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On Semi-open Sets With Respect To an Ideal

Friday Ifeanyi Michael

Department of Mathematics, Faculty of Science, The University of Western Ontario, London, Ontario, Canada

Abstract. We introduce a notion of semi-open sets in terms of ideals, which generalizes the usual notion of semi-open sets.

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1. Introduction

With the impetus given by Levine's introduction of semi-open sets and generalized closed sets [11, 12], there have been other attempts by some topologists to study closed sets - together with the accompanying topological notions - from different perspectives [see, for example, 1, 2, 3, 5, 6, 4]. Relevant to the present work is the idea of using topological ideals in describing topological notions, which, for some years now, has been an interesting subject for investigation [see some of the pioneering works in 7, 8, 9]. We recall here that an *ideal* $\mathscr{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X having the *heredity* property (that is, if $A \in \mathscr{I}$ and $B \subset A$, then $B \in \mathscr{I}$) and also satisfying *finite additivity* (that is, if $A, B \in \mathscr{I}$, then $A \cup B \in \mathscr{I}$).

In this paper, we define semi-open sets with respect to an ideal $\mathscr{I}$, and also study some of their properties. It turns out that our notion of semi-open sets with respect to a given ideal $\mathscr{I}$ generalizes both the usual notion of semi-openness [11] and the notion of semi-$\mathscr{I}$-openness considered in [5]; in particular, semi-$\mathscr{I}$-openness implies the usual semi-openness, which in turn implies semi-openness in our sense. Throughout we work with a topological space (X, τ) (or simply X), where no separation axioms are assumed. The usual notation cl(A) for the closure, and int(A) for the interior, of a subset A of a topological space (X, τ), will be used [see 3, 10, 4, for example].

Email address: `fmichael@uwo.ca`

2. Semi-openness With Respect To an Ideal

Let X be a topological space. Recall that a subset A of X is said to be *semi-open* [11] if there is an open set U such that $U \subseteq A \subset \mathrm{cl}(U)$. This motivates our first definition.

Definition 1. *A subset A of X is said to be* semi-open with respect to an ideal $\mathscr{I}$ *(written as $\mathscr{I}$-semi-open) if there exists an open set U such that $U - A \in \mathscr{I}$ and $A - \mathrm{cl}(U) \in \mathscr{I}$.*

If $A \in \mathscr{I}$, then it is easy to see that A is $\mathscr{I}$-semi-open. Moreover, every open set A is semi-open, and every semi-open set B is $\mathscr{I}$-semi-open, for any ideal $\mathscr{I}$ on X.

Example 1. *Consider a topological space (X, τ); $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{a, c\}, X\}$. Choose $\mathscr{I} = \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$, and observe that $\{b\}$ is $\mathscr{I}$-semi-open; however, $\{b\}$ is not semi-open in the sense of [11] as there is no open set U such that $U \subset \{b\} \subset \mathrm{cl}(U)$. Thus, if a set is $\mathscr{I}$-semi-open, it may not be semi-open in the usual sense.*

For an ideal $\mathscr{I}$ that is not countably additive, the concepts of semi-openness and $\mathscr{I}$-semi-openness coincide in the following case.

Theorem 1. *For an ideal $\mathscr{I}$ on a topological space X, the following are equivalent:*

1. *$\mathscr{I}$ is the minimal ideal on X, that is, $\mathscr{I} = \{\varnothing\}$;*
2. *The concepts of semi-openness and $\mathscr{I}$-semi-openness are the same.*

