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Coefficient Estimates for Certain Subclasses of Analytic Functions of Complex Order

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Abstract. In this paper, we introduce and investigate two interesting subclasses $\mathscr{H}_g(n,b,\lambda,\alpha,\delta)$ and $\mathscr{H}_g(n,b,\lambda,\alpha,\delta;u)$ of analytic functions of complex order in the open unit disk $\mathbb{U}$, which are defined by means of the familiar multiplier operator. Formfunctions belonging to the each of these subclasses, we obtain several results involving (for example) coefficient bounds. Then results presented here would generalize many known results.

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1. Introduce

Let $\mathbb{R} = (-\infty, +\infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers,

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

be the set of positive integers,

$$\mathbb{N}_2 = \{2, 3, 4, \ldots\}$$

and

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

We also let $\mathscr{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1}$$

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which are analytic in the open unit disc

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function $f(z) \in \mathscr{A}$ is said to belong to the class $S^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > \alpha \quad (z \in \mathbb{U};\ 0 \le \alpha < 1).$$

For functions $f(z)$ in the class $S^*(\alpha)$ given by (1), Robertson [11] proved some coefficient bounds which we recall here as Lemma 1 below.

Lemma 1. *If*

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in S^*(\alpha),$$

then

$$|a_j| \le \frac{\prod_{k=0}^{j-2}[k + 2(1-\alpha)]}{j!} \quad (j \in \mathbb{N}_2). \tag{2}$$

Nasr and Aouf [10] and Altintaş *et. al* [1–8] have extended the coefficient bounds (2) for the class of $S^*(\alpha)$ to hold true for various interesting subclasses of analytic functions of complex order.

For a function $f(z)$ in $\mathscr{A}$, the multiplier operator $D^n_{\alpha,\delta}f(z)$ was extended by Deniz and Orhan in [17] as follows:

$$\begin{aligned}
D^0_{\alpha,\delta}f(z) =& f(z)\\
D^1_{\alpha,\delta}f(z) =& D_{\alpha,\delta}f(z) = \alpha\delta z^2 f''(z) + (\alpha-\delta)zf'(z) + (1-\alpha+\delta)f(z)\\
&\cdots\\
D^n_{\alpha,\delta}f(z) =& D_{\alpha,\delta}(D^{n-1}_{\alpha,\delta}f(z))
\end{aligned}$$

where $\alpha \ge \delta \ge 0$ and n $\in \mathbb{N}_0$. If $f \in \mathscr{A}$ is given by (1) then from the definition of the operator $D^n_{\alpha,\delta}f(z)$, it is easy verity that $D^n_{\alpha,\delta}f(z) = z + \sum\limits_{k=2}^{\infty} \phi_k^n a_k z^k$, where $\phi_k = [1 + (\alpha\delta k + \alpha - \delta)(k-1)]$, $(\phi_k^n = [\phi_k]^n)$; $\alpha \ge \delta \ge 0$ and $n \in \mathbb{N}_0$.

Remark 1. *$D^n_{\alpha,\delta}f(z)$ is a generalization of many other linear operators considered earlier. In particular, for $f(z)$ in $\mathscr{A}$ we have the following :*

- *$D^n_{1,0}f(z) \equiv D^n f(z)$ the operator defined by Sălăgean (see [13]).*
- *$D^n_{\alpha,0}f(z) \equiv D^n_\alpha f(z)$ (see [16]).*

Recently, several authors have obtained many interesting results for various subclasses of analytic functions involving the Sălăgean derivative operator $D^n f(z)$. For example, Deng [9] defines a function class $\mathscr{B}(n,\lambda,\alpha,b)$ by

$$\Re(1+\frac{1}{b}[\frac{z[(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)]'}{(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)}-1])>\alpha$$
$$(0\leq\alpha<1;0\leq\lambda\leq1;n\in\mathbb{N}_0;b\in\mathbb{C}\backslash\{0\})$$

and also investigated the subclass $\mathscr{T}(n,\lambda,\alpha,b;u)$ of the analytic function class $\mathscr{A}$, which consists of functions $f(z)\in\mathscr{A}$ satisfying the following *nonhomogenous Cauchy-differential equation*:

$$z^2\frac{d^2w}{dz^2}+2(1+u)z\frac{dw}{dz}+u(1+u)w=(1+u)(2+u)h(z),$$

where

$$w=f(z)\in\mathscr{A},h(z)\in\mathscr{B}(n,\lambda,\alpha,b)\text{ and }u\in\mathbb{R}\backslash(-\infty,-1].$$

