EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 6, No. 4, 2013, 451-459
ISSN 1307-5543 – www.ejpam.com

On Integrability of Trigonometric Series with Special Type of Coefficients

Xhevat Z. Krasniqi

Department of Mathematics and Informatics, Faculty of Education, University of Prishtina "Hasan Prishtina", Avenue "Mother Theresa" 5, 10000 Prishtina, Republic of Kosovo

Abstract. In this paper some condition on integrability of cosine and sine trigonometric series with coefficients that keep their signs are obtained. The results extend some previous results of Telyakovskiĭ.

2010 Mathematics Subject Classifications: 42A16, 42A20

Key Words and Phrases: Trigonometric series, quasi-convex sequence, bounded variation sequence, integrability

1. Introduction and Preliminaries

Several mathematicians have studied the integrability conditions for trigonometric series with different types of coefficients. The first results pertaining to the trigonometric series of the form

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos kx \tag{1}$$

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{2}$$

considered the case of monotone coefficients. Later, some authors investigated the series (1) with quasi-monotone coefficients ($a_{n+1} \leq a_n(1+\alpha/n)$, $n \geq n_0$, $\alpha > 0$).

Many papers have been written on the series (1) when the sequence $\{a_k\}$ is a null-sequence and convex or quasi-convex, i.e. $\triangle^2 a_k \geq 0$ or

$$\sum_{k=1}^{\infty}(k+1)|\triangle^2 a_k| < \infty, \tag{3}$$

where $\triangle^2 a_k = \triangle\left(\triangle a_k\right)$, $\triangle a_k = a_k - a_{k+1}$.

Email address: `xhevat.krasniqi@uni-pr.edu`

Furthermore, when $\{a_k\}$ is a null-sequence of bounded variation, i.e. $\sum_{k=1}^{\infty}|\triangle a_k|<\infty$, is also considered.

We shall consider the series (1) and (2) whose coefficients tend to zero and satisfy any condition that provides their convergence on $(0,\pi]$. Let us denote their sums with $f(x)$ and $g(x)$ respectively.

If the coefficients a_k are quasi-convex, it is well-known that f is an integrable function on $[0,\pi]$ (see [1]), and the estimation

$$\int_0^{\pi}|f(x)|dx \le \pi\sum_{k=1}^{\infty}(k+1)|\triangle^2 a_k|$$

is valid.

In a similar direction, among others, S. A. Telyakovskiĭ [6] obtained some estimates of the integrals of the following form

$$\int_{\pi/(m+1)}^{\pi/\ell}|\phi(x)|dx, \quad 1\le \ell\le m, \quad (\ell,m\in\mathbb{N}), \tag{4}$$

expressed in terms of the coefficients a_k, where he used null-sequences of bounded variation of second order ($\sum_{k=1}^{\infty}|\triangle^2 a_k|<\infty$), instead of quasi-convex null-sequences. Here $\phi(x)$ is either $f(x)$ or $g(x)$.

It is obvious that the condition

$$\sum_{k=1}^{\infty}|\triangle^2 a_k|<\infty \tag{5}$$

is a weaker condition than the condition (3).

The following definition is introduced in [4]: A sequence $\{a_k\}$ is of bounded variation of integer order $p\ge 0$ if

$$\sum_{k=1}^{\infty}|\triangle^p a_k|<\infty, \tag{6}$$

where $\triangle^p a_k=\triangle\left(\triangle^{p-1}a_k\right)=\triangle^{p-1}a_k-\triangle^{p-1}a_{k+1}$, and we agree with $\triangle^0 a_k=a_k$.

In [4] an example is given to show that (6) is an effective generalization of the null sequences of bounded variation. This fact encouraged the present author to consider the series (1) with coefficients that satisfy the condition (6). The results are published in [2]. Also similar results the reader can find in [3, 4].

