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Some Results for Certain Subclasses of Functions with Differential Equation and Subordination

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Abstract. By applying the differential subordination theorem, we further investigate the subclass $H_{\lambda}^{n,\gamma}[\alpha,\beta]$ of functions which are analytic in the unit disk. Several subordination results on a convex function and a incomplete beta function are obtained. Moreover, the function that belongs to the $H_{\lambda}^{n,\gamma}[\alpha,\beta]$ with a Cauchy-Euler differential equation is also discussed on similar subject. Our results extend some earlier works.

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1. Introduction and Definition

Let $\mathscr{A}$ denote the class of all functions of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C},|z|<1\}$ and let S be the subclass of $\mathscr{A}$ consisting of univalent functions. $\mathscr{K}$ denotes the usual class of convex functions.

Suppose that the functions f and g are analytic in $\mathbb{U}$. We say that f is subordinate to g in $\mathbb{U}$ if there exists a functions ϕ analytic in $\mathbb{U}$ such that $\phi(0)=0$, $|\phi(z)|<1\,(|z|<1)$ and $f(z)=g(\phi(z))\,(|z|<1)$, written $f\prec g$.

Let be given two functions $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ and $g(z)=z+\sum_{k=2}^{\infty}b_kz^k$ analytic in the open unit disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$, then the Hadamard product(or convolution)$f*g$ of two functions f, g is defined by $f*g(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k$.

Let $(x)_k$ be the pochhammer symbol defined by

$$\begin{cases}1, & k=0, x\in\mathbb{C}/\{0\},\\ x(x+1)(x+2)\dots(x+k-1), & k\in\mathbb{N}=\{1,2,3,\dots\}, x\in\mathbb{C}.\end{cases}$$

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In [7], Ruscheweyh defined the incomplete beta function

$$h(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \quad |z| < 1, \tag{1}$$

where a is any real number and $c \neq \{0,-1,-2,\ldots\}$.

Now we recall the linear multiplier fractional differential operator $D_\lambda^{n,\gamma}$ introduced and studied by Al-Oboudi and Al-Amoudi [1] as follows:

$$D_\lambda^{0,0} f(z) = f(z),$$
$$D_\lambda^{1,\gamma} f(z) = \lambda z(\Omega^\gamma f(z))' + (1-\gamma)\Omega^\gamma f(z) = D_\lambda^\gamma f(z),$$
$$D_\lambda^{2,\gamma} f(z) = D_\lambda^\gamma (D_\lambda^{1,\gamma} f(z)),$$
$$\cdots\cdots$$
$$D_\lambda^{n,\gamma} f(z) = D_\lambda^\gamma (D_\lambda^{n-1,\gamma} f(z)),$$

for $n \in \mathbb{N}, \lambda \geqslant 0$ and $0 \leqslant \gamma < 1$, where $\Omega^\gamma f(z) = \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z)$ is an extension of the fractional derivative and fractional integral defined by Owa and Srivastava [6].

Suppose $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, in the light of the above definitions, it is easy to conclude that

$$D_\lambda^{n,\gamma} = z + \sum_{k=2}^{\infty} [\psi_k(\gamma,\lambda)]^n a_k z^k, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where

$$\psi_k(\gamma,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} [1+\lambda(k-1)] \quad (k = 2,3,\ldots). \tag{2}$$

Let T denote the subclass of S whose elements can be expressed in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad a_k \leqslant 0.$$

Using the differential operator $D_\lambda^{n,\gamma}$, Marouf [5] introduced and studied the class $H_\lambda^{n,\gamma}[\alpha,\beta]$. As a function $f(z) \in T$ is in the $H_\lambda^{n,\gamma}[\alpha,\beta]$ if and only if it satisfies

$$\Re\left\{ \alpha \frac{D_\lambda^{n+2,\gamma} f(z)}{D_\lambda^{n,\gamma} f(z)} + (1-\alpha) \frac{D_\lambda^{n+1,\gamma} f(z)}{D_\lambda^{n,\gamma} f(z)} \right\} > \beta \quad (\alpha \geqslant 0; 0 \leqslant \beta < 0).$$

In particular, the class $H_1^{0,0}[\alpha,\beta] \equiv \bar{H}[\alpha,\beta]$ was studied by Lashin [3] and the classes $H_1^{0,0}[0,\beta] \equiv T^*(\beta)$ and $H_1^{0,0}[1,\beta] \equiv C(\beta)$ were studied by Silverman [8].