In the same paper [9], coefficient bounds for the subclass $\mathscr{B}(n,\lambda,\alpha,b)$ and $\mathscr{T}(n,\lambda,\alpha,b,u)$ of analytic functions of complex order were obtained.

By using the multiplier differential operator $D^n_{\alpha,\delta}$, we now define the following new subclasses of functions belonging to the class $\mathscr{A}$.

Definition 1. *Let $g:\mathbb{U}\to\mathbb{C}$ be a convex function such that*

$$g(0)=1 \text{ and } \Re(g(z))>0\quad(z\in\mathbb{U}),$$

and f be an analytic function in $\mathbb{U}$ defined by (1). We say that $f\in\mathscr{H}_g(n,b,\lambda,\alpha,\delta)$ if it satisfies the following condition:

$$1+\frac{1}{b}[\frac{z[\mathscr{F}^n_{\lambda,\alpha,\delta}(z)]'}{\mathscr{F}^n_{\lambda,\alpha,\delta}(z)}-1]\in g(\mathbb{U})\quad(z\in\mathbb{U}),$$

where

$$\mathscr{F}^n_{\lambda,\alpha,\delta}(z)=(1-\lambda)D^n_{\alpha,\delta}f(z)+\lambda D^{n+1}_{\alpha,\delta}f(z)\quad(\alpha\geq\delta\geq0,0\leq\lambda\leq1;n\in\mathbb{N}_0;b\in\mathbb{C}\backslash\{0\}).$$

Definition 2. *A function $f(z)\in\mathscr{A}$ is said to be in the class $\mathscr{H}_g(n,b,\lambda,\alpha,\delta;u)$, if it satisfies the following nonhomogenous Cauchy-Euler differential equation:*

$$z^2\frac{d^2w}{dz^2}+2(1+u)z\frac{dw}{dz}+u(1+u)w=(1+u)(2+u)h(z) \tag{3}$$
$$(w=f(z)\in\mathscr{A},h(z)\in\mathscr{H}_g(n,b,\lambda,\alpha,\delta)\text{ and }u\in\mathbb{R}\backslash(-\infty,-1]).$$

Remark 2. *Their are many choices of the function g and the values of α, δ which would provide interesting subclasses of analytic functions of complex order. In particular, if we let*

$$g(z)=\frac{1+(1-2\beta)z}{1-z}(0\leq\beta<1;z\in\mathbb{U}),\alpha=1,and\delta=0,$$

it is easy to see g *is a convex function in* $\mathbb{U}$ *and satisfies the hypotheses of Definition 1. If* $f \in \mathscr{H}_g(n,b,\lambda,\alpha,\delta)$*, then*

$$\Re(1+\frac{1}{b}[\frac{z[(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)]'}{(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)}-1])>\beta(z\in\mathbb{U}),$$

that is

$$f \in \mathscr{B}(n,\lambda,\beta,b).$$

Remark 3. *In view of Remark 2, ifwe take*

$$g(z)=\frac{1+(1-2\beta)z}{1-z} \quad (0\le\beta<1; z\in\mathbb{U}), \alpha=1, \text{ and } \delta=0$$

in Definitions 1 and 2, it is easy to observe that the function classes

$$\mathscr{H}_g(n,b,\lambda,\alpha,\delta) \text{ and } \mathscr{H}_g(n,b,\lambda,\alpha,\delta;u)$$

become the aforementioned function classes

$$\mathscr{B}(n,\lambda,\alpha,b) \text{ and } \mathscr{T}(n,\lambda,\alpha,b;u),$$

respectively.

In our investigation, we shall use the principle of subordination between analytic functions, which is explained in Definition 3 below (see also [14, 15]).