For an integer non-negative number r and a sequence $\{a_k\}$ we write $\triangle_r a_k=a_k-a_{k+r}$ and $\triangle_r^2 a_k=\triangle_r\left(\triangle_r a_k\right)=a_k-2a_{k+r}+a_{k+2r}$. Note that for $r=1$ we obtain ordinary differences $\triangle a_k=a_k-a_{k+1}$ and $\triangle^2 a_k=\triangle\left(\triangle a_k\right)=a_k-2a_{k+1}+a_{k+2}$.

Let $r\in\mathbb{N}$, $k=1,2,\dots,r$, $n=0,1,2,\dots$,

$$B_{0,r,k}^{0}(x)=\frac{\sin((2k-r)x/2)}{2\sin(rx/2)}, \quad x\ne 2m\pi/r,\ m\in\mathbb{Z},$$

$$B^0_{n+1,r,k}(x)=\cos(k+nr)x,\quad BC^1_{n,r,0}(x)=\frac{\sin\left((2n+1)r\frac{x}{2}\right)}{2\sin\left(r\frac{x}{2}\right)}$$

$$B^1_{n,r,k}(x)=\sum_{m=0}^{n}B^0_{m,r,k}(x),\quad B^2_{n,r,k}(x)=\sum_{m=0}^{n}B^1_{m,r,k}(x),$$

$$BC^1_{n,r,k}(x)=\frac{\sin\left((2k+(2n+1)r)\frac{x}{2}\right)-\sin\left((2k-r)\frac{x}{2}\right)}{2\sin\left(r\frac{x}{2}\right)}$$

$$\overline{B}^1_{n+1,r,k}(x)=\sum_{m=0}^{n}\sin(k+mr)x=\frac{\cos\left((2k-r)\frac{x}{2}\right)-\cos\left((2k+(2n+1)r)\frac{x}{2}\right)}{2\sin\left(r\frac{x}{2}\right)}.$$

The following definition is introduced in [5]: A sequence $\{a_n\}$ keeps its sign if either $a_n\geq 0$ for all n, or $a_n\leq 0$ for all n.

Also in the same paper are proved some lemmas formulated below.

Lemma 1. *Let $r\in\mathbb{N}$, $k=1,2,\ldots,r$, $n=0,1,2,\ldots$.*

(a) If the sequence $\{\triangle_r a_{k+nr}\}$ keeps its sign separately for each k, then the series (1) *and* (2) *converge for almost all x. The function $g(x+m\frac{2\pi}{r})$ is almost everywhere representable in the form*

$$g(x+m\frac{2\pi}{r})=\sum_{k=1}^{r}\cos\left(km\frac{2\pi}{r}\right)\sum_{n=0}^{\infty}\triangle_r a_{k+nr}\overline{B}^1_{n+1,r,k}(x)$$
$$+\sum_{k=1}^{r-1}\sin\left(km\frac{2\pi}{r}\right)\sum_{n=0}^{\infty}\triangle_r a_{k+nr}BC^1_{n,r,k}(x).$$

(b) If the sequence $\{\triangle^2_r a_{k+nr}\}$ keeps its sign separately for each k, then $f(x)$ is almost everywhere representable in the form

$$f(x)=\sum_{k=1}^{r}\sum_{n=0}^{\infty}\triangle^2_r a_{k+(n-1)r}B^2_{n,r,k}(x).$$

Lemma 2. *Let $r\in\mathbb{N}$, $a_n\to 0$ for $n\to\infty$ and $\triangle_{2,r}a_n\geq 0$ for all n. Then $\triangle_r a_n\geq 0$ and $a_n\geq 0$ for all n.*

The aim of this paper is to achieve some results, similar to those of Telyakovskiĭ [6], for the series (1) and (2) with coefficients that satisfy the conditions: the sequences $\{\triangle_r a_{k+nr}\}$ and $\{\triangle^2_r a_{k+nr}\}$ keep their sign separately for each k.

We write $g(u)=O_r(h(u))$, $u\to 0$, if there exists a positive constant A_r, that depends only on r, such that $g(u)\leq A_r h(u)$ in a neighborhood of the point $u=0$. The constants A_r may be, in general, different in different estimates.

2. Main Results

We begin with the following result regarding to the cosine series.

