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On Prime, Minimal Prime and Annihilator Ideals in an Almost Distributive Lattice

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Abstract. Necessary and sufficient conditions for a prime ideal to be a minimal prime ideal and prime ideal to be a principal ideal in an ADL are furnished. And some properties of the special subsets of the set of all prime ideals in an ADL are studied.

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1. Introduction

An Almost Distributed Lattice(ADL) was introduced by U. M. Swamy and Rao. G .C [3]. After the Boole's axiomatization of the two valued propositional calculus as the Boolean algebra many generalizations of a Boolean algebra both ring theoretically and lattice theoretically, have come into being. With an idea of bringing common abstraction to most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra, the concept of an "Almost Distributive Lattice(ADL)" was introduced. An ADL is an algebra $(R, \vee, \wedge)$ of type $(2, 2)$ which satisfies almost all the properties of a distributive lattice except possibly the commutative of $\vee$, the commutative of $\wedge$ and the right distributivity of $\vee$ over $\wedge$. It was also observed that any one of these three properties converts an ADL into a distributive lattice.
The concept of an ideal was introduced in an ADL analogous to that in a distributive lattice [3]. If R is an ADL, then the set $PI(R)$ of all principal ideals of R form a distributive lattice. This enables to extend many existing concepts in Distributive lattices to the class of ADLs. Almost distributive lattice arise as a natural generalization of a distributive lattices and hence it is natural to consider the properties of prime ideals in an almost distributive lattice. It is

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interesting to note that the results which are valid for distributive lattices in verbatim, for ADLs, even though the techniques of the proofs in the case of ADLs are slightly different , for the reason that the operations $\vee$ and $\wedge$ are not commutative. If I is an ideal of R , the concept of the minimal prime ideal belonging to I is defined in [1].
In [2] the concept of Annihilator ideals in an ADL is introduced with suitable examples and proved some basic properties of the annihilator ideals, analogous to that in a distributive lattice. It is proved that the set $A(R)$ of all annihilator ideals of an ADL R with 0 can be made into a complete boolean algebra. The aim of this paper is to study some additional properties of prime, minimal prime and annihilators ideals in an ADL. This paper consists of four sections. In the second section we recall some basic definitions and results. Third section is devoted to prove several necessary and sufficient conditions for a prime ideal to be a minimal prime ideal and prime ideal to be a principal ideals in an ADL. Fourth section deals with some properties of the special subsets of the set of all prime ideals in an ADL.

2. Preliminaries

In this article we recall certain definitions and important results mostly from [1] and [2], that we need in sequel.
An Almost Distributive Lattice (ADL) is an algebra $(R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying the following axioms.

1. $a \vee 0 = a$,
2. $0 \wedge a = 0$,
3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
5. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
6. $(a \vee b) \wedge b = b$, for all a,b,c $\in R$.

Throughout this paper, R stands for an ADL $(R, \wedge, \vee, 0)$ with zero unless otherwise mentioned. For any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ or, equivalently, $a \vee b = b$, then $\leq$ is a partial ordering on R. An element $m \in R$ is called maximal element in the poset $(R, \leq)$. That is for any $a \in R$, $m \leq a \Rightarrow m = a$.
A non empty subset I of R is said to be an ideal (filter) of R, if $a \vee b \in I (a \wedge b \in I)$ and $a \wedge x \in I$ $(x \vee a \in I)$ whenever $a, b \in I$ and $x \in R$. If I is an ideal of R and $a, b \in R$, then $a \wedge b \in I \Leftrightarrow b \wedge a \in I$. A proper ideal P of R is said to be prime if for any $x, y \in R$, $x \wedge y \in P$ implies either $x \in P$ or $y \in P$. A prime ideal of P of R is said to be minimal if there exists no prime ideal Q of R such that $Q \subset R$. A proper ideal P of R is said to be maximal if, there is no proper ideal Q of R such that $P \subseteq Q$. Note that every maximal ideal of R is prime. Dually we can define prime filter, minimal prime filter and maximal filter. For any non-empty subset A of an ADL R, define $A^* = \{x \in R \,|\, a \wedge x = 0$, for all $a \in A\}$ and is called an annihilator ideal of

A. For $x \in R$, the annulate $(x]^*$ of x is defined as $(x]^* = \{y \in R \mid x \wedge y = 0\}$. Let $\wp$, Σ and $\mathfrak{M}$ denote the set of all prime ideals, maximal ideals and minimal prime ideals in R respectively. Now we quote some results

Result 1. *Let I be an ideal of R and $a \in R$ such that $a \notin I$. Then there exists a prime ideal P of R such that $I \subseteq P$ and $a \notin P$*

Result 2. *A prime ideal of an ADL R is minimal if and only if $a \in P \Rightarrow (a]^* \notin P$.*

Result 3. *Every prime ideal of R contains a minimal prime ideal.*

Result 4. *The set $\mathscr{I}(R)$ of all ideals of R is a complete distributive lattice with the least element $\{0\}$ and the greatest element R under set inclusion in which, for any $I, J \in \mathscr{I}(R), I \cap J$ is the infimum of I, J and the supremum is given by $I \vee J = \{i \vee j \mid i \in I, j \in J\}$.*

Result 5. *P is a minimal prime ideal of R if and only if $R \setminus P$ is a maximal filter of R.*

Result 6. *For any $a, b \in R$, we have the following :*

1. $(a] \vee (b] = (a \vee b] = (b \vee a]$
2. $(a] \wedge (b] = (a \wedge b] = (b \wedge a]$.