Definition 3. *For two functions* f *and* g *analytic in* $\mathbb{U}$*, we say that the function* $f(z)$ *is subordinate to* $g(z)$ *in* $\mathbb{U}$ *(written* $f \prec g$ $(z\in\mathbb{U})$*), if there exists a Schwarz function* $\omega(z)$ *analytic in* $\mathbb{U}$ *with*

$$\omega(0)=0 \text{ and } |\omega(z)|<1(z\in\mathbb{U}),$$

such that

$$f(z)=g(\omega(z)) \quad (z\in\mathbb{U}).$$

In particular, if the function g *is univalent in* $\mathbb{U}$*, the above subordination is equivalent to*

$$f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}).$$

In this paper, by use of the principle of subordination, we obtain coefficient bounds for functions in the subclasses

$$\mathscr{H}_g(n,b,\lambda,\alpha,\delta) \text{ and } \mathscr{H}_g(n,b,\lambda,\alpha,\delta;u)$$

of analytic functions of complex order, which we have introduce here. Our results would unify and extend the corresponding results obtained earlier by Nasr and Aouf [10], Altintaş *et. al* [1–8] and Deng [9].

2. Main Results and Their Proofs

In order to prove our main results(Theorems 1 and 2 below), we first recall the following lemma due to Rogosinski [12].

Lemma 2. *Let the function g given by*

$$g(z) = z + \sum_{k=1}^{\infty} g_k z^k$$

be convex $\mathbb{U}$. Also let the function f given by

$$f(z) = z + \sum_{k=1}^{\infty} a_k z^k$$

be holomorphic in $\mathbb{U}$. If

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

then

$$|a_k| \leq |g_1| \quad (k \in \mathbb{N}).$$

We now state and prove each of our main results given by Theorems 1 and 2 below.

Theorem 1. *Let the function $f \in \mathscr{A}$ be given by (1). If $f \in \mathscr{H}_g(n, b, \lambda, \alpha, \delta)$, then*

$$|a_j| \leq \frac{\prod_{k=0}^{j-2}(k + |g'(0)||b|)}{\phi_j^n[1 - \lambda + \lambda\phi_j](j-1)!} \quad (j \in \mathbb{N}_2).$$

Proof. By definition of $D^n_{\alpha,\delta} f(z)$ and $\mathscr{F}^n_{\lambda,\alpha,\delta}(z)$, we can write

$$\mathscr{F}^n_{\lambda,\alpha,\delta}(z) = z + \sum_{j=2}^{\infty} A_j z^j \quad (z \in \mathbb{U}), \tag{4}$$

where

$$\mathscr{A}_j = \phi_j^n(1 - \lambda + \lambda\phi_j) \quad (j \in \mathbb{N}_2). \tag{5}$$

From Definition 1, we thus have

$$1 + \frac{1}{b}\left[\frac{z[\mathscr{F}^n_{\lambda,\alpha,\delta}(z)]'}{\mathscr{F}^n_{\lambda,\alpha,\delta}(z)} - 1\right] \in g(\mathbb{U}).$$

By setting

$$p(z) = 1 + \frac{1}{b}\left[\frac{z[\mathscr{F}^n_{\lambda,\alpha,\delta}(z)]'}{\mathscr{F}^n_{\lambda,\alpha,\delta}(z)} - 1\right], \tag{6}$$

we also deduce that

$$p(0) = g(0) = 1 and p(z) \in g(\mathbb{U}) \quad (z \in \mathbb{U}).$$

Therefore, we have

$$p(z) \prec g(z) \quad (z \in \mathbb{U}).$$

According to Lemma 2, we obtain

$$|p_m| = \left|\frac{p^{(m)}(0)}{m!}\right| \leq |g'(0)| = |g_1|. \tag{7}$$

On the other hand, we find from (6) that

$$z[\mathscr{F}^n_{\lambda,\alpha,\delta}(z)]' = [1 + b(p(z) - 1)]\mathscr{F}^n_{\lambda,\alpha,\delta}(z) \quad (z \in \mathbb{U}). \tag{8}$$

Next, we suppose that

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \quad (z \in \mathbb{U}). \tag{9}$$