Theorem 1. *Let $r \in \mathbb{N}$, $k = 1,2,\dots,r$. If $a_n \to 0$ as $n \to \infty$ and the sequence $\{\triangle_r^2 a_{k+nr}\}$ keeps its sign separately for each k, then the series* (1) *converges for almost all x, and for $1 \le \ell \le m$, the sum function $f(x)$ satisfies*

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx = O\left(\frac{m+1-\ell}{m}\sum_{k=1}^{r}\sum_{n=0}^{\ell-1}\frac{n+1}{\ell}|\triangle_r a_{k+(n-1)r}|\right) + O\left(\sum_{k=1}^{r}\sum_{n=\ell}^{\infty}\min(n+1-\ell,\, m+1-\ell)|\triangle_r^2 a_{k+(n-1)r}|\right).$$

Proof. The convergence for almost all x of the series (1) has been proved in Lemma 1(a). Moreover, from Lemma 1(b) the sum function $f(x)$ is almost everywhere representable in the form

$$f(x) = \sum_{k=1}^{r}\sum_{n=0}^{\infty}\triangle_r^2 a_{k+(n-1)r}B_{n,r,k}^2(x). \tag{7}$$

Let i be a positive integer and $x \in \left(\frac{\pi}{i+1}, \frac{\pi}{i}\right]$. With agreement that $B_{-1,r,k}^2(x) \equiv 0$ and using the equality

$$\begin{aligned}
&\sum_{k=1}^{r}\sum_{n=0}^{i-1}\triangle_r^2 a_{k+(n-1)r}B_{n,r,k}^2(x) = \\
&= \sum_{k=1}^{r}\sum_{n=0}^{i-1}\left(\triangle_r a_{k+(n-1)r} - \triangle_r a_{k+nr}\right)B_{n,r,k}^2(x) \\
&= \sum_{k=1}^{r}\left(\sum_{n=0}^{i-1}\triangle_r a_{k+(n-1)r}\left(B_{n,r,k}^2(x) - B_{n-1,r,k}^2(x)\right) - \triangle_r a_{k+(i-1)r}B_{i-1,r,k}^2(x)\right) \\
&= \sum_{k=1}^{r}\left(\sum_{n=0}^{i-1}\triangle_r a_{k+(n-1)r}B_{n,r,k}^1(x) - \triangle_r a_{k+(i-1)r}B_{i-1,r,k}^2(x)\right),
\end{aligned}$$

from (7) we have

$$f(x) = \sum_{k=1}^{r}\left(\sum_{n=0}^{i-1}\triangle_r a_{k+(n-1)r}B_{n,r,k}^1(x) + \sum_{n=i}^{\infty}\triangle_r^2 a_{k+(n-1)r}\left(B_{n,r,k}^2(x) - B_{i-1,r,k}^2(x)\right)\right).$$

It is obvious that $|B_{n,r,k}^1(x)| \le n+1$, and since from (see [5, page 65])

$$B_{n,r,k}^2(x) = \frac{\sin^2\left((k+nr)\frac{x}{2}\right) - \sin^2\left((k-r)\frac{x}{2}\right)}{2\sin^2\left(r\frac{x}{2}\right)}$$

follows

$$|B^2_{n,r,k}(x) - B^2_{i-1,r,k}(x)| \le \frac{2}{\sin^2\left(r\frac{x}{2}\right)},$$

we have

$$\int_{\pi/(i+1)}^{\pi/i} |f(x)|dx = O\left\{\sum_{k=1}^{r}\left(\sum_{n=0}^{i-1}|\triangle_r a_{k+(n-1)r}|\frac{k+1}{i(i+1)} + \sum_{n=i}^{\infty}|\triangle_r^2 a_{k+(n-1)r}|\right)\right\}.$$

Now if we take the summation, when i goes from ℓ to m, to the both sides of the above equality we get

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx = O\left\{\sum_{k=1}^{r}\left(\sum_{i=\ell}^{m}\sum_{n=0}^{i-1}|\triangle_r a_{k+(n-1)r}|\frac{n+1}{i(i+1)} + \sum_{i=\ell}^{m}\sum_{n=i}^{\infty}|\triangle_r^2 a_{k+(n-1)r}|\right)\right\}. \quad (8)$$