Proof. First suppose that $\mathscr{I} = \{\varnothing\}$. It suffices to show that whenever a set A is $\mathscr{I}$-semi-open, then it is semi-open in the usual sense. Indeed, if A is $\mathscr{I}$-semi-open, then there is an open set U such that $U - A$, $A - \mathrm{cl}(U) \in \mathscr{I} = \{\varnothing\}$, and so $U \subset A \subset \mathrm{cl}(U)$, proving that A is semi-open. Conversely, suppose that whenever a set A is $\mathscr{I}$-semi-open, then it is semi-open. Let $B \in \mathscr{I}$. Then, B is $\mathscr{I}$-semi-open, and by assumption, B is semi-open. Thus, there is an open set V_1 such that $V_1 \subset B \subset \mathrm{cl}(V_1)$. Since $B \in \mathscr{I}$ and $V_1 \subset B$, we have that $V_1 \in \mathscr{I}$, and so $B \cup V_1 \in \mathscr{I}$. As $B \cup V_1$ is $\mathscr{I}$-semi-open, it is semi-open, so that there is an open set V_2 for which $V_2 \subset (B \cup V_1) \subset \mathrm{cl}(V_2)$. Similarly, there is an open set V_3 such that $V_3 \subset (B \cup V_1 \cup V_2) \subset \mathrm{cl}(V_3)$. Continuing in this way, we have an infinite collection of open sets $V_1, V_2, V_3, \ldots,$ such that $B \cup V_1 \cup V_2 \cup V_3 \cup \ldots \in \mathscr{I}$, which is impossible, as the ideal $\mathscr{I}$ is not closed under countable additivity. Thus, it must be the case that $V_1 = \varnothing$ (similarly for the other V_i's); therefore, $\mathrm{cl}(V_1) = \varnothing$, and the relations $V_1 \subset B \subset \mathrm{cl}(V_1)$ then give $B = \varnothing$, proving that $\mathscr{I} = \{\varnothing\}$.

Proposition 1. *Let $\mathscr{I}$ and $\mathscr{I}'$ be two ideals on a topological space X.*

1. *If $\mathscr{I} \subset \mathscr{I}'$, then every $\mathscr{I}$-semi-open set A is $\mathscr{I}'$-semi-open;*
2. *If A is $(\mathscr{I} \cap \mathscr{I}')$-semi-open, then it is simultaneously $\mathscr{I}$-semi-open and $\mathscr{I}'$-semi-open.*

Corollary 1. *For a subset A of X and an ideal $\mathscr{I}$ on X, recall that $\mathscr{I}_A = \{A \cap S | S \in \mathscr{I}\}$ is also an ideal on X.*

1. *If a set B is $\mathscr{I}_A$-semi-open, then it is $\mathscr{I}$-semi-open.*

2. *If $A = \emptyset$, then $\mathscr{I}_A = \mathscr{I}_\emptyset = \{\emptyset\}$, the minimal ideal. Thus, if a set C is $\mathscr{I}_\emptyset$-semi-open, then C is also $\mathscr{I}$-semi-open.*

Proposition 2. *If A and B are both $\mathscr{I}$-semi-open, then so is their union $A \cup B$.*

Proof. Let the given conditions hold. To show that $A \cup B$ is $\mathscr{I}$-semi-open, we need to produce an open set U such that $U - (A \cup B) \in \mathscr{I}$ and $(A \cup B) - \mathrm{cl}(U) \in \mathscr{I}$. Since A and B are both $\mathscr{I}$-semi-open, there are open sets U_1 and U_2 such that

$$U_1 - A \in \mathscr{I},\ A - \mathrm{cl}(U_1) \in \mathscr{I},\ U_2 - B \in \mathscr{I},\ B - \mathrm{cl}(U_2) \in \mathscr{I}.$$

Choose $U = U_1 \cup U_2$, and observe that

$$(U_1 \cup U_2) - (A \cup B) = ((U_1 - A) - B) \cup ((U_2 - B) - A) \in \mathscr{I}.$$

Also,

$$(A \cup B) - \mathrm{cl}(U_1 \cup U_2) = ((A - \mathrm{cl}(U_1)) - \mathrm{cl}(U_2)) \cup ((B - \mathrm{cl}(U_2)) - \mathrm{cl}(U_1)) \in \mathscr{I}.$$

Therefore, by definition, $A \cup B$ is $\mathscr{I}$-semi-open.