Result 7. *Intersection of all minimal prime ideals of R is $\{0\}$.*

Result 8. *For any non-empty subset A of R, A^* is an ideal of R.*

Result 9. *For any non-empty subset A of R, $A^* \cap A = \emptyset$.*

Result 10. *For any non-empty subset A of R, $A^* = \cap\{M \in \mathfrak{M} \mid A \nsubseteq M\}$.*

Result 11. *Let I be an ideal and S be a multiplicatively closed subset of R such that $I \cap S = \emptyset$. Then there is a prime ideal M of R such that $I \subseteq M$ and $M \cap S = \emptyset$.*

Result 12. *For any ideal I of R, we have, $I = \cap\{P \mid P$ is a prime ideal of $R, I \subseteq P\}$.*

Result 13. *Every maximal ideal is prime in R.*

3. Prime Ideals

Addition to the properties of prime, minimal prime and annihilator ideals in an ADL carried out in [1] and [2], we study some more properties of these ideals in an ADL in this article. Two ideals I and J of R are said to be co-maximal if $I \vee J = R$.
If the ideals $(a]$ are $(a]^*$ are co- maximal, then we have

Theorem 1. *For any $a \in R$ if the ideals $(a]$ are $(a]^*$ are co- maximal, then $(a] = (a]^{**}$ and the ideals $(a]^*$ and $(a]^{**}$ are co- maximal and conversely.*

Proof. For $a \in R$ let $R = (a] \vee (a]^*$. We have

$$(a]^{**} = (a]^{**} \cap R = (a]^{**} \cap \left[(a] \vee (a]^*\right] = \left((a]^{**} \cap (a]\right) \vee \left((a]^{**} \cap (a]^*\right) = (a]$$

since $(a] \subseteq (a]^{**})$. Hence in this case $R = (a] \vee (a]^* = (a]^* \vee (a]^{**}$. Proof of converse is obvious.

Now we prove necessary and sufficient conditions for every prime ideal to be minimal in R.

Theorem 2. *Every prime ideal in R is minimal prime if and only if the ideals $(a]$ are $(a]^*$ are co-maximal for each $a \in R$.*

Proof. Let every prime ideal in R be minimal prime. Let if possible there exists $a \in R$ such that $(a] \vee (a]^* \subset R$. Select $x \in R$ such that $x \notin (a] \vee (a]^*$. Hence by Result 1, there exists a prime ideal, say P, in R such that $[(a] \vee (a]^*] \subseteq P$ and $P \cap [x) = \emptyset$. P being minimal by assumption, $(a]$ and $(a]^*$ can not be contained in P simultaneously (see Result 2) . This in turn shows that $R = (a] \vee (a]^*$ for each $a \in R$.
Conversely, let $R = (a] \vee (a]^*$ for each $a \in R$. Let if possible, there exists a prime ideal P in R which is not minimal. By Result 3, there exists a minimal prime ideal, say M, in R such that $M \subset P$. Select $x \in P \setminus M$. As $x \notin M, (x]^* \subseteq M$ (see Result 2). But then $(x] \subseteq P$ and $(x]^* \subseteq P$ will give $R = (x] \vee (x]^* \subseteq P$; a contradiction. Hence every prime ideal in R must be minimal prime.

Using Theorem 1 and Theorem 2, we have

Corollary 1. *Following statements are equivalent in R*

1. *Every prime ideal in R is minimal prime.*
2. *The ideals $(a]$ and $(a]^*$ are co-maximal, for each $a \in R$.*
3. *$(a] = (a]^{**}$ and the ideals $(a]^*$ and $(a]^{**}$ are co-maximal, for each $a \in R$.*

We know that P is a minimal prime ideal of R if and only if $R \setminus P$ is a maximal filter of R (see Result 5). Using this relation between minimal prime ideals and maximal filters of R, we get

Corollary 2. *Following statements are equivalent in R with maximal elements.*

1. *Every prime ideal in R is minimal prime.*
2. *Every prime filter in R is maximal.*
3. *Every prime filter in R is minimal prime.*

It is well known that a proper ideal in R need not be prime. A sufficient condition for a proper ideal in R to be prime is proved in the following theorem.

Theorem 3. *A proper ideal P in R is prime, if the set $\{I \in \mathscr{I}(R) | P \subseteq I\}$ is a totally ordered subset of $\mathscr{I}(R)$.*

Proof. Let a proper ideal P in R be such that the set $\{I \in \mathscr{I}(R) | P \subseteq I\}$ is a totally ordered subset of $\mathscr{I}(R)$. Let P be not prime. Then there exists $a, b \in R$ such that $a \wedge b \in P$ with $a \notin P$ and $b \notin P$. As $P \vee (a] \supset P$ and $P \vee (b] \supset P$ by assumption $P \vee (a] \subseteq P \vee (b]$ or $P \vee (b] \subseteq P \vee (a]$. Let us assume without loss of generality $P \vee (a] \subseteq P \vee (b]$. As $a \wedge b \in P$, we get

$$\begin{aligned} P = & P \vee (a \wedge b] = P \vee [(a] \wedge (b]] && \text{(by Result 6)} \\ = & [P \vee (a]] \wedge [P \vee (b]] \\ = & P \vee (a] && \text{(since } P \vee (a] \subseteq P \vee (b]) \end{aligned}$$

This shows that $a \in P$; a contradiction. Hence P must be a prime ideal.