For the first term in the parentheses of the right-hand side of (8) we have

$$\begin{aligned}
&\sum_{i=\ell}^{m}\sum_{n=0}^{i-1}|\triangle_r a_{k+(n-1)r}|\frac{n+1}{i(i+1)} = \\
&\quad= \sum_{i=\ell}^{m}\sum_{n=0}^{\ell-1}|\triangle_r a_{k+(n-1)r}|\frac{n+1}{i(i+1)} + \sum_{i=\ell+1}^{m}\sum_{n=\ell}^{i-1}|\triangle_r a_{k+(n-1)r}|\frac{n+1}{i(i+1)} \\
&\quad= \sum_{n=0}^{\ell-1}(n+1)|\triangle_r a_{k+(n-1)r}|\left(\frac{1}{\ell} - \frac{1}{m+1}\right) \\
&\quad\quad + \sum_{n=\ell}^{m-1}(n+1)|\triangle_r a_{k+(n-1)r}|\left(\frac{1}{n+1} - \frac{1}{m+1}\right) \\
&\quad\quad \le \frac{m+1-\ell}{m}\sum_{n=0}^{\ell-1}\frac{n+1}{\ell}|\triangle_r a_{k+(n-1)r}| + \sum_{n=\ell}^{m}\sum_{j=n}^{\infty}|\triangle_r^2 a_{k+(j-1)r}|. \qquad (9)
\end{aligned}$$

But the second term in (9) can be written as

$$\begin{aligned}
\sum_{i=\ell}^{m}\sum_{n=i}^{\infty}|\triangle_r^2 a_{k+(n-1)r}| &= \sum_{i=\ell}^{m}\sum_{n=i}^{m}|\triangle_r^2 a_{k+(n-1)r}| + \sum_{i=\ell}^{m}\sum_{n=i}^{\infty}|\triangle_r^2 a_{k+(n-1)r}| \\
&= \sum_{n=\ell}^{m}(n+1-\ell)|\triangle_r^2 a_{k+(n-1)r}| \\
&\quad + (m+1-\ell)\sum_{n=m+1}^{\infty}|\triangle_r^2 a_{k+(n-1)r}|. \qquad (10)
\end{aligned}$$

The proof of theorem follows from (8), (9) and (10). □

Now we shall prove an estimation of the integral in Theorem 1 only in terms of second order difference of the sequence $\{a_{k+(n-1)r}\}$.

Corollary 1. *If the coefficients of the series* (1) *satisfy conditions of the Theorem 1, then*

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx = O\left(\frac{m+1-\ell}{m}\sum_{k=1}^{r}\sum_{n=0}^{\infty}\min\left(\frac{(n+1)^2}{\ell}, n+1, m\right)|\triangle_r^2 a_{k+(n-1)r}|\right).$$

Proof. To deduce the required estimation we use the identity

$$\triangle_r a_{k+(n-1)r} = \sum_{i=n}^{\infty}\left(\triangle_r^2 a_{k+(i-1)r}\right).$$

We have

$$\begin{aligned}\sum_{n=0}^{\ell-1}\frac{n+1}{\ell}|\triangle_r a_{k+(n-1)r}| &\le \sum_{n=0}^{\ell-1}\frac{n+1}{\ell}\sum_{i=n}^{\infty}|\triangle_r^2 a_{k+(i-1)r}| \\ &= \sum_{i=0}^{\ell-1}\sum_{n=0}^{i}\frac{n+1}{\ell}|\triangle_r^2 a_{k+(i-1)r}| + \sum_{i=\ell}^{\infty}\sum_{n=0}^{\ell-1}\frac{n+1}{\ell}|\triangle_r^2 a_{k+(i-1)r}| \\ &\le \sum_{i=0}^{\ell-1}\frac{(i+1)^2}{\ell}|\triangle_r^2 a_{k+(i-1)r}| + \ell\sum_{i=\ell}^{\infty}|\triangle_r^2 a_{k+(i-1)r}|. \qquad (11)\end{aligned}$$

If $k < m$, then we can estimate the second term in the estimation of the Theorem 1 by means of the fact that

$$n+1-\ell \le n+1-\ell\frac{n+1}{m} = \frac{m-\ell}{m}(n+1).$$

Finally, from the above and (11) along with the estimate of the Theorem 1 we immediately obtain the required estimation. □

In the following we shall deal with trigonometric series of the form (2).