Proposition 3. *Let X be a topological space in which there is an open singleton subset $\{a\}$ satisfying $\mathrm{cl}(\{a\}) = X$. For any ideal $\mathscr{I}$ on X with $\{a\} \in \mathscr{I}$, we have that:*

1. *Every singleton subset of X is $\mathscr{I}$-semi-open;*

2. *Every finite subset of X is $\mathscr{I}$-semi-open.*

Proof. Let the given conditions hold. Suppose that $\{s\}$ is a singleton subset of X. Since ($\{a\}$ is open and) $\{a\} - \{s\} = \{a\} \in \mathscr{I}$, and $\{s\} - \mathrm{cl}(\{a\}) = \{s\} - X = \emptyset \in \mathscr{I}$, it follows that $\{s\}$ is $\mathscr{I}$-semi-open; this proves *(1)*. To see that *(2)* holds, let $A = \{s_1, s_2, s_3, \ldots, s_n\}$ be a finite subset of X. Since $A = \{s_1\} \cup \{s_2\} \cup \{s_3\} \cup \ldots \cup \{s_n\}$, the result follows from the fact that each singleton subset $\{s_i\}$ $(i = 1, 2, 3, \ldots, n)$ is $\mathscr{I}$-semi-open and a repeated use of Proposition 2 above.

Proposition 3 does not hold for any choice of ideal.

Example 2. *Consider $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$, and observe that $\mathrm{cl}(\{a\}) = X$. If we choose the minimal ideal $\mathscr{I} = \{\emptyset\}$ on X, then the singleton subset $\{b\}$ is not $\mathscr{I}$-semi-open, as there is no open set U satisfying $U - \{b\} \in \mathscr{I}$ and $\{b\} - \mathrm{cl}(U) \in \mathscr{I}$ simultaneously.*

Proposition 4. *Let A and B be subsets of a topological space X such that A is open, $A \subset B$, and A is dense in B (that is, $B \subset \mathrm{cl}(A)$). Then B is $\mathscr{I}$-semi-open for any ideal $\mathscr{I}$ on X. In particular, the conclusion holds in the special case when $B = \mathrm{cl}(A)$.*

Remark 1. *If two sets A and B are $\mathscr{I}$-semi-open, then their intersection $A \cap B$ need not be $\mathscr{I}$-semi-open. For example, let $X = \{a, b, c\}$ be equipped with a topology $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. Note that $\mathrm{cl}(\{a\}) = \{a, b\}$ and $\mathrm{cl}(\{c\}) = \{b, c\}$; moreover, the subsets $\{a, b\}$ and $\{b, c\}$ are semi-open with respect to the minimal ideal $\mathscr{I} = \{\emptyset\}$, in view of Proposition 4 above. However, the singleton subset $\{b\} = \{a, b\} \cap \{b, c\}$ is not semi-open with respect to the minimal ideal $\mathscr{I} = \{\emptyset\}$.*

Obviously, if $A, B \in \mathscr{I}$, then $A \cap B$ will be semi-open with respect to the ideal $\mathscr{I}$. For those subsets that are not members of the ideal $\mathscr{I}$, one rather strong condition for their intersection to be semi-open with respect to the ideal $\mathscr{I}$ is given below.

Proposition 5. *Let $\mathscr{I}$ be an ideal on a topological space X, where every non-empty open subset of X is dense, and the collection of open subsets of X satisfies the finite intersection property.*

1. *If A is $\mathscr{I}$-semi-open and $A \subset B$, then B is $\mathscr{I}$-semi-open;*
2. *If A is $\mathscr{I}$-semi-open, then so is $A \cup B$, for any subset B of X;*
3. *If both A and B are $\mathscr{I}$-semi-open, then so is their intersection $A \cap B$.*

Proof. (1) Suppose that A is $\mathscr{I}$-semi-open, and that $A \subset B$. There is an open set U such that $U - A \in \mathscr{I}$ and $A - \text{cl}(U) \in \mathscr{I}$. Notice that such an open set U is necessarily non-empty, since we are dealing with those subsets of X that do not belong to the ideal $\mathscr{I}$. Since $A \subset B$, we have that $U - B \subset U - A \in \mathscr{I}$; moreover, $B - \text{cl}(U) = B - X = \varnothing \in \mathscr{I}$. Thus, B is $\mathscr{I}$-semi-open.

(2) Since $A \subset B \Longleftrightarrow A \cup B = B$, *(2)* immediately follows from *(1)*.

(3) Suppose that both A and B are $\mathscr{I}$-semi-open. Without loss of generality, suppose that $A \cap B \neq \varnothing$; otherwise, $A \cap B$ will be trivially $\mathscr{I}$-semi-open. By assumption, there are open sets U and V such that $U - A$, $A - \text{cl}(U) \in \mathscr{I}$ and $V - B$, $B - \text{cl}(V) \in \mathscr{I}$. Consider the open set $U \cap V$, which is non-empty (by the finite intersection property). Since $(U \cap V) - (A \cap B) = ((U - A) \cap V) \cup (U \cap (V - B)) \in \mathscr{I}$ and $(A \cap B) - \text{cl}(U \cap V) = (A \cap B) - X = \varnothing \in \mathscr{I}$, it follows that $A \cap B$ is $\mathscr{I}$-semi-open.