For a special subset of the set $A_0(R)$ of all annulets of R, we have

Theorem 4. *Let X be a non-empty subset of R such that $0 \notin X$. Then*

$$\bigcup \left\{ \{a\}^* \,|\, a \in X \right\} = \bigcap \{ M \in \mathfrak{M} \,|\, M \cap X = \emptyset \} = \bigcap \left\{ P \in \wp \,|\, P \cap X = \emptyset \right\}.$$

Proof. Let $x \in \bigcup \{\{a\}^* | a \in X\}$. Then $x \wedge a = 0$ for some $a \in X$. Now, $0 = x \wedge a \in M$ for $M \in \mathfrak{M}$ with $M \cap X = \emptyset$ implies $x \in M$. Hence we get $x \in \bigcap \{M \in \mathfrak{M} | M \cap X = \emptyset\}$. This shows $\bigcup \{\{a\}^* | a \in X\} \subseteq \bigcap \{M \in \mathfrak{M} | M \cap X = \emptyset\}$.
Conversely, let if possible, there exists $x \in \bigcap \{M \in \mathfrak{M} | M \cap X = \emptyset\}$ such that $x \notin \bigcup \{\{a\}^* | a \in X\}$. Then $x \wedge a \neq 0$ for each $a \in X$. Let $\overline{X} = \{x \wedge a | a \in x\}$. Then as $0 \notin \overline{X}$, $[\overline{X})$ is a proper filter of R. Hence it must be contained in some maximal filter say F in R . Define $M = R \setminus F$. Then $M \in \mathfrak{M}$ (see Result 5) and $\overline{X} \cap M = \emptyset$. Thus $M \in \bigcap \{M \in \mathfrak{M} | M \cap X = \emptyset\}$. Hence by the choice of x, $x \in M$; a contradiction. This shows that $\bigcap \{M \in \mathfrak{M} | M \cap X = \emptyset\} \subseteq \bigcup \{\{a\}^* | a \in X\}$. Combining both the inclusions, we get $\bigcup \{\{a\}^* | a \in X\} = \bigcap \{M \in \mathfrak{M} | M \cap X = \emptyset\}$. As $\bigcap \{M \in \mathfrak{M} | M \cap X = \emptyset\} = \bigcap \left\{P \in \wp | P \cap X = \emptyset\right\}$ holds always, the result follows.

Recall that an ideal I of R is said to be an annihilator ideal if $I = I^{**}$. The set of all annihilator ideals in R is denoted by $A(R)$. Further we have

Theorem 5. *For any prime ideal P in R, consider the following statements.*

1. *For any $I \in \mathscr{I}(R), I$ and P are comparable.*
2. *For any $N \neq R$ in $A(R)$, $N \subseteq P$.*
3. *For any $M \in \mathfrak{M}, M \subseteq P$.*
4. *For any $x \notin P, \{x\}^* = \{0\}$.*

Then $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

Proof. $1 \Rightarrow 2$
Let if possible, there exist $N \neq R$ in $A(R)$ such that $N \nsubseteq P$. Hence by (1) $P \subset N$. Select $x \in N \setminus P$. As P is a prime ideal and $x \notin P$, we get $N^* \subseteq \{x\}^* \subseteq P \subseteq N$. This in turn implies that $N^* = \{0\}$; and hence $N = N^{**} = R$ contradicting the fact that $N \neq R$. Hence $N \subseteq P$ for each $N \neq R$ in $N(R)$.
$2 \Rightarrow 3$
Let $M \in \mathfrak{M}$. Define $X = L \setminus M$. Then by Theorem 4 we have

$$\bigcup\{\{a\}^* \,|\, a \in X\} = \bigcap\{M \in \mathfrak{M} \,|\, M \cap X = \emptyset\}.$$

Hence $\bigcup\{\{a\}^* \,|\, a \notin M\} = M$. Now $a \notin M \Rightarrow \{a\}^* \neq R$. Hence by (2), $\{a\}^* \subseteq P$ for each $a \notin M$. This gives $M = \bigcup\{\{a\}^* \,|\, a \notin M\} \subseteq P$ and the implication follows.
$3 \Rightarrow 4$
Let $a \notin P$. By assumption 7, $M \subseteq P$ for each $M \in \mathfrak{M}$. Hence $a \notin M$ for each $M \in \mathfrak{M}$. But then $\{a\}^* \subseteq M$ for each $M \in \mathfrak{M}$ (see Result 2) will give $\{a\}^* \cap \{M \,|\, M \in \mathfrak{M}\} = \{0\}$ (see Result 7). Thus we get $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

Theorem 6. *The statements of Theorem 5 are equivalent if R satisfies following condition (*).*
(*) For any I
$vertJ$, $I, J \in \mathscr{I}(R)$, there exists $x \in I \setminus J$ and $y \in J \setminus I$ such that $x \wedge y = 0$.