Theorem 2. *Let* $r \in \mathbb{N}$, $k = 1, 2, \dots, r$. *If* $a_n \to 0$ *as* $n \to \infty$ *and the sequence* $\{\triangle_r a_{k+nr}\}$ *keeps its sign separately for each* k, *then the series* (2) *converges for almost all* x, *and for* $1 \le \ell \le m$, *the sum function* $g(x)$ *satisfies*

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell}|f(x)|dx &= \sum_{k=1}^{r}\sum_{i=\ell}^{m}\frac{d_{i,r,k}}{k}|a_{k+ir}| \\ &+ O_r\left(\frac{m+1-\ell}{m}\sum_{k=1}^{r}\sum_{n=1}^{\ell-1}\frac{n^2}{\ell^2}|\triangle_r a_{k+nr}| + \sum_{k=1}^{r}\sum_{n=\ell}^{m}\sum_{j=n}^{\infty}|\triangle_r^2 a_{k+jr}|\right),\end{aligned}$$

where

$$d_{i,r,k} := \ln\frac{\sin\frac{r\pi}{2i}}{\sin\frac{r\pi}{2(i+1)}} + \cos\frac{k\pi}{2i(i+1)}\cos\frac{k\pi(2i+1)}{2i(i+1)}.$$

Proof. Under assumptions of the theorem and Lemma 1(a) the series (2) converges for almost all x and for $m=0$ the sum function $g(x)$ is almost everywhere representable in the form

$$g(x)=\sum_{k=1}^{r}\sum_{n=0}^{\infty}\triangle_r a_{k+nr}\overline{B}^1_{n+1,r,k}(x).$$

Let us denote

$$\varphi_{n,r,k}(x):=-\frac{\cos(2k+(2n+1)r)\frac{x}{2}}{2\sin\left(r\frac{x}{2}\right)},$$

$$\psi_{n,r,k}(x):=\sum_{s=0}^{n}\varphi_{s,r,k}(x)=\frac{\sin(k+nr)x+\sin kx}{4\sin^2\left(r\frac{x}{2}\right)}.$$

Let $i\in\mathbb{N}$ be such that $i\geq r$. From definition of $\overline{B}^1_{n+1,r,k}(x)$ we can write

$$\begin{aligned}
g(x)=&\sum_{k=1}^{r}\sum_{n=0}^{i-1}\triangle_r a_{k+nr}\overline{B}^1_{n+1,r,k}(x)+\sum_{k=1}^{r}\sum_{n=i}^{\infty}\triangle_r a_{k+nr}\overline{B}^1_{n+1,r,k}(x)\\
=&\sum_{k=1}^{r}\sum_{n=0}^{i-1}\triangle_r a_{k+nr}\overline{B}^1_{n+1,r,k}(x)+\sum_{k=1}^{r}\sum_{n=i}^{\infty}\triangle_r a_{k+nr}\frac{\cos(2k-r)\frac{x}{2}}{2\sin\left(r\frac{x}{2}\right)}\\
&-\sum_{k=1}^{r}\sum_{n=i}^{\infty}\triangle_r a_{k+nr}\frac{\cos(2k+(2n+1)r)\frac{x}{2}}{2\sin\left(r\frac{x}{2}\right)}\\
=&\sum_{k=1}^{r}a_{k+ir}\frac{\cos(2k-r)\frac{x}{2}}{2\sin\left(r\frac{x}{2}\right)}+\sum_{k=1}^{r}\sum_{n=0}^{i-1}\triangle_r a_{k+nr}\overline{B}^1_{n+1,r,k}(x)\\
&+\sum_{k=1}^{r}\sum_{n=i}^{\infty}\triangle_r a_{k+nr}\varphi_{n,r,k}(x):=h_0(x)+h_1(x)+h_2(x).
\end{aligned}\tag{12}$$