Since $A_1 = 1$, in view of (4), (8), (9), we deduce that

$$(j-1)A_j = (p_1 A_{j-1} + p_2 A_{j-2} + \ldots + p_{j-1})b \quad (j \in \mathbb{N}_2). \tag{10}$$

In view of (7) and (10), for $j = 2, 3, 4$, we obtain

$$\begin{aligned}
|A_2| &\leq |g'(0)||b|, \\
|A_3| &\leq \frac{|g'(0)||b|(1 + |g'(0)||b|)}{2!} \\
|A_4| &\leq \frac{|g'(0)||b|(1 + |g'(0)||b|)(2 + |g'(0)||b|)}{3!},
\end{aligned}$$

respectively. Also, making use of the principle of mathematical induction, we can obtain

$$|A_j| \leq \frac{\prod_{k=0}^{j-2}(k + |g'(0)||b|))}{(j-1)!} \quad (j \in \mathbb{N}_2).$$

From (5), we can easily obtain

$$|a_j| \leq \frac{\prod_{k=0}^{j-2}(k + |g'(0)||b|)}{\phi_j^n[1 - \lambda + \lambda\phi_j](j-1)!} \quad (j \in \mathbb{N}_2),$$

as asserted by Theorem 1. This completes the proof of Theorem 1. □

Theorem 2. *Let the function $f \in \mathscr{A}$ be given by (1). If $f \in \mathscr{H}_g(n, b, \lambda, \alpha, \delta; u)$, then*

$$|a_j| \leq \frac{(1+u)(2+u)\prod_{k=0}^{j-2}(k+|g'(0)||b|)}{(j+u)(j+u+1)\phi_j^n[1-\lambda+\lambda\phi_j](j-1)!} \quad (j \in \mathbb{N}_2; u \in \mathbb{R}\backslash(-\infty, -1]).$$

Proof. Let the function $f \in \mathscr{A}$ be given by (1). Also let

$$h(z) = z + \sum_{j=2}^{\infty} h_j z^j \in \mathscr{H}_g(n, b, \lambda, \alpha, \delta).$$

Thus, from (3), we deduce that

$$a_j = \frac{(1+u)(2+u)h_j}{(j+u)(j+u+1)} \quad (j \in \mathbb{N}_2; u \in \mathbb{R}\backslash(-\infty, -1]).$$

Using Theorem 1, we obtain

$$|a_j| \leq \frac{(1+u)(2+u)\prod_{k=0}^{j-2}(k+|g'(0)||b|)}{(j+u)(j+u+1)\phi_j^n[1-\lambda+\lambda\phi_j](j-1)!} \quad (j \in \mathbb{N}_2; u \in \mathbb{R}\backslash(-\infty, -1]),$$

as claimed in Theorem 2. This completes the proof of Theorem 2. □

3. Corollaries and Consequences

In view of Remark 2, if we set

$$g(z) = \frac{1+(1-2\beta)z}{1-z} \quad (0 \leq \beta < 1; z \in \mathbb{U}), \ \alpha = 1, \text{ and } \delta = 0$$

in Theorems 1 and 2, respectively, we can easily deduce the following two corollaries, which we merely state here without proofs.

Corollary 1. *Let the function $\in \mathscr{A}$ be given by (1). If $f \in \mathscr{B}(n, \lambda, \beta, b)$, then*

$$|a_j| \leq \frac{\prod_{k=0}^{j-2}[k+2|b|(1-\beta)]}{j^n(1-\lambda+\lambda j)(j-1)!} \quad (j \in \mathbb{N}_2).$$

Corollary 2. *Let the function $f \in \mathscr{A}$ be given by (1). If $f \in \mathscr{T}(n, \lambda, \beta, b; u)$, then*

$$|a_j| \leq \frac{(1+u)(2+u)\prod_{k=0}^{j-2}[k+2|b|(1-\beta\pi)]}{j^n(1-\lambda+\lambda j)(j-1)!(j+u)(j+1+u)} \quad (j \in \mathbb{N}_2; u \in \mathbb{R}\backslash(-\infty, -1]).$$

Remark 4. *Corollaries 1 and 2 were obtained by Deng [9]. However, by use of Theorems 1 and 2, we are able to derive these results much more easily.*

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