To prove our results we shall need the following Definition and Lemma:

Definition 1. *[See 9] An infinite sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers will be called a subordinating factor sequence if whenever $f \in \mathscr{K}$, we have the subordination given by*

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \quad (z \in \mathbb{U}, a_1 = 1).$$

Lemma 1. *[See 9] The sequence* $\{b_n\}_{n=1}^{\infty}$ *is subordinating factor sequence if and only if*

$$\Re\{1+2\sum_{n=1}^{\infty} b_n z^n\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 2. *[See 7] Let* $0 < a \leqslant c$*. If* $c \geqslant 2$ *or* $a + c \geqslant 3$*, then the function*

$$h(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \quad (z \in \mathbb{U})$$

belongs to the class $\mathscr{K}$ *of convex functions.*

In [5], Marouf proved the sufficient and necessary condition on a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in T$ to be $H_{\lambda}^{n,\gamma}[\alpha,\beta]$, which is equivalent to the following Lemma:

Lemma 3. *[See 5] A function* $f(z) \in T$ *is in the* $H_{\lambda}^{n,\gamma}[\alpha,\beta]$ *if and only if*

$$\sum_{k=2}^{\infty}[(\alpha\psi_k(\gamma,\lambda)+1)(\psi_k(\gamma,\lambda)-1)+1-\beta][\psi_k(\gamma,\lambda)]^n|a_k| \leqslant 1-\beta \tag{3}$$

which $\psi_k(\gamma,\lambda)$ *is defined as* (2).

Lemma 4. *[See 4] If the functions* $f(z)$ *and* $g(z)$ *are analytic in* $\mathbb{U}$ *with* $g(z) \prec f(z)$*, then for* $s > 0$ *and* $z = re^{i\theta}$ $(0 < r < 1)$*, we have*

$$\int_0^{2\pi} |f(re^{i\theta})|^s \leqslant \int_0^{2\pi} |g(re^{i\theta})|^s.$$

2. Some Results on the Class $H_{\lambda}^{n,\gamma}[\alpha,\beta]$

We begin with the following theorem:

Theorem 1. *If* $f \in H_{\lambda}^{n,\gamma}[\alpha,\beta]$ *in* $\mathbb{U}$ *and* $s > 0$*,* $0 < |z| = r < 1$*, then for function* $g \in \mathscr{K}$

$$\frac{\Phi(2)}{\Phi(2)+1-\beta} f * g(z) \prec 2g(z) \tag{4}$$

and

$$\frac{\Phi(2)}{\Phi(2)+1-\beta}\int_0^{2\pi} |f * g(re^{i\theta})|^s d\theta \leqslant 2\int_0^{2\pi} |g(re^{i\theta})|^s d\theta \tag{5}$$

where $\Phi(2) = [(\alpha\psi_2(\gamma,\lambda)+1)(\psi_2(\gamma,\lambda)-1)+1-\beta][\psi_2(\gamma,\lambda)]^n$.

