EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 5, No. 3, 2012, 302-316
ISSN 1307-5543 – www.ejpam.com

A Comparison on Metric Dimension of Graphs, Line Graphs, and Line Graphs of the Subdivision Graphs

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Abstract. The *line graph* $L(G)$ of a simple graph G is the graph whose vertices are in one-to-one correspondence with the edges of G; two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. If $S(G)$ is the *subdivision graph* of a graph G, then the *para-line graph* G^* of G is $L(S(G))$. The *metric dimension* $\dim(G)$ of a graph G is the minimum cardinality of a set of vertices such that every vertex of G is uniquely determined by its vector of distances to the chosen vertices. In this paper, we study metric dimension of para-line graphs; we also compare metric dimension of graphs, line graphs, and para-line graphs. First, we show that $\lceil \log_2 \Delta(G) \rceil \le \dim(G^\star) \le n-1$, for a simple and connected graph G of order $n \ge 2$ with the maximum degree $\Delta(G)$, where both bounds are sharp. Second, we determine the metric dimension of para-line graphs for some classes of graphs; further, we give an example of a graph G such that $\max\{\dim(G), \dim(L(G)), \dim(G^\star)\}$ equals $\dim(G)$, $\dim(L(G))$, and $\dim(G^\star)$, respectively. We conclude this paper with some open problems.

2010 Mathematics Subject Classifications: 05C12

Key Words and Phrases: line graph, para-line graph, line graph of the subdivision graph, distance, resolving set, metric dimension

1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple, undirected, and connected graph of order $|V(G)| = n \ge 2$. For a graph G and $W \subseteq V(G)$, we denote by $\langle W \rangle$ the subgraph induced by W. For a vertex $v \in V(G)$, the *open neighborhood of* v is the set $N_G(v) = \{u \mid uv \in E(G)\}$, and the *closed neighborhood of* v is the set $N_G[v] = N_G(v) \cup \{v\}$; for $S \subseteq V(G)$, the *open neighborhood of* S is the set $N_G(S) = \cup_{v \in S} N_G(v)$. The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the the number of edges incident to the vertex v in G; an *end-vertex* is a vertex of degree one. We denote by $\Delta(G)$ the maximum degree of a graph G. The *distance* between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is the length of the shortest path in G between u and v; we omit G when ambiguity is not a concern. The diameter, $diam(G)$, of a graph G is given by $\max\{d(u, v) \mid u, v \in V(G)\}$. We denote by K_n, C_n, and P_n the complete graph, the

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cycle, and the path on n vertices, respectively. For other terminologies in graph theory, refer to [6].

The *subdivision graph* $S(G)$ of a graph G is obtained from G by deleting every edge uv of G and replacing it by a vertex w of degree 2 that is joined to u and v [see p.151 of 6]. The *line graph* $L(G)$ of a simple graph G is the graph whose vertices are in one-to-one correspondence with the edges of G; two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent [see 2, 18, 28, 29]. Following [25], we define the *para-line graph* of G to be $L(S(G))$, which we will denote by $G^{\star}$. Alternatively, we can construct $G^{\star}$ from G as follows:

(i) Replace each vertex $u \in V(G)$ by $K_{(u)}$, the complete graph on $\deg_G(u)$ vertices;

(ii) There is an edge joining a vertex of $K_{(u_1)}$ and a vertex of $K_{(u_2)}$ in $G^{\star}$ if and only if there is an edge joining u_1 and u_2 in G;

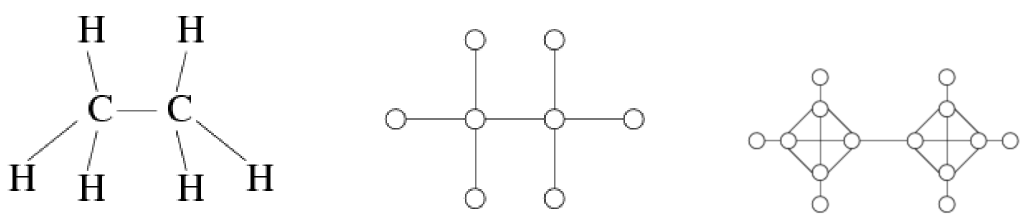
(iii) For each vertex v of $K_{(u)}$, $\deg_{G^{\star}}(v) = \deg_G(u)$.

A vertex $x \in V(G)$ *resolves* a pair of vertices $u, v \in V(G)$ if $d(u,x) \neq d(v,x)$. A set of vertices $S \subseteq V(G)$ *resolves* G if every pair of distinct vertices of G is resolved by some vertex in S; then S is called a *resolving set* of G. For an ordered set $S = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ of distinct vertices, the *metric code* (or *code*, for short) of $v \in V(G)$ with respect to S, denoted by $code_S(v)$, is the k-vector $(d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G. Slater [26, 27] introduced the concept of a resolving set for a connected graph under the term *locating set*. He referred to a minimum resolving set as a *reference set*, and the cardinality of a minimum resolving set as the *location number* of a graph. Independently, Harary and Melter [13] studied these concepts under the term *metric dimension*. Since metric dimension is suggestive of the dimension of a vector space in linear algebra, sometimes a minimum resolving set of G is called a *basis* of G. Metric dimension as a graph parameter has numerous applications, among them are robot navigation [17], sonar [27], combinatorial optimization [23], and pharmaceutical chemistry [5]. It is noted in [12] that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been heavily studied; for surveys, see [1] and [7]. For more articles on metric dimension in graphs, see [3, 4, 8, 9, 14, 16, 19, 24].

In this paper, we study metric dimension of para-line graphs; we also compare metric dimension of graphs, line graphs, and para-line graphs. For a simple and connected graph G of order $n \geq 2$, we show that $\lceil \log_2 \Delta(G) \rceil \leq \dim(G^{\star}) \leq n-1$, and the bounds are sharp. We also determine the metric dimension of para-line graphs for some classes of graphs; further, we give an example of a graph G such that $\max\{\dim(G), \dim(L(G)), \dim(G^{\star})\}$ equals $\dim(G)$, $\dim(L(G))$, and $\dim(G^{\star})$, respectively. We conclude this paper with some open problems.