Proof. To prove that conditions are equivalent in R, it is enough to prove that $4 \Rightarrow 1$ under the condition (*). Let there exist an ideal $I \in \mathscr{I}(R)$ such that I
$vertP$, By condition (*) select $x \in I \setminus P$ and $y \in P \setminus I$ such that $x \wedge y = 0$. Then $x > 0, y > 0$ and $y \in \{x\}^*$. Again by assumption, $x \in P$ implies $\{x\}^* = \{0\}$; a contradiction. Hence I and P must be comparable for each $I \in \mathscr{I}(R)$.

Theorem 7. *In an ADL R, if an ideal $I \neq \{0\}$ is a totally ordered subset of R, then I^* is a minimal prime ideal in R.*

Proof. Claim 1: $I^* = \{a\}^*$ for any $0 < a \in I$.
Let $0 < a \in I$. Then $I^* \subseteq \{a^*\}$ always. Let if possible, $I^* \subset \{a\}^*$. Select $x \in \{a\}^* \setminus I^*$. Then $x > 0, x \wedge a = 0$ and $x \wedge b \neq 0$ for some $b \in I$. As I is totally ordered, either $x \wedge b \leq a$ or $a \leq x \wedge b$. If $x \wedge b \leq a$, then $x \wedge b = (x \wedge b) \wedge a = x \wedge (b \wedge a) = x \wedge 0$ (since $a \wedge b = 0 \Rightarrow b \wedge a = 0$). Hence $x \wedge b = 0$; a contradiction. If $a \leq x \wedge b$, then $a = a \wedge (x \wedge b) = x \wedge (a \wedge b) = x \wedge 0 = 0$; a contradiction. Thus $I^* = \{a\}^*$ for any $a \in R$.
Claim 2: I^* is a prime ideal in R.
I^* is an ideal in R (see Result 8). Let there exist $a, b \in R$ such that $a \wedge b \in I^*$ with $a \notin I^*$ and $b \notin I^*$. But then $a \wedge x > 0$ and $b \wedge y > 0$ for some $x, y \in I$. Now,

$$(a \wedge x) \wedge (b \wedge y) = a \wedge [x \wedge b \wedge y] = a \wedge [b \wedge x \wedge y] = (a \wedge b) \wedge (x \wedge y) = 0$$

(as $x \wedge y \in I$ and $a \wedge b \in I^*$). But this shows that $b \wedge y \in \{a \wedge x\}^*$. Thus $b \wedge y \in I \cap I^* = \{0\}$ (see Result 9). Hence $b \wedge y = 0$; a contradiction. Hence I^* is a prime ideal.

Claim 3: I^* is a minimal prime ideal in R.
We know that $I^* = \bigcap\{M \in \mathfrak{M} \mid I \nsubseteq M\}$ (by Result 10)...(I). Let if possible there exists a minimal prime ideal M in R such that $M \subset I^*$ (see Result 3). If $I \nsubseteq M$, then there exists $x \in I$ such that $x \notin M$.

$$\begin{aligned}
& x \in I \text{ and } x \notin M \Rightarrow x > 0. \\
& x \in I \text{ implies } I = \{x\}^* && \text{by claim 1} \\
& x \notin M, M \text{ is minimal } \Rightarrow \{x\}^* \subseteq M \\
& \Rightarrow I \subseteq M \\
& \Rightarrow I = I \cap M \subseteq I \cap I^* = \{0\} \\
& \Rightarrow I = \{0\}, \text{ a contradiction} && (I^* \subseteq M \text{ by Result 9})
\end{aligned}$$

Hence $I^* \in \mathfrak{M}$

Necessary and sufficient conditions for any prime ideal in R to be a principal ideal are given in the following theorem.

Theorem 8. *Let $\{P_\alpha \mid \alpha \in \Delta\}$ (Δ any indexing set) be any family prime ideals in R. Then following statements are equivalent:*

1. *For any ideal I in R if $I \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$, then $I \subseteq P_\alpha$ for some $\alpha \in \Delta$.*
2. *For any prime ideal I in R if $P \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$, then $P \subseteq P_\alpha$ for some $\alpha \in \Delta$.*
3. *Every (proper) ideal in R is a principal.*
4. *Every prime ideal in R is a principal.*

Proof. This implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$ are obviously true.
$(2) \Rightarrow (1)$
Let $I \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$, I an ideal on R. Define $M = R \setminus \bigcup_{\alpha\in\Delta} P_\alpha$. Then $M \neq \emptyset$. Let $x, y \in M$. Then $x, y \notin \bigcup_{\alpha\in\Delta} P_\alpha$ imply $x \notin P_\alpha$ and $y \notin P_\alpha$ for each $\alpha \in \Delta$. P_α being a prime ideal, $x \wedge y \notin P_\alpha$ for each $\alpha \in \Delta$. But then $x \wedge y \in M$. This shows that M is closed for $\wedge$. Further $I \cap M = \emptyset$. Hence by Result 11 there exist a prime ideal P in R such that $I \subseteq P$ and $P \cap M = \emptyset$. But then $P \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$ will imply $P \subseteq P_\alpha$ for some $\alpha \in \Delta$, by the condition (2). As $I \subseteq P$ we get $I \subseteq P_\alpha$ and the implication follows.
$(2) \Rightarrow (3)$
Suppose R satisfies the condition (2). Assume that there exists a prime ideal P in R which is not principal. Hence $P \neq (y]$ for any $y \notin P$. As $(y] = \bigcap\{P \mid P \text{ is a prime ideal and } y \in P\}$ (see Result 12) we get $P \neq \bigcap\{P_y \mid P_y \text{ is a prime ideal in } R \text{ containing } y\}$. If $P \subseteq P_y$ for each prime ideal P_y containing y, then $(y] \subseteq \{P_y \mid y \in P\} = (y]$ will imply $P = (y]$; a contradiction. Hence for each $y \in R$ there exists a prime ideal P_y in R such that $y \in P_y$ and $P \nsubseteq P_y$. Again $P \subseteq \bigcup\{P_y \mid y \in P\}$ implies $P \subseteq P_y$ for some $y \in P$; a contradiction. Hence our assumption is wrong. Therefore every prime ideal in R is principal.
$(3) \Rightarrow (4)$