Proof. Suppose we take $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in H_{\lambda}^{n,\gamma}[\alpha, \beta]$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathscr{K}$, then

$$\frac{\Phi(2)}{2\Phi(2) + 2(1-\beta)} f * g(z) = \frac{\Phi(2)}{2\Phi(2) + 2(1-\beta)} z + \sum_{k=2}^{\infty} \frac{\Phi(2)}{2\Phi(2) + 2(1-\beta)} a_k b_k z^k.$$

If we can know

$$\Re\left\{1 + 2\sum_{k=2}^{\infty} \frac{\Phi(2)}{2\Phi(2) + 2(1-\beta)} a_k z^k\right\} > 0$$

From Lemma 1, it implies that the sequence

$$\left\{\frac{\Phi(2)}{2\Phi(2) + 2(1-\beta)} a_k\right\}_1^{\infty}$$

is a subordination factor sequence, with $a_1 = 1$. Now

$$\begin{aligned}
&\Re\left\{1 + 2\sum_{k=2}^{\infty} \frac{\Phi(2)}{2\Phi(2) + 2(1-\beta)} a_k z^k\right\} = \Re\left\{1 + \sum_{k=2}^{\infty} \frac{\Phi(2)}{\Phi(2) + 1-\beta} a_k z^k\right\} \\
&= \Re\left\{1 + \frac{\Phi(2)}{\Phi(2) + 1-\beta} z + \frac{1}{\Phi(2) + 1-\beta} \sum_{k=2}^{\infty} \Phi(2) a_k z^k\right\} \\
&\geqslant 1 - \frac{\Phi(2)}{\Phi(2) + 1-\beta} r - \frac{1}{\Phi(2) + 1-\beta} \sum_{k=2}^{\infty} \Phi(2)|a_k| r^k. \qquad (6)
\end{aligned}$$

since

$$\Phi(k) = [(\alpha\psi_k(\gamma, \lambda) + 1)(\psi_k(\gamma, \lambda) - 1) + 1 - \beta][\psi_k(\gamma, \lambda)]^n \quad (k = 2, 3, \ldots)$$

and

$$\psi_k(\gamma, \lambda) = \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}[1 + \lambda(k-1)] \quad (k = 2, 3, \ldots)$$

is a increasing function of k, so $0 < \Phi(2) \leqslant \Phi(k) \quad (k = 2, 3, \ldots)$.

Following (6), we can write

$$\geqslant 1 - \frac{\Phi(2)}{\Phi(2) + 1-\beta} r - \frac{1}{\Phi(2) + 1-\beta} \sum_{k=2}^{\infty} \Phi(k)|a_k| r^k.$$

As $0 < r < 1$, it can make sure

$$\geqslant 1 - \frac{\Phi(2)}{\Phi(2) + 1-\beta} r - \frac{r}{\Phi(2) + 1-\beta} \sum_{k=2}^{\infty} \Phi(k)|a_k|. \qquad (7)$$

Using Lemma 3 in (3) and following (7), we obtain

$$\Re\left\{1+2\sum_{k=2}^{\infty}\frac{\Phi(2)}{2\Phi(2)+2(1-\beta)}a_k z^k\right\} \geqslant 1-\frac{\Phi(2)}{\Phi(2)+1-\beta}r-\frac{1-\beta}{\Phi(2)+1-\beta}r=1-r>0,$$

In the light of Definition 1, we have

$$\frac{\Phi(2)}{2\Phi(2)+2(1-\beta)}f*g(z)=\sum_{k=1}^{\infty}\frac{\Phi(2)}{2\Phi(2)+2(1-\beta)}b_k c_k z^k \prec g(z),$$

Furthermore, it is easy to deduce the result in (5) by using (4) and Lemma 4.

Corollary 1. *If* $f(z)=z+\sum_{k=2}^{\infty}a_k z^k \in H_{\lambda}^{n,\gamma}[\alpha,\beta]$ *and* $F(z)=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k$, *then*

$$\frac{\Phi(2)}{\Phi(2)+1-\beta}F(z)\prec 2h(a,c;z) \tag{8}$$

and

$$\Re f(z)>\frac{\beta-1-\Phi(2)}{\Phi(2)}, \tag{9}$$

where $\Phi(2)=[(\alpha\psi_2(\gamma,\lambda)+1)(\psi_2(\gamma,\lambda)-1)+1-\beta][\psi_2(\gamma,\lambda)]^n$, *and* $h(a,c;z)$ *is the incomplete beta function defined in* (1) *with* $0<a\leqslant c$, $c\geqslant 2$ *or* $a+c\geqslant 3$.