2. Para-Line Graphs: Applications and Basic Properties

Para-line graphs are chemically relevant, though little considered in the surge of chemical graph theory of the last few decades. Often a (usually "organic") molecule is represented by a graph whose vertices correspond to atoms other than hydrogen, and whose edges correspond to bonded pairs of such atoms. For example, for stable hydrocarbon molecules, it is understood that the carbon atoms have valence 4 counting connections to H atoms, so that this H-deleted graph determines the molecular structure (see Figure 1). But there are other ways to represent molecules in terms of graphs, say in terms of a minimal set of localized "orbitals" each taken as a vertex, with edges identified to stronger interactions between pairs of orbitals. Indeed such a graph was implicit in several early quantum chemical works presaging the first-principles quantum chemistry mediated by way of large computers. This now standard quantum chemical approach has been tremendously successful in treating molecules one by one. But the simpler orbital model offers a potential advantage of general meaningful theorems applying to whole classes of molecules. One can view para-line graphs with vertices corresponding to atomic "hybrid" orbitals, and edges corresponding to stronger interactions between pairs of such orbitals. Given a traditional molecular structural formula for a hydrocarbon without multiple bonds, there corresponds a graph G with one vertex corresponding to each atom (either C or H), and an edge corresponding to each bond.



(a) Hydrocarbon C_2H_6 (b) Hydrocarbon graph G (c) its para-line graph $G^\star$

Figure 1

If we let $E^\star_{in} = \cup_{u\in V(G)} E(K_{(u)})$, then the usual chemical bonds are manifested in $E^\star_{bond} = E(G^\star) - E^\star_{in}$. For a hydrocarbon, each C atom is represented by a tetrahedral quartet of interconnected sites (a 4-vertex clique), each vertex corresponding to a different (so-called sp) hybrid orbital; each H atom by a single atomic (1s) orbital; each edge of $E^\star_{bond}$ corresponding to an interatomic chemical molecular bond; and each edge of $E^\star_{in}$ by an intra-atomic interaction. Often, the vertices of the resulting para-line graph are assigned different weights corresponding to C or H atoms; also, it's typically given different edge weights corresponding to the C−C and C−H bonds. There are also the intra-atomic interactions between different hybrid orbitals on the same atom, and then the intra-atomic edge weights would be somewhat less than the inter-atomic weights. For articles on chemical graphs, see [11, 20, 21, 22].

Next, we recall the following

Theorem 1. *[29] Let G_1 and G_2 be connected graphs with $G_1 \ncong G_2$. Then $L(G_1) \cong L(G_2)$ if and only if $\{G_1, G_2\} = \{C_3, K_{1,3}\}$, where $K_{1,3}$ is the star on 4 vertices.*

In [15], it is shown that $G^\star_1 \cong G^\star_2$ if and only if $G_1 \cong G_2$. In order to be self-contained, we include a proof here.

Theorem 2. *[15] Let G_1 and G_2 be connected graphs. Then $G_1^\star \cong G_2^\star$ if and only if $G_1 \cong G_2$.*

Proof. ($\Longleftarrow$) It is obvious.

($\Longrightarrow$) Let $S(G)$ denote the subdivision graph of a graph G; notice that $|E(S(G))| = 2|E(G)|$, an even number. Then $G_1^\star \cong G_2^\star$ implies $L(S(G_1)) \cong L(S(G_2))$. If $S(G_1) \ncong S(G_2)$, then $S(G_1) = C_3$ or $S(G_1) = K_{1,3}$ by Theorem 1; in each case, $|E(S(G_1)| = 3$, an odd number, which is impossible. Thus, $S(G_1) \cong S(G_2)$. It remains to show that $G_1 \cong G_2$. Notice that $S(G_i)$ is a bi-partite graph with a unique bi-partition of the vertex set of $S(G_i)$ by the connectedness of $S(G_i)$, where $i = 1, 2$. Let $V_{i,1} = V(G_i)$ and $V_{i,2} = V(S(G_i)) - V(G_i)$ be the bi-partite sets of $S(G_i)$, where $i = 1, 2$. For each P_3 of $S(G_i)$, say uvu', such that $u, u' \in V_{i,1}$ and $v \in V_{i,2}$ ($i = 1, 2$), we replace uvu' by uu'. The resulting graph is $G_1 \cong G_2$. □

3. Bounds of Metric Dimension on para-Line Graphs

In this section, we obtain general bounds of the metric dimension of para-line graphs. First, we recall the bounds of the metric dimension of graphs and its line graphs.

Theorem 3. *[5] If G is a connected graph of order $n \geq 2$ and diameter d, then*

$$f(n, d) \leq \dim(G) \leq n - d,$$

where $f(n, d)$ is the least positive integer k for which $k + d^k \geq n$.

A generalization of Theorem 3 has been given in [14] by Hernando et al.

Theorem 4. *[14] Let G be a graph of order n, diameter $d \geq 2$, and metric dimension k. Then*

$$n \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i - 1)^{k-1}.$$

Theorem 5. *[10] If G is a connected graph of order $n \geq 5$, then*

$$\lceil \log_2 \Delta(G) \rceil \leq \dim(L(G)) \leq n - 2.$$

Next, we recall the following definitions that are stated in [14]. Two distinct vertices u, v of a graph G are *adjacent twins* if $N_G[u] = N_G[v]$, and *non-adjacent twins* if $N_G(u) = N_G(v)$. Observe that if u, v are adjacent twins then $uv \in E(G)$, and if u, v are non-adjacent twins then $uv \notin E(G)$. If u, v are adjacent or non-adjacent twins, then u, v are twins; if S is a resolving set of G, then $u \in S$ or $v \in S$.

Theorem 6. *Let G be a connected graph of order $n \geq 2$. Then*

$$\lceil \log_2 \Delta(G) \rceil \leq \dim(G^\star) \leq n - 1, \tag{1}$$

and both bounds are sharp.

Proof. First, we consider for $2 \le n \le 4$. If $n = 2$, then $\Delta(G) = 1$ and $\dim(G^\star) = 1$. If $n = 3$, then $\Delta(G) = 2$ and $1 \le \dim(G^\star) \le 2$. If $n = 4$, then $2 \le \Delta(G) \le 3$ and $1 \le \dim(G^\star) \le 3$; here, $\dim(G^\star) = 1$ if and only if $G = P_4$, and $\dim(G^\star) = 3$ if and only if $G = K_4$. So, (1) holds for $2 \le n \le 4$. Next, we consider $n \ge 5$.