Let a prime ideal $P \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$, ($\Delta$ any indexing set). By assumption, $P = (x]$ for some $x \in R$. Then $(x] \subseteq \bigcup_{\alpha\in\Delta} P_\alpha$ implies $x \in P_\alpha$ for some $\alpha \in \Delta$. But then $P = (x] \subseteq P_\alpha$ and we are through. (4) $\Rightarrow$ (3)

Suppose the statement (3) is false. Then there exists a proper non principal ideal in R. Let $\mathfrak{A}$ denote the non-empty collection of non-principal proper ideals of R. It is clear that $\mathfrak{A}$ is closed under the formation of unions of chains in $\mathfrak{A}$. So, by Zorn's lemma we get a maximal element M in $\mathfrak{A}$ which is not principal. Since M is proper, $R \neq M$. As M is not prime, there exist elements $a, b \in M$ such that $a \wedge b \in M$. Now as M is a maximal element in $\mathfrak{A}$, $M \vee (a]$ and $M \vee (b]$ are principal ideals. Let $(M \vee (a]) = (x]$ and $(M \vee (b]) = (y]$. Hence $M = (M \vee (a]) \wedge (M \vee (b]) = (x] \wedge (y] = (x \wedge y]$ (by result 6); contradicting that M is not principal. Hence the implication.

Thus as (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) result follows.

For $a \in R$, an ideal in R which is maximal w.r.t not containing the element a is called a-maximal ideal. Interestingly, we have

Theorem 9. *Any a-maximal ideal in R is prime.*

Proof. Let M be a-maximal ideal in R. Then $a \notin M$. Suppose there exist $x, y \in R$ such that $x \wedge y \in M$ with $x \notin M$ and $y \notin M$. But then $a \in M \vee (x]$ and $y \vee M \vee (y]$ will imply $a = m_1 \vee (t \wedge x)$ and $y = m_2 \vee (s \wedge y)$, for some $t, s \in R$. Thus

$$\begin{aligned} a =& \left[m_1 \vee (t \wedge x)\right] \wedge \left[m_2 \vee (s \wedge y)\right] \\ =& m_1 \wedge \left[m_2 \vee (s \wedge y)\right] \vee (t \wedge x) \wedge \left[m_2 \vee (s \wedge y)\right] \\ =& (m_1 \wedge m_2) \vee \left[m_1 \wedge (s \wedge y)\right] \vee (t \wedge x) \wedge m_2) \vee (t \wedge x) \wedge (s \wedge y) \ldots \end{aligned} \tag{1}$$

$m_1 \wedge m_2 \in M$ and $m_1 \wedge (s \wedge y) \in M$ (since $m_1 \in M$). Further

$$m_2 \in M \Rightarrow m_2 \wedge (t \wedge x) \in M \Rightarrow (t \wedge x) \wedge m_2 \in M$$

(since M is an ideal). Again $(t \wedge x) \wedge (s \wedge y) = t \wedge (x \wedge s \wedge y) = t \wedge (s \wedge x \wedge y) = (t \wedge s) \wedge (x \wedge y)$. As $x \wedge y \in M$,we get $(x \wedge y) \wedge (t \wedge s) \in M$ and hence $(t \wedge s) \wedge (x \wedge y) \in M$. But then by (1), we have $a \in M$; a contradiction. Hence M is prime.

An ideal J in R is meet irreducible if $J = \bigcap_{\lambda\in\Delta} I_\lambda$ where $\{I_\lambda\}_{\lambda\in\Delta}$ is a family of ideals in R (Δ is any indexing family), then $J = I_\lambda$ for some $\lambda \in \Delta$. In the following theorem we furnish some characterizations of a-maximal ideals in R.