Proof. Since $0<a\leqslant c$, $c\geqslant 2$ or $a+c\geqslant 3$, using Lemma 2, we can know that

$$h(a,c;z)=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k \in \mathscr{K}.$$

Taking $g(z)=h(a,c;z)$ and $g(z)=\frac{z}{1-z}$ in Theorem 1, respectively, the results (8) and (9) are obtained.

Corollary 2. *If* $f\in\bar{H}[\alpha,\beta]$ *in* $\mathbb{U}$ *and* $s>0$, $0<|z|=r<1$, *then for function* $g\in\mathscr{K}$

$$\frac{2(\alpha+1)-\beta}{2(\alpha-\beta)+3}f*g(z)\prec 2g(z)$$

and

$$\frac{[2(\alpha+1)-\beta]}{2(\alpha-\beta)+3}\int_0^{2\pi}|f*g(re^{i\theta})|^s d\theta\leqslant 2\int_0^{2\pi}|g(re^{i\theta})|^s d\theta.$$

Proof. By taking $n=0$, $\gamma=0$ and $\lambda=1$ in Theorem 1, Corollary 2 is given.

Corollary 3. *If $f \in T^*(\beta)$ in $\mathbb{U}$ and $s > 0$, $0 < |z| = r < 1$, then for function $g \in \mathscr{K}$*

$$\frac{2-\beta}{3-2\beta} f * g(z) \prec 2g(z)$$

and

$$\frac{2-\beta}{3-2\beta} \int_0^{2\pi} |f * g(re^{i\theta})|^s d\theta \leqslant 2 \int_0^{2\pi} |g(re^{i\theta})|^s d\theta.$$

Proof. By taking $\alpha = 0$ in Corollary 2, Corollary 3 is given.

Corollary 4. *If $f \in C(\beta)$ in $\mathbb{U}$ and $s > 0$, $0 < |z| = r < 1$, then for function $g \in \mathscr{K}$*

$$\frac{4-\beta}{5-2\beta} f * g(z) \prec 2g(z)$$

and

$$\frac{4-\beta}{5-2\beta} \int_0^{2\pi} |f * g(re^{i\theta})|^s d\theta \leqslant 2 \int_0^{2\pi} |g(re^{i\theta})|^s d\theta.$$

Proof. By taking $\alpha = 1$ in Corollary 2, Corollary 4 is given.

3. Some Results on the Class $H_\lambda^{n,\gamma}[\alpha,\beta]$ with Fixed Equation

In this section, we shall obtain several interesting results on the functions which are defined by the class $H_\lambda^{n,\gamma}[\alpha,\beta]$ with the following nonhomogeneous Cauchy-Euler differential equation:

$$z^2 \frac{d^2L}{dz^2} + 2(\mu+1)z\frac{dL}{dz} + \mu(\mu+1)L = (1+\mu)(2+\mu)f(z) \tag{10}$$

where $L(z) \in T$, $f(z) \in H_\lambda^{n,\gamma}[\alpha,\beta]$, $\mu + 1 > 0$, $\mu \in R$.

The cauchy-Euler differential equation was introduced earlier to study the distortion inequalities and neighborhoods problems of the other class of functions by O. Altintaş et al. [2].

Theorem 2. *If the function $L(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k \in T$ satisfy the equation (10) with $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in H_\lambda^{n,\gamma}[\alpha,\beta]$, then for function $g(z) \in \mathscr{K}$,*

$$\frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2) + (\mu+1)(1-\beta)} L * g(z) \prec 2g(z) \tag{11}$$

and

$$\frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2) + (\mu+1)(1-\beta)} \int_0^{2\pi} |L * g(re^{i\theta})|^s d\theta \leqslant 2 \int_0^{2\pi} |g(re^{i\theta})|^s d\theta, \tag{12}$$

where $\Phi(2) = [(\alpha\psi_2(\gamma,\lambda)+1)(\psi_2(\gamma,\lambda)-1)+1-\beta][\psi_2(\gamma,\lambda)]^n$, $0 < |z| = r < 1$, $s > 0$.