The lower bound of (1) follows by Theorem 5, since $\Delta(G) = \Delta(S(G))$. For the sharpness of the lower bound, take $G = P_n$; then $\Delta(G) = 2$ and $\dim(G^\star) = 1$ (see (a) of Theorem 8).

Next, we prove the upper bound of (1). Let $V(G) = \{v_1, v_2, \dots, v_n\}$, and let $\deg_G(v_i) = d_i$ $(1 \le i \le n)$. Following the construction of $G^\star$ from G, let each vertex v_i be replaced by $K_{(v_i)} \cong K_{d_i}$; here, we denote by $\mathscr{U}_i$ the vertex set $V(K_{(v_i)}) = \{u_{i,1}, u_{i,2}, \dots, u_{i,d_i}\} \subseteq V(G^\star)$ for each i $(1 \le i \le n)$. Let $S = \{u_{1,a_1}, u_{2,a_2}, \dots, u_{n-1,a_{n-1}}\}$ with $|S| = n - 1$ such that $|S \cap \mathscr{U}_i| = 1$ for each i $(1 \le i \le n-1)$ and that no two vertices in S are adjacent in $G^\star$. We will show that S is a resolving set for $G^\star$. It suffices to show that, for any two vertices $u_x, u_y \in V(G^\star) - S$,

$$d_{G^\star}(u_x, u_{i,a_i}) \ne d_{G^\star}(u_y, u_{i,a_i}) \text{ for some } u_{i,a_i} \in S. \tag{2}$$

We consider two cases.

Case 1: *$u_x, u_y \in \mathscr{U}_i$ for some i $(1 \le i \le n)$.* We consider two subcases.

Subcase 1.1: $1 \le i \le n-1$. First, suppose that $N_{G^\star}(u_x) \cap \mathscr{U}_n = \emptyset = N_{G^\star}(u_y) \cap \mathscr{U}_n$. Since at most one vertex of $\mathscr{U}_i - \{u_{i,a_i}\}$ is adjacent to at most one vertex of $\mathscr{U}_k$ for $k \ne i$ $(1 \le k \le n-1)$, say $N_{G^\star}(u_x) \cap \mathscr{U}_k \ne \emptyset$, we have $d_{G^\star}(u_x, u_{k,a_k}) < d_{G^\star}(u_y, u_{k,a_k})$. Second, suppose that $N_{G^\star}(u_x) \cap \mathscr{U}_n \ne \emptyset$ or $N_{G^\star}(u_y) \cap \mathscr{U}_n \ne \emptyset$, say the former; notice that not both u_x and u_y can have a neighbor in $\mathscr{U}_n$ by the construction of $G^\star$. Then $d_{G^\star}(u_x, u_{k,a_k}) > d_{G^\star}(u_y, u_{k,a_k})$ for some $k \ne i$ $(1 \le k \le n-1)$ satisfying $N_{G^\star}(u_y) \cap \mathscr{U}_k \ne \emptyset$. So, (2) holds for each case.

Subcase 1.2: $i = n$. Since $d_{G^\star}(u_x, u_{k,a_k}) < d_{G^\star}(u_y, u_{k,a_k})$ for some k $(1 \le k \le n-1)$ satisfying $N_{G^\star}(u_x) \cap \mathscr{U}_k \ne \emptyset$, (2) holds.

Case 2: *$u_x \in \mathscr{U}_i$ and $u_y \in \mathscr{U}_j$ for $i \ne j$ $(1 \le i, j \le n)$.* We consider two subcases.

Subcase 2.1: $1 \le i, j \le n-1$. If $u_{i,a_i}u_y \notin E(G^\star)$, then $d_{G^\star}(u_x, u_{i,a_i}) = 1$ and $d_{G^\star}(u_y, u_{i,a_i}) \ge 2$, and thus (2) holds. So, we consider $u_{i,a_i}u_y \in E(G^\star)$; notice that $d_{G^\star}(u_x, u_{i,a_i}) = 1 = d_{G^\star}(u_y, u_{i,a_i})$. But $d_{G^\star}(u_y, u_{j,a_j}) = 1$ and $d_{G^\star}(u_x, u_{j,a_j}) \ge 2$, and thus (2) holds.

Subcase 2.2: *$1 \le i \le n-1$ and $j = n$, or $1 \le j \le n-1$ and $i = n$, say the former.* If $N_{G^\star}(u_y) \cap \mathscr{U}_i = \emptyset$, then $d_{G^\star}(u_x, u_{i,a_i}) = 1 < d_{G^\star}(u_y, u_{i,a_i})$, and thus (2) holds. So, suppose that $N_{G^\star}(u_y) \cap \mathscr{U}_i \ne \emptyset$. If $u_y u_{i,a_i} \notin E(G^\star)$, then $d_{G^\star}(u_x, u_{i,a_i}) = 1$ and $d_{G^\star}(u_y, u_{i,a_i}) = 2$, and thus (2) holds. If $u_y u_{i,a_i} \in E(G^\star)$, then $d_{G^\star}(u_x, u_{i,a_i}) = 1 = d_{G^\star}(u_y, u_{i,a_i})$; further, one can easily verify that $code_S(u_x) = code_S(u_y)$ implies that v_i and v_n are adjacent twins in G. If every pair of vertices in G are twins in G, then $G \cong K_n$, and $\dim(K_n^\star) = n - 1$ (see Theorem 12).

In each case, S forms a resolving set for $G^\star$ with $|S| = n-1$. So, the upper bound of (1) follows. For the sharpness of the upper bound, take $G = K_n$; then $\dim(G^\star) = n-1$ by Theorem 12. □

The authors of [5] characterized connected graphs of order n with metric dimension 1, $n-2$, and $n-1$, respectively.