Theorem 10. *Following statements are equivalent in R.*

1. *M is a-maximal ideal for some $a \in R$.*
2. *M is meet irreducible*
3. *$M \subset M' = \cap\{I \in \mathscr{I}(R) \mid I \supset M\}$*
4. *M is x-maximal for some $x \in M' \setminus M$.*

Proof. $(1) \Rightarrow (2)$
Let $M = \bigcap_{\lambda \in \Delta} I_\lambda$ where $\{I_\lambda\}_{\lambda \in \Delta}$ is a family of ideals in R and Δ is any indexing family. M is a-maximal $\Rightarrow a \notin M \Rightarrow a \notin I_{\lambda_0}$ for some $\lambda_0 \in \Delta$. M being a- maximal ideal, we get $I_{\lambda_0} = M$ as $M \subseteq I_{\lambda_0}$.
$(2) \Rightarrow (3)$
Let if possible $M \subset M' = \cap\{I \in \mathscr{I}(R) \,|\, I \supset M\}$. By assumption (2), $M = I$ for some $I \supset M$, a contradiction. Hence $M \subset M'$.
$(3) \Rightarrow (4)$
By (3), $M \subset M' = \cap\{I \in \mathscr{I}(R) \,|\, I \supset M\}$. Select $x \in M' \setminus M$. Thus $x \in I$ for each $\mathscr{I} \in I(R)$ with $I \supset M$. If M is not x-maximal, then there exists an ideal say J properly containing M and not containing x (see Result 1). But then $J \in \{I \in \mathscr{I}(R) \,|\, I \supset M\}$. Hence $x \in J$; a contradiction. Therefore M is x-maximal for any $x \in M' \setminus M$.
$(4) \Rightarrow (1)$ being obviously true, all the statements are equivalent.

Similar to the result in ring-theory, we have (see [reticulated rings])

Theorem 11. *For each element x and a prime ideal P of R, the following are equivalent*

(i) $\{x\}^* \subseteq P$

(ii) There is some $M \in \mathfrak{M}$ with $M \subseteq P$ and $x \notin M$.

Proof. $(i) \Rightarrow (ii)$
Define $S = \{a \wedge x \,|\, a \notin P\}$. If $0 \in S$, then $a \wedge x = 0$ for some $a \notin P$ will imply $a \in \{x\}^* \subseteq P$ (by (i)); a contradiction. Hence $0 \notin S$. Again for a maximal element m in R, $m \wedge x = x$ and $m \notin P$ will give $x \in S$. Hence S is non-empty. Further $a \wedge x, b \wedge y \in S$ for $a, b \in P$ implies $(a \wedge x) \wedge (b \wedge y) = a \wedge (x \wedge b \wedge y) = a \wedge (b \wedge x \wedge y) = (a \wedge b) \wedge (x \wedge y) \in S$ as $a \wedge b \notin P$ (P being a prime ideal in R). Thus S is a multiplicatively closed subset of R not containing 0. Hence by Result 12 there exists a prime ideal P in R with $P \cap S = \emptyset$. Hence $x \notin P$. As every prime ideal in R contains a minimal prime ideal, select $M \in \mathfrak{M}$ such that $M \subseteq P$, and the implication follows.
$(ii) \Rightarrow (i)$ follows by Result 2 .

4. The Properties of the Set U(I)

For any ideal I in R, define $U(I) = \{P \in \wp \,|\, I \nsubseteq P\}$ and for any $a \in R$ define
$U(a) = \{P \in \wp \,|\, a \notin P\}$. Note that $U(a) = U((a])$ for any $a \in R$. The aim of this article is to study some properties of the sets $U(I)$. For any prime ideal P in R we define
$S_P = \bigcap\{M \in \wp \,|\, M \subseteq P\}$.

Theorem 12. *For a prime ideal P in R, $S_P = \{a \in R \,|\, a = 0$ or $P \| Q$ for each $Q \in U(a)\}$*

Proof. Let $\mathscr{K} = \{a \in R \,|\, a = 0$ or $P\|Q$ for each $Q \in U(a)\}$. To prove that $S_P = \mathscr{K}$. Let $S_P \neq \mathscr{K}$. Select $0 < a \in S_P$ such that $a \notin \mathscr{K}$. By the definition of $\mathscr{K}$, there exist $Q \in U(a)$ such that P and Q are comparable. If $P \subseteq Q$, then $S_P \subseteq P \Rightarrow S_P \subseteq Q$. If $Q \subseteq P$, then $S_P \subseteq S_Q$

and $S_Q \subseteq Q$ imply $S_P \subseteq Q$. Thus in either the case $S_P \subseteq Q$. But then $a \in Q$; a contradiction. Therefore $S_P \subseteq \mathscr{K} \ldots$ (I)
Let if possible $\mathscr{K} \nsubseteq S_P$. Pick $0 < b \in \mathscr{K} \setminus S_P$. As $b \notin S_P$ there exists $M \in \mathfrak{M}$ such that $M \subseteq P$ but $b \notin M$. But then $M \in X(b)$. As $b \in \mathscr{K}$ and $b > 0$, we get M
$vertP$; a contradiction. Hence $\mathscr{K} \subseteq S_P \ldots$ (II)
From (I) and (II) we get $\mathscr{K} = S_P$ and the result follows.