Proof. Suppose $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathscr{K}$, then

$$\frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)} L * g(z) = \frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)} z + \sum_{k=2}^{\infty} \frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)} b_k c_k z^k.$$

If we show that

$$\Re\{1 + 2\sum_{k=2}^{\infty} \frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)} c_k z^k\} > 0$$

Then from Lemma 1, we say that the sequence

$$\left\{ \frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)} c_k \right\}_1^{\infty}$$

is a subordination factor sequence, with $c_1 = 1$. Now

$$\begin{aligned}
&\Re\left\{1 + 2\sum_{k=2}^{\infty} \frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)} c_k z^k\right\} \\
&= \Re\left\{1 + \sum_{k=2}^{\infty} \frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} c_k z^k\right\} \\
&= \Re\left\{1 + \frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} z + \frac{(\mu+3)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} \sum_{k=2}^{\infty} \Phi(2) c_k z^k\right\} \\
&\geqslant 1 - \frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} r - \frac{(\mu+3)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} \sum_{k=2}^{\infty} \Phi(2)|c_k| r^k \quad (13)
\end{aligned}$$

Because $L(z)$ satisfies the differential equation with the $f(z) \in H_{\lambda}^{n,\gamma}[\alpha,\beta]$, so

$$c_k = \frac{(\mu+1)(\mu+2)}{(k+\mu)(k+\mu+1)} a_k$$

Following (13), we have

$$\begin{aligned}
&\geqslant 1 - \frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} r - \frac{(\mu+3)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} \sum_{k=2}^{\infty} \Phi(2) \frac{(\mu+1)(\mu+2)}{(k+\mu)(k+\mu+1)} |a_k| r^k \\
&\geqslant 1 - \frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} r - \frac{(\mu+3)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} \sum_{k=2}^{\infty} \Phi(2) \frac{(\mu+1)(\mu+2)}{(2+\mu)(\mu+3)} |a_k| r^k \\
&\geqslant 1 - \frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} r - \frac{(\mu+1)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)} \sum_{k=2}^{\infty} \Phi(2)|a_k| r^k \qquad (14)
\end{aligned}$$

Since

$$\Phi(k)=[(\alpha\psi_k(\gamma,\lambda)+1)(\psi_k(\gamma,\lambda)-1)+1-\beta][\psi_k(\gamma,\lambda)]^n \quad (k=2,3,\ldots)$$

and

$$\psi_k(\gamma,\lambda)=\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}[1+\lambda(k-1)] \quad (k=2,3,\ldots)$$

is a increasing function of k, so $0<\Phi(2)\leqslant\Phi(k) \quad (k=2,3,\ldots)$.
Following (14), we can write

$$\geqslant 1-\frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}r-\frac{(\mu+1)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}\sum_{k=2}^{\infty}\Phi(k)|a_k|r^k.$$

As $0<r<1$, it can make sure

$$\geqslant 1-\frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}r-\frac{(\mu+1)r}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}\sum_{k=2}^{\infty}\Phi(k)|a_k|. \tag{15}$$

Since $f(z)=z+\sum\limits_{k=2}^{\infty}\in H_{\lambda}^{n,\gamma}[\alpha,\beta]$, using Lemma 3 and following (15), we obtain

$$\begin{aligned}&\Re\{1+2\sum_{k=2}^{\infty}\frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)}c_kz^k\}\\ &\quad\geqslant 1-\frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}r-\frac{(1-\beta)(\mu+1)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}r=1-r>0.\end{aligned}$$

In the light of Definition 1, we have

$$\frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)}L*g(z)=\sum_{k=1}^{\infty}\frac{(\mu+3)\Phi(2)}{2(\mu+3)\Phi(2)+2(\mu+1)(1-\beta)}b_kc_kz^k\prec g(z).$$

Furthermore, it is easy to deduce the result in (12) by using (11) and Lemma 4.