Theorem 7. *[5] Let G be a connected graph of order $n \geq 2$. Then*

(a) $\dim(G) = 1$ *if and only if* $G = P_n$*,*

(b) $\dim(G) = n-1$ *if and only if* $G = K_n$*,*

(c) for $n \geq 4$, $\dim(G) = n-2$ *if and only if* $G = K_{s,t}$ $(s, t \geq 1)$*,* $G = K_s + \overline{K}_t$ $(s \geq 1, t \geq 2)$*, or* $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$*; here, $A+B$ denotes the graph obtained from the disjoint union of graphs A and B by joining every vertex of A with every vertex of B, and $\overline{H}$ denotes the graph whose vertex set is $V(H)$ and $uv \in E(\overline{H})$ if and only if $uv \notin E(H)$ for $u, v \in V(H)$.*

Next, we characterize graphs G satisfying $\dim(G^\star) = 1$ and $\dim(G^\star) = |V(G)| - 1$, respectively.

Theorem 8. *Let G be a connected graph of order $n \geq 2$. Then*

(a) $\dim(G^\star) = 1$ *if and only if* $G = P_n$*,*

(b) for $n \geq 4$, $\dim(G^\star) = n-1$ *if and only if* $G = K_n$*.*

Proof. Let G be a connected graph of order $n \geq 2$.

(a) ($\Longleftarrow$) If $G = P_n$, then $G^\star = P_{2n-2}$, and thus $\dim(G^\star) = 1$ by (a) of Theorem 7.
($\Longrightarrow$) Suppose $\dim(G^\star) = 1$. Then $\Delta(G) \leq 2$; otherwise, $G^\star$ contains K_3 as a subgraph, and thus $\dim(G^\star) \geq 2$ by (a) of Theorem 7. Since $\Delta(G) \leq 2$, G is either a path or a cycle. Since $C_n^\star \cong C_{2n}$ and $\dim(C_n^\star) = 2$, $G \cong P_n$.

(b) ($\Longleftarrow$) See Theorem 12.
($\Longrightarrow$) Suppose that $G \neq K_n$ for $n \geq 4$, and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $\deg_G(v_i) = d_i$ $(1 \leq i \leq n)$. Without loss of generality, assume that $v_1v_2 \notin E(G)$. Following the construction of $G^\star$ from G, let each vertex v_i be replaced by $K_{(v_i)} \cong K_{d_i}$; here, we denote by $\mathcal{U}_i$ the vertex set $V(K_{(v_i)}) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,d_i}\} \subseteq V(G^\star)$ for each i $(1 \leq i \leq n)$. Let $S = \{u_{2,a_2}, u_{3,a_3}, \ldots, u_{n-1,a_{n-1}}\}$ with $|S| = n-2$ such that $|S \cap \mathcal{U}_i| = 1$ for each i $(2 \leq i \leq n-1)$ and that no two vertices in S are adjacent in $G^\star$. We will show that S is a resolving set for $G^\star$. It suffices to show that, for any two vertices $u_x, u_y \in V(G^\star) - S$, $d_{G^\star}(u_x, u_{i,a_i}) \neq d_{G^\star}(u_y, u_{i,a_i})$ for some $u_{i,a_i} \in S$. As in the proof of Theorem 6, S forms a resolving set for $\langle \cup_{i=1}^{n-1} mathcalU_i \rangle \subseteq G^\star$ and $\langle \cup_{i=2}^{n} \mathcal{U}_i \rangle \subseteq G^\star$. So, it remains to show that, for $u_x \in \mathcal{U}_1$ and $u_y \in \mathcal{U}_n$,
$d_{G^\star}(u_x, u_{i,a_i}) \neq d_{G^\star}(u_y, u_{i,a_i})$ for some $u_{i,a_i} \in S$. If $u_xu_{i,a_i} \in E(G^\star)$ or $u_yu_{i,a_i} \in E(G^\star)$ for some i $(2 \leq i \leq n-1)$, say the former, then $d_{G^\star}(u_x, u_{i,a_i}) = 1 < d_{G^\star}(u_y, u_{i,a_i})$. If $u_xu_{i,a_i} \notin E(G^\star)$ and $u_yu_{i,a_i} \notin E(G^\star)$ for each i $(2 \leq i \leq n-1)$, then $code_S(u_x) = code_S(u_y)$ implies that v_1 and v_n are non-adjacent twins in G. If there exist a pair of vertices in G that are not non-adjacent twin

in G, we are done. Otherwise, every pair of vertices must be non-adjacent twins in G, but this is impossible: if w_1 and w_2 are non-adjacent twins in G satisfying $w_k \in N_G(w_1) \cap N_G(w_2)$, then $w_1 w_k \in E(G)$. So, S is a resolving set for $G^\star$ with $|S| = n-2$, and hence $\dim(G^\star) \le n-2$. □

4. Metric Dimension of para-Line Graphs for Trees, Complete Graphs, Complete Bi-partite Graphs, Wheel Graphs, and Bouquet of Circles

In this section, we determine the metric dimension of some classes of para-line graphs. First, we determine the metric dimension of $T^\star$ for a tree T that is not a path.

The following definitions are stated in [5]. Fix a graph G. A vertex of degree at least three is called a *major vertex*. An end-vertex u is called *a terminal vertex of a major vertex* v if $d(u,v) < d(u,w)$ for every other major vertex w. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v is an *exterior major vertex* if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G, and let $ex(G)$ denote the number of exterior major vertices of G.

Theorem 9. *[5, 17, 19] If T is a tree that is not a path, then* $\dim(T) = \sigma(T) - ex(T)$.

Theorem 10. *[10] If T is a tree that is not a path, then* $\dim(L(T)) = \sigma(T) - ex(T)$.

As an immediate consequence of Theorem 10, we have the following

Corollary 1. *If T is a tree that is not a path, then* $\dim(T^\star) = \sigma(T) - ex(T)$.

Second, we determine the metric dimension of $K_n^\star$ for the complete graph K_n of order $n \ge 2$. We first recall the metric dimension of $L(K_n)$.