Let $< P, \leq >$ be a bounded poset. A non empty subset F of P is a semi filter (or dual semi ideal) in P if $a \in F, b \in P$ and $a \leq b$ imply $b \in F$ [see 4]. For $M \in \Sigma$, we define $\widehat{W_M} = \{P \in \wp \mid P \subseteq M\}$ and $W_M = \bigcap \widehat{W_M}$
Using the concept of semi filter in the poset of prime idals $(\wp, \subseteq)$ in R, we have

Theorem 13. *Let $I \in \mathscr{I}(R)$ be such that $U(I)$ is a semi filter in the poset $(\wp, \subseteq)$. Then following properties hold in R.*

1. *If $I \subseteq M \in \Sigma$ and $P \in \wp, P \subseteq M$. Then $I \subseteq P$*
2. *If $I \subseteq M \in \Sigma$ imply $I \subseteq W_M$*
3. *$I = \bigcap_{M\in\Sigma}\{W_M \mid I \subseteq W_M\}$*
4. *$U(I) = \wp \setminus \bigcup_{M\in\mathscr{K}} \widehat{W_M} = \wp \setminus \bigcup_{M\in\mathscr{K}}\{P \in \wp \mid P \subseteq M\}$*
5. *If $a \in I$, we have $(a]^* \vee I = R$.*

Proof. 1. Suppose that $I \nsubseteq P$. Then $P \in U(I)$. But as $U(I)$ is a semi filter and $P \subseteq M$ we get $M \in U(I)$. So $I \nsubseteq M$, which is absurd. Therefore $I \subseteq P$.
2. Select $P \in \wp$ such that $P \subseteq M$. If $I \nsubseteq P$, then $P \in U(I)$. As $U(I)$ is a semi filter in $(\wp, \subseteq)$ and $P \subseteq M$ we get $M \in U(I)$ (since $M \in \wp$). But then $I \nsubseteq M$; a contradiction. Hence $I \subseteq P$. This shows that $I \subseteq P$ for each $P \in \wp$ with $P \subseteq M$. Hence $I \subseteq \bigcap\{P \in \wp \mid P \subseteq M\} = W_M$.
3. Obviously, $I \subseteq \bigcap_{M\in\Sigma}\{W_M \mid I \subseteq W_M\}$. Hence to prove that $\bigcap_{M\in\Sigma}\{W_M \mid I \subseteq W_M\} \subseteq I$. Let if possible $\bigcap_{M\in\Sigma}\{W_M \mid I \subseteq W_M\} \nsubseteq I$. Select $x \in \bigcap_{M\in\Sigma}\{W_M \mid I \subseteq W_M\}$ such that $x \notin I$. By Result 11, there exists a prime ideal Q in R such that $I \subseteq Q$ and $x \notin Q$. As Q is a proper ideal, Q must be contained in some maximal ideal say M in R (by Result 3). But then $I \subseteq M$ will imply $I \subseteq W_M$ (by property 1). Therefore $x \in W_M$. As $W_M = \{P \in \wp \mid P \subseteq M\}$, we get $W_M \subseteq Q$. But then $x \in Q$; which is absurd. Hence $\bigcap_{M\in\Sigma}\{W_M \mid I \subseteq W_M\} \subseteq I$. Combining both the inclusions, we get $I = \bigcap_{M\in\Sigma}\{W_M \mid I \subseteq W_M\}$.
4. Let $Q \in \bigcup_{M\in\mathscr{K}}\{P \in \wp \mid P \subseteq M\}$. Then $Q \in \wp, Q \subseteq M$ and $I \subseteq M$. Hence $I \subseteq Q$ (see property 1). Therefore $Q \notin U(I)$. This shows that $U(I) \subseteq \wp \setminus \bigcup_{M\in\mathscr{K}}\{P \in \wp \mid P \subseteq M\}$. Now, let $Q \notin U(I)$. Then $I \subseteq Q$. Let M denote a maximal ideal containing Q (by Result 3). As $I \subseteq M$, $M \in \mathscr{K}$. This shows that $Q \in \bigcup_{M\in\mathscr{K}}\{P \in \wp \mid P \subseteq M\}$. Therefore
$Q \notin \wp \setminus \bigcup_{M\in\mathscr{K}}\{P \in \wp \mid P \subseteq M\}$. This shows that $\wp \setminus \bigcup_{M\in\mathscr{K}}\{P \in \wp \mid P \subseteq M\} \subseteq U(I)$. Combining both the inclusions $U(I) = \wp \setminus \bigcup_{M\in\mathscr{K}}\{P \in \wp \mid P \subseteq M\}$.
5. Let $I \in \mathscr{I}(R)$ be such that $U(I)$ is a semi filter in $(\wp, \subseteq)$ but $(a]^* \vee I \neq R$ for some $a \in I$, i.e. $(a]^* \vee I \subset R$. Then $(a]^* \vee I$ is a proper ideal of R. Then by Theorem 11 there exists a maximal ideal M of R such that $(a]^* \vee I \subseteq M$ (see Result 5). Then we have $(a]^* \subseteq M$. Then

by Result 13 there exists a minimal prime ideal $Q \subseteq M$ and $a \notin Q$. As $a \in I$ and $a \notin Q$ we have $I \nsubseteq Q$, which means that $Q \in U(I)$. Since $Q \in U(I)$ and $Q \subseteq M$ and $U(I)$ is semi filter, we conclude that $M \in U(I)$, i.e. $I \nsubseteq M$. This is a contradiction since $I \subseteq (a]^* \vee I \subseteq M$. Hence $(a]^* \vee I = R$.

In the following theorem we prove sufficient condition on an ideal I for $U(I)$ to be a semi filter in $(\wp, \subseteq)$.