For $x\in\left(\frac{\pi}{i+1},\frac{\pi}{i}\right]$, $i=1,2,\dots$, we have

$$\begin{aligned}
\int_{\pi/(m+1)}^{\pi/\ell}|h_1(x)|dx\leq&\sum_{i=\ell}^{m}\int_{\pi/(i+1)}^{\pi/i}\sum_{k=1}^{r}\sum_{n=0}^{i-1}|\triangle_r a_{k+nr}||\overline{B}^1_{n+1,r,k}(x)|dx\\
=&\sum_{k=1}^{r}\sum_{i=\ell}^{m}\sum_{n=0}^{\ell-1}|\triangle_r a_{k+nr}|\int_{\pi/(i+1)}^{\pi/i}|\overline{B}^1_{n+1,r,k}(x)|dx\\
&+\sum_{k=1}^{r}\sum_{i=\ell+1}^{m}\sum_{n=\ell}^{i-1}|\triangle_r a_{k+nr}|\int_{\pi/(i+1)}^{\pi/i}|\overline{B}^1_{n+1,r,k}(x)|dx\\
=&\sum_{k=1}^{r}\sum_{n=0}^{\ell-1}|\triangle_r a_{k+nr}|\int_{\pi/(m+1)}^{\pi/\ell}|\overline{B}^1_{n+1,r,k}(x)|dx
\end{aligned}$$

$$+\sum_{k=1}^{r}\sum_{n=\ell}^{m-1}|\triangle_r a_{k+nr}|\int_{\pi/(m+1)}^{\pi/(n+1)}|\overline{B}^{1}_{n+1,r,k}(x)|dx. \tag{13}$$

Since

$$|\overline{B}^{1}_{n+1,r,k}(x)|\leq\sum_{m=0}^{n}(k+mr)x\leq 4rn^2x,$$

then from (13) it follows that

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell}|h_1(x)|dx \leq & C_r\sum_{k=1}^{r}\sum_{n=1}^{\ell-1}n^2|\triangle_r a_{k+nr}|\left(\frac{1}{\ell^2}-\frac{1}{(m+1)^2}\right)\\ &+C_r\sum_{k=1}^{r}\sum_{n=\ell}^{m-1}n^2|\triangle_r a_{k+nr}|\left(\frac{1}{(n+1)^2}-\frac{1}{(m+1)^2}\right)\\ \leq & C_r\frac{m+1-\ell}{m}\sum_{k=1}^{r}\sum_{n=1}^{\ell-1}\frac{n^2}{\ell^2}|\triangle_r a_{k+nr}|+C_r\sum_{k=1}^{r}\sum_{n=\ell}^{m}\sum_{j=n}^{\infty}|\triangle_r^2 a_{k+jr}|. \end{aligned}\tag{14}$$

Now we shall estimate the integral of the function $|h_2(x)|$ for $x\in\left(\frac{\pi}{i+1},\frac{\pi}{i}\right]$. Indeed, the summation by parts gives

$$\begin{aligned}h_2(x)=&\lim_{p\to\infty}\sum_{k=1}^{r}\sum_{n=i}^{p}\triangle_r a_{k+nr}\varphi_{n,r,k}(x)\\ =&\sum_{k=1}^{r}\lim_{p\to\infty}\left(\sum_{n=i}^{p-1}\triangle_r^2 a_{k+nr}\sum_{s=0}^{n}\varphi_{s,r,k}(x)\right.\\ &\left.-\triangle_r a_{k+ir}\sum_{s=0}^{i-1}\varphi_{s,r,k}(x)+\triangle_r a_{k+pr}\sum_{s=0}^{p}\varphi_{s,r,k}(x)\right)\\ =&\sum_{k=1}^{r}\left(\sum_{n=i}^{\infty}\triangle_r^2 a_{k+nr}\psi_{n,r,k}(x)-\triangle_r a_{k+ir}\psi_{i-1,r,k}(x)\right)\\ =&\sum_{k=1}^{r}\sum_{n=i}^{\infty}\triangle_r^2 a_{k+nr}\left(\psi_{n,r,k}(x)-\psi_{i-1,r,k}(x)\right),\end{aligned}$$