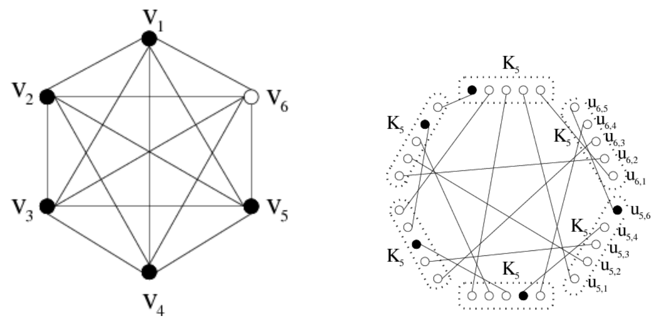
Theorem 11. *[1] For the complete graph K_n of order $n \ge 6$,* $\dim(L(K_n)) = \lceil \frac{2n}{3} \rceil$.

Theorem 12. *If K_n is the complete graph of order $n \ge 2$, then* $\dim(K_n^\star) = n-1$.

Proof. Let $G = K_n$ for $n \ge 2$. If $n = 2$, then $G \cong P_2$; thus $\dim(G^\star) = 1$. If $n = 3$, then $G \cong C_3$; thus $\dim(G^\star) = 2$. If $n = 4$, then $\dim(G^\star) \ge 3$ by Theorem 3, since $diam(G^\star) = 3$ and $|V(G^\star)| = 12$; thus $\dim(G^\star) = 3$ by Theorem 6. So, let $n \ge 5$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ with $\deg_G(v_i) = n-1$ $(1 \le i \le n)$. Following the construction of $G^\star$ from G, let each vertex v_i be replaced by $K_{(v_i)} \cong K_{n-1}$; here, we denote by $\mathscr{U}_i$ the vertex set $V(K_{(v_i)}) = \{u_{i,1}, u_{i,2}, \dots, u_{i,n}\} - \{u_{i,i}\} \subseteq V(G^\star)$ for each i $(1 \le i \le n)$. See Figure 2 for the labelings of K_n and $K_n^\star$; here, the solid vertices form a minimum resolving set for K_6 and $K_6^\star$, respectively. Let S be a resolving set for $K_n^\star$. By Theorem 6, $\dim(K_n^\star) \le n-1^\dagger$. It remains to show that $\dim(K_n^\star) \ge n-1$. Assume, to the contrary, that $\dim(K_n^\star) \le n-2$; then $|S| \le n-2$ and there are two or more $\mathscr{U}_i$'s satisfying $S \cap \mathscr{U}_i = \emptyset$. We may assume that $|S \cap \mathscr{U}_1| \ge |S \cap \mathscr{U}_2| \ge \ldots \ge |S \cap \mathscr{U}_{n-2}| \ge 0$ and that $S \cap \mathscr{U}_i = \emptyset$ for $i \in \{n-1, n\}$, by relabeling if necessary. If $|S \cap \mathscr{U}_1| \ge n-3$, then one can easily see that S fails to resolve $K_n^\star$. So,

†One can readily check that $S = \{u_{i,i+1} \mid 1 \le i\ len-1\}$ is a resolving set for $K_n^\star$

$|S \cap \mathscr{U}_i| \leq n-4$ each i $(1 \leq i \leq n)$, and we may assume that $S \cap \{u_{1,n-2}, u_{1,n-1}, u_{1,n}\} = \emptyset$ by relabeling if necessary. Then $code_S(u_{1,n-1}) = code_S(u_{1,n})$, contradicting the assumption that S is a resolving set for $K_n^\star$. Thus, any resolving set S for $K_n^\star$ must satisfy $|S| \geq n-1$. Therefore, $\dim(K_n^\star) = n-1$. □



(a) K_6 (b) $K_6^\star$

Figure 2: Labeling of K_n and $K_n^\star$

Third, we consider the metric dimension of $K_{s,t}^\star$ for the complete bi-partite graph $K_{s,t}$ of order $s+t \geq 4$, where $s,t \geq 2$. We recall the metric dimension of $L(K_{s,t})$ first.

Theorem 13. *[4] Let $K_{s,t}$ be the complete bi-partite graph, where $t \geq s \geq 1$. Then*

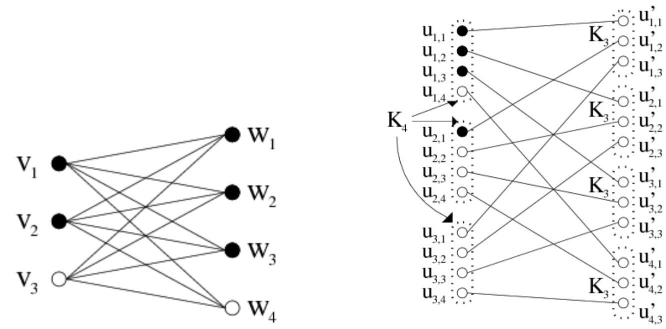
$$\dim(L(K_{s,t})) = \begin{cases} \lfloor \frac{2(s+t-1)}{3} \rfloor & \text{if } s \leq t \leq 2s-1, \\ t-1 & \text{if } t \geq 2s. \end{cases}$$

Since $K_{2,2} \cong C_4$ and $C_4^\star \cong C_8$, $\dim(K_{2,2}^\star) = 2$. So, we consider for $s,t \geq 2$ excluding $s=t=2$.

Theorem 14. *Let $K_{s,t}$ be the complete bi-partite graph of order $s+t \geq 4$. For $s,t \geq 2$, excluding $s=t=2$, $\dim(K_{s,t}^\star) \leq s+t-3$.*