Remark 2. *In Example 2, we saw that the singleton subset $\{b\}$ was not semi-open with respect to the minimal ideal $\mathscr{I} = \{\varnothing\}$. Notice that the set $\{a, b\} = \{a\} \cup \{b\}$ is semi-open with respect to $\mathscr{I} = \{\varnothing\}$, simply because the non-empty open singleton subset $\{a\}$ is dense in X; this is an instance of Proposition 5(2) above.*

Proposition 6. *Under the conditions of Proposition 5, we have that A is $\mathscr{I}$-semi-open if and only if* $\text{cl}(A)$ *is $\mathscr{I}$-semi-open.*

Proof. If A is $\mathscr{I}$-semi-open, then - because $A \subset \text{cl}(A)$ - so is $\text{cl}(A)$, by Proposition 5*(2)*. Conversely, suppose that $\text{cl}(A)$ is $\mathscr{I}$-semi-open. Then there is an open set U such that $U - \text{cl}(A) \in \mathscr{I}$ and $\text{cl}(A) - \text{cl}(U) \in \mathscr{I}$. Notice that U is necessarily non-empty; otherwise, we would have $\text{cl}(U) = \varnothing$, which forces $A \in \mathscr{I}$, which we don't want (as we're dealing with those subsets that do not belong to the ideal $\mathscr{I}$). To show that A is $\mathscr{I}$-semi-open, consider the open set $V = U - \text{cl}(A) = U \cap (\text{cl}(A))^c \in \mathscr{I}$, by assumption. We have that $V - A = U \cap (\text{cl}(A))^c \cap A^c \in \mathscr{I}$, because of the heredity property; moreover, $A - \text{cl}(V) = A - \text{cl}(U \cap (\text{cl}(A))^c) = A - X = \varnothing \in \mathscr{I}$. This shows that A is $\mathscr{I}$-semi-open.

Theorem 2. *The following are equivalent for a subset A of X:*

1. *$X - A$ is $\mathscr{I}$-semi-open.*

2. *There exists a closed set F such that* $\text{int}(F) - A \in \mathscr{I}$ *and* $A - F \in \mathscr{I}$.

Proof. First suppose that $X - A$ is $\mathscr{I}$-semi-open. Then there exists an open set U such that $U - (X - A) \in \mathscr{I}$ and $(X - A) - \text{cl}(U) \in \mathscr{I}$. Since $U - (X - A) = A - (X - U)$ and $(X - A) - \text{cl}(U) = \text{int}(X - U) - A$, we have *(2)* by choosing the closed set $X - U$ as F. Conversely, if we suppose that *(2)* holds, then the choice of the open set $U = X - F$ shows that $X - A$ is $\mathscr{I}$-semi-open.

Definition 2. *A subset A of X is said to be* semi-closed with respect to $\mathscr{I}$ *(written as $\mathscr{I}$-semi-closed) if and only if $X - A$ is $\mathscr{I}$-semi-open.*

Proposition 7. *If both A and B are $\mathscr{I}$-semi-closed, then so is their intersection $A \cap B$.*

Proof. Let the given conditions hold. There are closed sets F_1 and F_2 such that $\text{int}(F_1) - A, A - F_1 \in \mathscr{I}$ and $\text{int}(F_2) - B, B - F_2 \in \mathscr{I}$. With $F = F_1 \cap F_2$, we have that

$$\text{int}(F_1 \cap F_2) - (A \cap B) = ((\text{int}(F_1) - A) \cap \text{int}(F_2)) \cup (\text{int}(F_1) \cap (\text{int}(F_2) - B)) \in \mathscr{I},$$

and

$$(A \cap B) - (F_1 \cap F_2) = ((A - F_1) \cap B) \cup (A \cap (B - F_2)) \in \mathscr{I};$$

therefore, $A \cap B$ is $\mathscr{I}$-semi-closed.

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