where $\psi_{n,r,k}(x)$ are defined as above. So, we have

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell}|h_2(x)|dx \leq & C_r\sum_{k=1}^{r}\sum_{i=\ell}^{m}\int_{\pi/(i+1)}^{\pi/i}\sum_{n=i}^{\infty}|\triangle_r^2 a_{k+nr}|\frac{dx}{x^2}\\ \leq & C_r\sum_{k=1}^{r}\sum_{i=\ell}^{m}\sum_{n=i}^{\infty}|\triangle_r^2 a_{k+nr}|.\end{aligned}\tag{15}$$

It is clear that for $i \geq r$

$$\begin{aligned}\int_{\pi/(i+1)}^{\pi/i}\left|\frac{\cos(2k-r)\frac{x}{2}}{2\sin\left(r\frac{x}{2}\right)}\right|dx =&\frac{1}{2}\int_{\pi/(i+1)}^{\pi/i}\left|\cos kx\cot\left(\frac{rx}{2}\right)+\sin kx\right|dx\\ \leq&\frac{1}{2}\int_{\pi/(i+1)}^{\pi/i}\cot\left(\frac{rx}{2}\right)dx+\frac{1}{2}\int_{\pi/(i+1)}^{\pi/i}\sin kx dx\\ \leq&\frac{1}{k}\left(\ln\frac{\sin\frac{r\pi}{2i}}{\sin\frac{r\pi}{2(i+1)}}+\cos\frac{k\pi}{2i(i+1)}\cos\frac{k\pi(2i+1)}{2i(i+1)}\right)\\ =&\frac{d_{i,r,k}}{k}.\end{aligned} \tag{16}$$

Therefore from (16) we have

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell}|h_0(x)|dx \leq&\sum_{k=1}^{r}\sum_{i=\ell}^{m}|a_{k+ir}|\int_{\pi/(i+1)}^{\pi/i}\left|\frac{\cos(2k-r)\frac{x}{2}}{2\sin\left(r\frac{x}{2}\right)}\right|dx\\ \leq&\sum_{k=1}^{r}\sum_{i=\ell}^{m}\frac{d_{i,r,k}}{k}|a_{k+ir}|.\end{aligned} \tag{17}$$

Finally, the proof of the theorem is an immediate result of relations (12), (14), (15) and (17).

References

[1] A. N. Kolmogorov. Sur l'ordre de grandeur des coefficients de la série de Fourier-Lebesgue. *Bulletin de l'Academie Polonaise*, 83–86, 1923.

[2] Xh. Z. Krasniqi. Integrability of cosine trigonometric series with coefficients of bounded variation of order p. *Applied Mathematics E-Notes*, 11:61–66, 2011.

[3] Xh. Z. Krasniqi. On the first derivative of the sums of trigonometric series with quasi-convex coefficients of higher order. *Acta et Commentationes Universitatis Tartuensis de Mathematica*, 14:53–63, 2010.

[4] Xh. Z. Krasniqi. Integrability of double cosine trigonometric series with coefficients of bounded variation of second order. *Commentationes Mathematicae*, 51:125-139, 2011.

[5] B. V. Simonov. Trigonometric series in the Orlicz–Lorentz spaces. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 51:61–74, 2007.

[6] S. A. Telyakovskiĭ. Localizing the conditions of integrability of trigonometric series (Russian). Theory of functions and differential equations, Collection of articles. In honor of the ninetieth birthday of Academician S. M. Nikolskii(Russian), *Trudy Matematicheskogo Instituta Imeni V. A. Steklova*, 210:264–273, 1995.