Proof. For $t \geq s \geq 2$ excluding $s=t=2$, let $G = K_{s,t}$. Let V and W be the bi-partite sets of G, where $V = \{v_1, v_2, \ldots, v_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Following the construction of $G^\star$ from G, let each vertex v_i (w_j, respectively) be replaced by $K_{(v_i)} \cong K_t$ ($K_{(w_j)} \cong K_s$, respectively); here, we denote by $\mathscr{U}_i$ the vertex set $V(K_{(v_i)}) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,t}\} \subseteq V(G^\star)$ and we denote by $\mathscr{U}'_j$ the vertex set $V(K_{(w_j)}) = \{u'_{j,1}, u'_{j,2}, \ldots, u'_{j,s}\} \subseteq V(G^\star)$ for each i,j $(1 \leq i \leq s$ and $1 \leq j \leq t)$. Let $u_{i,k}u'_{k,i} \in E(G)$ for $u_{i,k} \in \mathscr{U}_i$ and $u'_{k,i} \in \mathscr{U}'_k$, where $1 \leq i \leq s$ and $1 \leq k \leq t$. See Figure 3 for the labelings of $K_{s,t}$ and $K_{s,t}^\star$; here, the solid vertices form a resolving set for $K_{3,4}$ and $K_{3,4}^\star$, respectively. We will show that $S = \{u_{1,a} \mid 1 \leq a \leq t-1\} \cup (\cup_{b=2}^{s-1}\{u_{b,1}\})$ forms a resolving set for $K_{s,t}^\star$ with $|S| = s+t-3$, and thus $\dim(K_{s,t}^\star) \leq s+t-3$. It suffices to show that, for any two vertices $u_x, u_y \in V(G^\star) - S$,

$$d_{G^\star}(u_x, z) \neq d_{G^\star}(u_y, z) \text{ for some } z \in S. \tag{3}$$



(a) $K_{3,4}$ (b) $K^{\star}_{3,4}$

Figure 3: Labeling of $K_{s,t}$ and $K^{\star}_{s,t}$

We consider three cases.

Case 1: *$u_x \in \mathscr{U}_i$ and $u_y \in \mathscr{U}_j$, where $1 \le i, j \le s$.* We consider two subcases.

Subcase 1.1: *$i \neq j$.* If $i = 1$ or $j = 1$, say the former, then $1 = d_{G^\star}(u_x, u_{1,1}) < d_{G^\star}(u_y, u_{1,1})$. If $i = s$ or $j = s$, say the former, then $d_{G^\star}(u_y, u_{j,1}) = 1 < d_{G^\star}(u_x, u_{j,1})$. If $2 \le i, j \le s-1$, then $1 = d_{G^\star}(u_x, u_{i,1}) < d_{G^\star}(u_y, u_{i,1})$. So, (3) holds in each case.

Subcase 1.2: *$i = j$. Notice that $2 \le i \le s$ in this case.* Write $u_x = u_{i,\alpha}$ and $u_y = u_{i,\beta}$ for $\alpha \neq \beta$ $(1 \le \alpha, \beta \le t)$; then $d_{G^\star}(u_{i,\alpha}, u_{1,\alpha}) = 3$ and $d_{G^\star}(u_{i,\beta}, u_{1,\alpha}) = 4$ for each α $(1 \le \alpha \le t-1)$, and thus (3) holds.

Case 2: *$u_x \in \mathscr{U}_i$ and $u_y \in \mathscr{U}'_j$, where $1 \le i \le s$ and $1 \le j \le t$.* If $i = 1$, then $1 = d_{G^\star}(u_x, u_{1,1}) < d_{G^\star}(u_y, u_{1,1})$ or $1 = d_{G^\star}(u_x, u_{1,2}) < d_{G^\star}(u_y, u_{1,2})$. If $i = s$, then $3 \le d_{G^\star}(u_x, u_{1,1}) \le 4$ and $1 \le d_{G^\star}(u_y, u_{1,1}) \le 3$; for $d_{G^\star}(u_x, u_{1,1}) = 3$ (i.e., $u_x = u_{s,1}$), $d_{G^\star}(u_x, u_{1,2}) = 4 > d_{G^\star}(u_y, u_{1,2})$. If $2 \le i \le s-1$, then $d_{G^\star}(u_x, u_{i,1}) = 1 < d_{G^\star}(u_y, u_{i,1})$ for $u_y \neq u'_{1,i}$; $d_{G^\star}(u_x, u_{1,1}) = 4 > d_{G^\star}(u_y, u_{1,1})$ for $u_y = u'_{1,i}$. So, (3) holds in each case.

Case 3: *$u_x \in \mathscr{U}'_i$ and $u_y \in \mathscr{U}'_j$, where $1 \le i, j \le t$.* We consider two subcases.

Subcase 3.1: *$i \neq j$.* If $u_x \in N_{G^\star}(S)$ or $u_y \in N_{G^\star}(S)$, say the former, then $d_{G^\star}(u_x, z) = 1 < d_{G^\star}(u_x, z)$ for $z \in S$. So, suppose that $u_x \notin N_{G^\star}(S)$ and $u_y \notin N_{G^\star}(S)$; we write $u_x = u'_{i,\alpha}$ and $u_y = u'_{j,\beta}$, where $2 \le \alpha, \beta \le s$. If $i = 1$ or $j = 1$, say the former, then $u_x = u'_{1,s}$: $d_{G^\star}(u'_{1,s}, u_{1,1}) = 2$ and $d_{G^\star}(u'_{j,\beta}, u_{1,1}) = 3$. So, we consider $2 \le i, j \le t$. First, suppose $d_{G^\star}(u'_{i,\alpha}, z) = 2$ for some $z \in S$. Then $u'_{i,1} \in N(S)$ (i.e., $z = u_{1,i}$ for $i \neq t$) or $z = u_{\alpha,1}$, where $2 \le i, \alpha \le s-1$: if $z = u_{1,i}$, then $d_{G^\star}(u'_{j,\beta}, u_{1,i}) = 3$; if $z = u_{\alpha,1}$, then $d_{G^\star}(u'_{j,\beta}, u_{\alpha,1}) = 2$ implies $\alpha = \beta$, but $d_{G^\star}(u'_{i,\alpha}, u_{1,i}) = 2$ and $d_{G^\star}(u'_{j,\alpha}, u_{1,i}) = 3$. Second, suppose $d_{G^\star}(u'_{i,\ alpha}, z) = 3$ for all $z \in S$. Then $u'_{i,\alpha} = u'_{t,s}$, the unique vertex of $\cup_{j=1}^{t} \mathscr{U}'_j$ with 3 in each entry of its code. So, (3) holds in each case.