Corollary 5. *If the function* $L(z)=z+\sum\limits_{k=2}^{\infty}c_kz^k\in T$ *satisfy the equation* (10) *with* $f(z)=z+\sum\limits_{k=2}^{\infty}a_kz^k\in H_{\lambda}^{n,\gamma}[\alpha,\beta]$ *and* $F(z)=z+\sum\limits_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}c_kz^k$*, then*

$$\frac{(\mu+3)\Phi(2)}{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}F(z)\prec 2h(a,c;z) \tag{16}$$

and

$$\Re L(z)>-\frac{(\mu+3)\Phi(2)+(\mu+1)(1-\beta)}{(\mu+3)\Phi(2)}, \tag{17}$$

where $\Phi(2)=[(\alpha\psi_2(\gamma,\lambda)+1)(\psi_2(\gamma,\lambda)-1)+1-\beta][\psi_2(\gamma,\lambda)]^n$*, and* $h(a,c;z)$ *is the incomplete beta function with* $0<a\leqslant c$*,* $c\geqslant 2$ *or* $a+c\geqslant 3$ *and* $0<|z|=r<1$*,* $s>0$*.*

Proof. Since $0 < a \leqslant c$, $c \geqslant 2$ or $a + c \geqslant 3$, using Lemma 2, we can know that

$$h(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \in \mathscr{K}.$$

Taking $g(z) = h(a,c;z)$ and $g(z) = \frac{z}{1-z}$ in Theorem 2, respectively, the results (16) and (17) are obtained.

Corollary 6. *If the function* $L(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k \in T$ *satisfy the equation* (10) *with* $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in \bar{H}[\alpha, \beta]$, *then for function* $g(z) \in \mathscr{K}$,

$$\frac{(\mu+3)[2(\alpha+1)-\beta]}{(\mu+3)[2(\alpha+1)-\beta]+(\mu+1)(1-\beta)} L * g(z) \prec 2g(z)$$

and

$$\frac{(\mu+3)[2(\alpha+1)-\beta]}{(\mu+3)[2(\alpha+1)-\beta]+(\mu+1)(1-\beta)} \int_0^{2\pi} |L * g(re^{i\theta})|^s d\theta \leqslant 2 \int_0^{2\pi} |g(re^{i\theta})|^s d\theta.$$

Proof. By taking $n = 0$, $\gamma = 0$ and $\lambda = 1$ in Theorem 2, Corollary 6 is given.

Corollary 7. *If the function* $L(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k \in T$ *satisfy the equation* (10) *with* $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in T^*(\beta)$, *then for function* $g(z) \in \mathscr{K}$,

$$\frac{(\mu+3)(2-\beta)}{(\mu+3)(2-\beta)+(\mu+1)(1-\beta)} L * g(z) \prec 2g(z)$$

and

$$\frac{(\mu+3)(2-\beta)}{(\mu+3)(2-\beta)+(\mu+1)(1-\beta)} \int_0^{2\pi} |L * g(re^{i\theta})|^s d\theta \leqslant 2 \int_0^{2\pi} |g(re^{i\theta})|^s d\theta.$$

Proof. By taking $\alpha = 0$ in Corollary 6, Corollary 7 is given.

Corollary 8. *If the function* $L(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k \in T$ *satisfy the equation* (10) *with* $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in C(\beta)$, *then for function* $g(z) \in \mathscr{K}$,

$$\frac{(\mu+3)(4-\beta)}{(\mu+3)(4-\beta)+(\mu+1)(1-\beta)} L * g(z) \prec 2g(z)$$

and

$$\frac{(\mu+3)(4-\beta)}{(\mu+3)(4-\beta)+(\mu+1)(1-\beta)} \int_0^{2\pi} |L * g(re^{i\theta})|^s d\theta \leqslant 2 \int_0^{2\pi} |g(re^{i\theta})|^s d\theta.$$

Proof. By taking $\alpha = 1$ in Corollary 6, Corollary 8 is given.

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