Theorem 14. *Let $M_1, M_2, \dots M_n$ (n-finite) be maximal ideals in R. Let $I = \bigcap_{i=1}^n W_{M_i}$. Then $U(I)$ is a semi filter in poset $(\wp, \subseteq)$.*

Proof. Claim 1: $U(I) = \wp\wp \setminus \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\}$.

$$\begin{aligned}
\text{Select} P \in U(I) &\Rightarrow I \nsubseteq P, P \in \wp \\
&\Rightarrow \bigcup_{i=1}^{n} W_i \nsubseteq P, P \in \wp \\
&\Rightarrow W_{M_i} \nsubseteq P, P \in \wp && \text{for all } i, 1 \le i \le n. \text{ But this gives} \\
&\bigcap \{P \in \wp \mid P \subseteq M_i\} \nsubseteq P, && \text{for all } i, 1 \le i \le n
\end{aligned}$$

Thus $P \notin \{P \in \wp \mid P \subseteq M\}$ for all $i, 1 \le i \le n$. Hence $P \notin \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\}$. i.e. $P \in \wp \setminus \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\}$. Thus $U(I) \subseteq \wp \setminus \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\} \dots$ (I)
Now, select $P \in \wp$ such that $P \notin \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\}, P \in \wp$. But then $\bigcap \{P \in \wp \mid P \subseteq M_i\} \nsubseteq P$ for each i, $1 \le i \le n$, $P \in \wp$ implies $I \nsubseteq P\ P \in \wp$. Hence $P \in U(I)$.
Thus $\wp \setminus \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\} \subseteq U(I) \dots$ (II)
Combining both the inclusions we get $U(I) = \wp \setminus \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\}$
Claim 2: $U(I)$ is a semi filter in $(\wp, \subseteq)$
Let $P, Q \in \wp$ with $P \subseteq Q$ and $P \in U(I)$. $P \in U(I)$ gives $P \notin \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\}$, (by claim 1). As $P \subseteq Q$, obviously, $Q \notin \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\}$. But then
$Q \in \wp \setminus \bigcup_{i=1}^n \{P \in \wp \mid P \subseteq M_i\} = U(I)$. This shows that $U(I)$ is a semi filter in $(\wp, \subseteq)$ and the result follows.

Theorem 15. *Let I be an ideal of R such that $(x]^*$ and I are co-maximal ideals for each $x \in R$. Then $U(I)$ is a semi filter in $(\wp, \subseteq)$.*

Proof. Let $P \in U(I), P \subseteq Q$ and $Q \in \wp$. We must show that $Q \in U(I)$. Assume on the contrary that $Q \notin U(I)$. This implies $I \subseteq Q$. As P is prime ideal we have some $J \in \mathfrak{M}$ such that $J \subseteq P$ (by Result 2). If $I \subseteq J \subseteq P$ then $P \notin U(I)$, which is a contradiction. Hence $I \nsubseteq J$. Select $x \in I$ such that $x \notin J$. If $(x]^* \nsubseteq J$, then there exists $t \in (x]^*, t \notin J$. As $t \in (x]^*$ gives that $t \wedge x = 0 \in J$. But as $x \notin J$ and $t \notin J$ will give $t \wedge x \notin J$ i.e. $0 \notin J$ which is impossible. Hence $(x]^* \subseteq J$. But $J \subseteq P \subseteq Q$ implies $(x]^* \subseteq Q$. Since $I \subseteq Q$, $(x]^* \vee I \subseteq Q$. Thus $R = Q$ which is absurd. Hence we conclude that $Q \in U(I)$. From this it follows that $U(I)$ is a semi filter.

Necessary and sufficient for $U(I)$ to be a semi filter $(I \in \mathscr{I}(R))$ in $(\wp, \subseteq)$ is proved in the following theorem.

Theorem 16. *For $I \in \mathscr{I}(R)$, $U(I)$ is a semi filter in $(\wp, \subseteq)$ if and only if $U(I) = \bigcup_{a\in I} V((a]^*)$.*

Proof. Let $U(I) = \bigcup_{a\in I} V((a]^*), P \in U(I), Q \in \wp$ and $P \subseteq Q$. Then $I \nsubseteq P$. Therefore there exists $x_1 \in I \setminus P$. But then $(x_1]^* \subseteq P \subseteq Q$ shows that $Q \in V\left((x_1]^*\right) \subseteq \bigcup_{a\in I} V((a]^*) = U(I)$. This in turn shows that $U(I)$ is a semi filter.
Now $U(I)$ is a semi filter in $(\wp, \subseteq)$. To prove that $U(I) = \bigcup_{a\in I} V((a]^*)$.
Let $P \in U(I)$. But then $I \nsubseteq P$. Therefore there exists an element $x_1 \in I \setminus P$. But then $(x_1]^* \subseteq P$. i.e $P \in V\left((x_1]^*\right) \subseteq \bigcup_{a\in I} V((a]^*)$. Hence $U(I) \subseteq \bigcup_{a\in I} V((a]^*)$. let $P \in \bigcup_{a\in I} V((a]^*)$ then there exists an element $y \in I$ such that $P \in V(y]^*$ but then $(y]^* \subseteq P$. If $I \subseteq P$, then $(y]^* \vee I \subseteq P$. Hence by Theorem 13 $R \subseteq P$ a contradiction. Therefore we must we have $I \nsubseteq P$. Therefore $P \in U(I)$. Thus $\bigcup_{a\in I} V((a]^*) \subseteq U(I)$. Combining both the inclusions we get $U(I) = \bigcup_{a\in I} V((a]^*)$.

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