Subcase 3.2: $i = j$. If $u_x \in N_{G^\star}(S)$ or $u_y \in N_{G^\star}(S)$, say the former, then $d_{G^\star}(u_x, z) = 1 < d_{G^\star}(u_x, z)$ for $z \in S$. So, suppose that $u_x \notin N_{G^\star}(S)$ and $u_y \notin N_{G^\star}(S)$; notice that $2 \le j \le t$ in this case. We write $u_x = u'_{j,\alpha}$ and $u_y = u'_{j,\beta}$. Then $d_{G^\star}(u'_{j,\alpha}, u_{\alpha,1}) = 2$ and $d_{G^\star}(u'_{j,\beta}, u_{\alpha,1}) = 3$ and for $\alpha \ne \beta$, where $1 \le \alpha \le s-1$ and $1 \le \beta \le s$. In each case, (3) holds.

□

Next, we determine the metric dimension of $W^\star_{1,n}$ for the wheel graph $W_{1,n} = K_1 + C_n$, where $n \ge 3$. We recall the metric dimension of the wheel graph and its line graph.

Theorem 15. *[3, 24] For $n \ge 3$, let $W_{1,n} = K_1 + C_n$ be the wheel graph on $n+1$ vertices. Then*

$$\dim(W_{1,n}) = \begin{cases} 3 & \text{if } n = 3 \text{ or } n = 6, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

Theorem 16. *[8] For $n \ge 3$,*

$$\dim(L(W_{1,n})) = \begin{cases} 3 & \text{if } n = 3,4, \\ 4 & \text{if } n = 5, \\ n - \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Theorem 17. *For $n \ge 3$,*

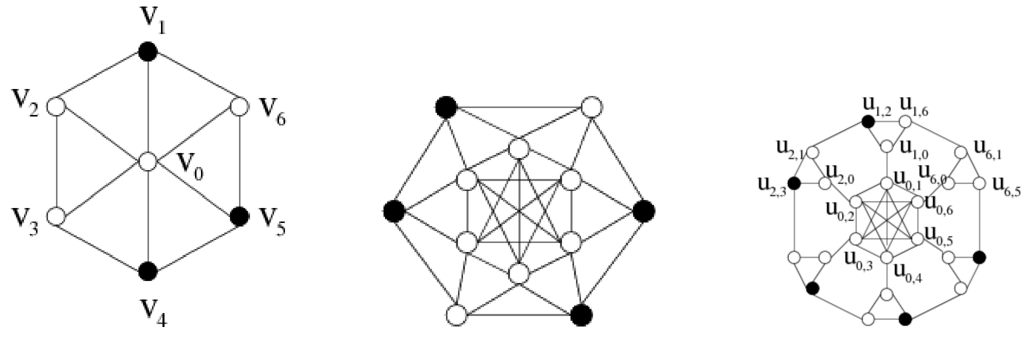
$$\dim(W^\star_{1,n}) = \begin{cases} 3 & \text{if } n = 3, \\ n-1 & \text{if } n \ge 4. \end{cases}$$

Proof. For $n \ge 3$, let $G = W_{1,n}$, and let $V(G) = \{v_0, v_1, v_2, \ldots, v_n\}$ with $\deg_G(v_0) = n$. Following the construction of $G^\star$ from G, let the vertex v_0 be replaced by $K_{(v_0)} \cong K_n$ and let each vertex v_i $(1 \le i \le n)$ be replaced by $K_{(v_i)} \cong K_3$; here, we denote by $\mathscr{U}_0$ the vertex set $V(K_{(v_0)}) = \{u_{0,1}, u_{0,2}, \ldots, u_{0,n}\} \subseteq V(G^\star)$ and we denote by $\mathscr{U}_i$ $(1 \le i \le n)$ the vertex set $V(K_{(v_i)}) = \{u_{i,0}, u_{i,i-1}, u_{i,i+1}\} \subseteq V(G^\star)$, where the subscript of u is taken modulo n if the subscript is bigger than n except when $i = 1$ (we take $u_{1,n}$ in place of $u_{i,i-1}$ if $i = 1$). See Figure 4 for the labelings of $W_{1,n}$ and $W^\star_{1,n}$; here, the solid vertices form a minimum resolving set for $W_{1,6}$, $L(W_{1,6})$, and $W^\star_{1,6}$, respectively.

If $n = 3$, noting that $W_{1,3} \cong K_4$, $\dim(W^\star_{1,3}) = 3$ by Theorem 12. So, we consider $n \ge 4$; let S be a resolving set for $W^\star_{1,n}$. We make the following

Claim: For $n \ge 4$, there exists at most one i such that $S \cap (\mathscr{U}_i \cup \{u_{0,i}\}) \ne \emptyset$, where $1 \le i \le n$.

Proof of Claim. Assume, to the contrary, that $S \cap (\mathscr{U}_x \cup \{u_{0,x}\}) = \emptyset = S \cap (\mathscr{U}_y \cup \{u_{0,y}\})$ for two distinct x, y, where $1 \le x, y \le n$. Then $code_S(u_{0,x}) = code_S(u_{0,y})$, contradicting the assumption that S is a resolving set for $W^\star_{1,n}$. So, there exists at most one i such that $S \cap (\mathscr{U}_i \cup \{u_{0,i}\}) \ne \emptyset$. □

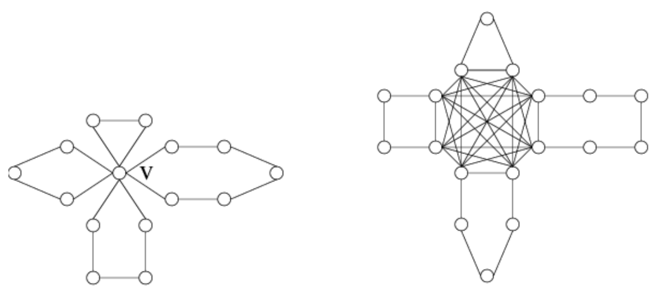


(a) $W_{1,6}$ (b) $L(W_{1,6})$ (c) $W^{\star}_{1,6}$

Figure 4: Labelings of $W_{1,n}$ and $W^{\star}_{1,n}$; $\dim(W_{1,6})=3, \dim(L(W_{1,6}))=4, \dim(W^{\star}_{1,6})=5$

By Claim, $|S| \geq n-1$, and thus $\dim(W^{\star}_{1,n}) \geq n-1$ for $n \geq 4$. On the other hand, $\dim(W^{\star}_{1,n}) \leq n-1$ by (b) of Theorem 8[‡]. Therefore, $\dim(W^{\star}_{1,n}) = n-1$ for $n \geq 4$. □

It is well known that $\dim(C_n)=2$ for $n \geq 3$. Let $B_n=(k_1,k_2,\ldots,k_n)$ be a bouquet of $n \geq 2$ circles C^1, C^2, …, C^n, with a cut-vertex, where k_i is the number of vertices of C^i $(1 \leq i \leq n)$. See Figure 5 for $B_4=(3,4,5,6)$ and its line graph. We recall the metric dimension of a bouquet of circles and its line graph.



