem 4. *Assume that there exist functions* $L(t, U(t))$ *and* $g(t, w)$ *satisfying the following conditions*

(i) $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ *and* $g(t, 0) \equiv 0$*;*

(ii) $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ *where* $B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\}$*;* $L(t, \theta) \equiv 0$ *and* $L(t, U)$ *is positive definite and locally Lipschitzian in* U*;*

(iii) for $t > t_0$ *and* $U \in E_1$

$$D_{-}L(t, U(t)) \leq g(t, L(t, U(t)))$$

then the stability of the zero solution of

$$w' = g(t,w) \quad w(t_0) = w_0 \tag{22}$$

implies the stability of the zero solution of

$$D_H U(t) = (QU)(t) \quad U_{t_0} = \chi_0 \tag{23}$$

where $\Phi_o = \psi_0 + \chi_0$.

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Since $L(t,U)$ is positive definite, it follows that there exists a function $b \in K$ such that

$$b(\tilde{D}[U,\theta]) \leq L(t,U) \text{ for } (t,U) \in \mathbb{R}_+ \times B \tag{24}$$

Suppose that the solution of the system (22) is stable. Then, given $b(\epsilon) > 0$, $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that when ever $w_0 < \delta$, we have $w(t,t_0,w_0) < b(\epsilon)$, $t \geq t_0$, where $w(t,t_0,w_0)$ is any solution of the system (22). Choose $w_0 = L(t_0, \chi_0)$.

Since$L(t,U)$ is continuous and $L(t,\theta) \equiv 0$, there exist function $\delta_1 = \delta_1(t_0,\epsilon) > 0$ such that $D_1[\chi_0,\theta] \leq \delta_1$ and $L(t_0,\chi_0) \leq \delta$ holds simultaneously.

We claim that if $D_1[\chi_0,\theta] \leq \delta_1$, then $\tilde{D}[U(t),\theta] < \epsilon$ for all $t \geq t_0$.
Suppose this is not true. Then there exists a solution $U(t) = U(t,t_0,\chi_0)$ of the system (23) satisfying the properties $\tilde{D}[U(t_2),\theta] = \epsilon$ and $\tilde{D}[U(t),\theta] < \epsilon$ for $t_0 < t < t_2 < \infty$. From (24)

$$b(\epsilon) \leq L(t_2, U(t_2)) \tag{25}$$

Furthermore, $U(t) \in B$ for $t \in [t_0,t_2]$. Hence, the choice of $w_0 = L(t_0,\chi_0)$ and condition (iii) gives, as a consequence of Theorem 1, the estimate

$$L(t,U(t)) \leq r(t), \quad t \in [t_0,t_2], \tag{26}$$

where $r(t) = r(t,t_0,w_0)$ is the maximal solution of the comparison equation (22). Now from equation (25)

$$b(\epsilon) \leq L(t_2,U(t_2)) \leq r(t_2) < b(\epsilon),$$

which is a contradiction. Hence the zero solution of the system (23) is stable.

The following theorem provides sufficient conditions for asymptotic stability of the system (23).

Theorem 5. *Assume that*

(i) there exist functions $L(t,U), g(t,w)$ *satisfying the conditions of Theorem 4;*

(ii) there exists a function $\alpha(t)$ *such that* $\alpha(t) > 0$ *is continuous for* $t \in \mathbb{R}_+$ *and* $\alpha(t) \longrightarrow \infty$ *as* $t \longrightarrow \infty$*. Further assume that the relation*

$$\alpha(t)D_-L(t,U(t)) + L(t,U(t))D_-\alpha(t) \leq w(t,L(t,U(t))\alpha(t))$$

for $t \geq t_0, U \in E_\alpha$ then, if the zero solution of the system (22) is stable then the zero solution of the system (23) is asymptotically stable.

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given.Set $\alpha_0 = min_{t\in\mathbb{R}_+}\alpha(t)$, then $\alpha_0 > 0$ follows from the assumption (ii). Since $L(t,U)$ is positive definite, there exists $b \in K$ such that equation (24) holds.

$$\epsilon_1 = \alpha_0 b(\epsilon), \tag{27}$$

where $\epsilon > 0$. Then the stability of the zero solution (22) implies that, given $\epsilon_1 > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta = \delta(\epsilon_1, t_0)$ such that $w_0 < \delta \implies w(t,t_0,w_0) < \epsilon_1$, $t \geq t_0$, where $w(t,t_0,w_0)$ is any solution of (23). Choose $W_0 = L(t_0,\chi_0)$, then proceeding as in the Theorem 4 with ϵ_1 insted of $b(\epsilon)$, we can prove that the zero solution of (23) is stable.
Let $U(t,t_0,\chi_0)$ be any solution of the system (23) such that

$$D_1[\chi_0,\theta] \leq \delta_0$$

where $\delta_0 = \delta(t_0, \frac{1}{2}\rho)$. Since the zero solution of the system (23) is stable, it follows that

$$\tilde{D}[U(t),\theta] < \frac{1}{2}\rho,\ t \geq t_0.$$

since $\alpha(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, there exists a number $T = T(t_0,\epsilon) > 0$ such that

$$b(\epsilon)\alpha(t) > \epsilon_1, \quad t \geq t_0 + T \tag{28}$$

Now from the Theorem 2 and the relation (24), we get

$$\alpha(t)b(\tilde{D}[U,\theta]) \leq \alpha(t)L(t,U) \leq r(t), \quad t \geq t_0$$

Where $U(t) = U(t,t_0,\chi_0)$ is any solution of (23) such that $D_1[\chi_0,\theta] \leq \delta_0$.

If the zero solution of the system (23) is not asymptotically stable, then there exists a sequence $\{t_k\}$, $t_k \geq t_0 + T$ and $t_k \longrightarrow \infty$ as $k \to \infty$ such that $\tilde{D}[U(t_k),\theta] \geq \epsilon$ for some solution $U(t)$ satisfying $D_1[\chi_0,\theta] \leq \delta_0$.
Now

$$\begin{aligned} b(\epsilon)\alpha(t) &\leq b(\tilde{D}[U(t_k),\theta])\alpha(t) \\ &\leq \alpha(t_k)L(t_k,U(t_k)) \\ &\leq r(t_k) \\ &< \epsilon_1 \text{ for } t_k \geq t_0 + T \end{aligned}$$

which is a contraduction. Hence the solution of IVP (23) is asymptotically stable and the proof is complete.

The next theorem gives sufficient conditions for the uniformly asymptotic stablity of IVP (23). As the proofs are similar to that in [1] we omit them.

Theorem 6. *Assume that there exists a function $L(t,U)$ satisfies the following properties*

(i) $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ *where* $B = B(\theta,\rho)$, $L(t,U)$ *is positive definite, decrescent and locally Lipschitzian in* U,

(ii)

$$D_{-}L(t,U(t)) < -c(\tilde{D}[U(t),\theta]) \tag{29}$$

for $t \geq t_0$, $U \in E_0$ *and* $c \in K$ *then the zero solution of (23) is uniformly assymptotically stable.*

Our final stability result is a general result which offers various stability criteria in a single setup. The proof of this theorem, which can be obtained using the comparison results given in Theorem 3, is omitted.

Theorem 7. *Assume that there exists a function $L(t,U)$ satisfying properties (i), (ii) and (iii) of the Theorem 3 then the stability properties of the zero solution of IVP (22) implies the corresponding properties of the zero solution of IVP (23).*

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