(a) $B_4=(3,4,5,6)$ (b) $L(B_4)$

Figure 5: A bouquet of four circles $B_4=(3,4,5,6)$ and its line graph

Theorem 18. *[16] Let $B_n=(k_1,k_2,\ldots,k_n)$ be a bouquet of $n \geq 2$ circles with a cut-vertex. If x is the number of even cycles of B_n, then*

$$\dim(B_n)=\begin{cases} n & \text{if } x=0 \\ n+x-1 & \text{if } x \geq 1. \end{cases}$$

Theorem 19. *[8] Let $B_n=(k_1,k_2,\ldots,k_n)$ be a bouquet of $n \geq 2$ circles with a cut-vertex. Then*

$$\dim(L(B_n))=2n-1.$$

As an immediate consequence of Theorem 19, we have the following

‡One can readily check that $S=\{u_{i,i+1} \mid 1 \leq i \leq n-1\}$ forms a resolving set for $W^{\star}_{1,n}$.

Corollary 2. *Let $B_n = (k_1, k_2, \ldots, k_n)$ be a bouquet of $n \geq 2$ circles with a cut-vertex. Then*

$$\dim(B_n^{\star}) = 2n - 1.$$

5. Metric Dimension of Graphs, Line Graphs, and para-Line Graphs

In this section, we compare metric dimension of a connected graph G, its line graph $L(G)$, and its para-line graph $G^{\star}$: we give an example of a graph G such that $\max\{\dim(G), \dim(L(G)), \dim(G^{\star})\}$ equals $\dim(G)$, $\dim(L(G))$, and $\dim(G^{\star})$, respectively.

Remark 1. *There exists a graph G with* $\max\{\dim(G), \dim(L(G)), \dim(G^{\star})\} = \dim(G)$. *If $G = K_{s,2s}$ for $s \geq 2$, then* $\dim(G) = 3s - 2$, $\dim(L(G)) = 2s - 1$, *and* $\dim(G^{\star}) \leq 3s - 3$*; further, notice that* $\dim(G) - \dim(L(G))$ *can be arbitrarily large.*

Remark 2. *There exists a graph G with* $\max\{\dim(G), \dim(L(G)), \dim(G^{\star})\} = \dim(L(G))$. *If $G = B_n$, a bouquet of $n \geq 2$ circles, containing no even cycles, then* $\dim(G) = n$ *and* $\dim(L(G)) = 2n-1 = \dim(G^{\star})$*; further, notice that* $\dim(L(G)) - \dim(G)$ *and* $\dim(G^{\star}) - \dim(G)$ *can be arbitrarily large.*

Remark 3. *There exists a graph G with* $\max\{\dim(G), \dim(L(G)), \dim(G^{\star})\} = \dim(G^{\star})$. *If $G = W_{1,n}$ for $n \geq 7$, then* $\dim(G) = \lfloor \frac{2n+2}{5} \rfloor$ *and* $\dim(L(G)) = n - \lceil \frac{n}{3} \rceil$, *and* $\dim(G^{\star}) = n - 1$*; further, notice that* $\dim(G^{\star}) - \dim(G)$ *and* $\dim(G^{\star}) - \dim(L(G))$ *can be arbitrarily large.*

6. Summary and Open Problems

In Table 1, we summarize metric dimension of some graphs G, the line graphs $L(G)$, and the para-line graphs $G^{\star}$. Here, x denotes the number of even cycles of a bouquet of circles B_n; for the complete bi-partite graphs $K_{s,t}$ we consider $s, t \geq 2$ excluding $s = t = 2$.

We conclude this paper with some open problems.

Problems. For a connected graph G, let $L(G)$ be the line graph of G and let $G^{\star}$ be the para-line graph of G.

Q1. [10] Can we characterize graphs G such that $\dim(G) = \dim(L(G))$?

Q2. Can we characterize graphs G such that $\dim(G) = \dim(G^{\star})$?

Q3. Can we characterize graphs G such that $\dim(L(G)) = \dim(G^{\star})$?

Table 1: Metric Dimension of Some Graphs G, Line Graphs $L(G)$, and para-Line Graphs $G^\star$.

G	$\dim(G)$	$\dim(L(G))$	$\dim(G^\star)$
P_n, $n \geq 2$	1	1	1
C_n, $n \geq 3$	2	2	2
$T(\neq P_n)$	$\sigma(T) - ex(T)$	$\sigma(T) - ex(T)$	$\sigma(T) - ex(T)$
K_n, $n \geq 6$	$n-1$	$\lceil \frac{2n}{3} \rceil$	$n-1$
$K_{s,t}$	$s+t-2$	$\begin{cases} \lfloor \frac{2(s+t-1)}{3} \rfloor & \text{if } s \leq t \leq 2s \\ t-1 & \text{if } t \geq 2s \end{cases}$	$\leq s+t-3$
$W_{1,n}$, $n \geq 3$	$\begin{cases} 3 & \text{if } n = 3,6 \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise} \end{cases}$	$\begin{cases} 3 & \text{if } n = 3,4 \\ 4 & \text{if } n = 5 \\ n - \lceil \frac{n}{3} \rceil & \text{if } n \geq 6 \end{cases}$	$\begin{cases} 3 & \text{if } n = 3 \\ n-1 & \text{if } n \geq 4 \end{cases}$
B_n, $n \geq 2$	$\begin{cases} n & \text{if } x = 0 \\ n+x-1 & \text{if } x \geq 1 \end{cases}$	$2n-1$	$2n-1$

ACKNOWLEDGEMENTS The first author is supported by the Welch Foundation of Houston, Texas, grant BD-0894. The authors thank Cong X. Kang for his helpful comments and suggestions, especially on Theorem 6.

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