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Nil(n)-Modules Bifibred over Groups

Hasan Atik

Department of Mathematics, Science Faculty, İstanbul Medeniyet University, İstanbul, Turkey

Abstract. In this work, we defined a functor from category of nil(n)-modules to that of groups. Then we showed by direct calculation that the functor is both fibration and cofibration of categories.

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1. Introduction

Crossed modules were defined by Whitehead [12] as a model for homotopy connected 2-types. Some universal constructions for crossed modules, for example, the notions of pullback and induced crossed modules have been worked in [4–6]. Furthermore, for Lie algebra cases of these constructions see [8], and for commutative algebras see [10]. Induced crossed modules allow detailed computations of non-abelian information on second relative homotopy calculations. By extending these constructions for two dimensional case of crossed modules, Arslan, Arvasi and Onarli in [1], have defined the notions of pullback and induced 2-crossed module.

Baues [3] defined nil(n)-modules as a model for homotopy 2-types and studied some properties of nil(2)-modules which forms a base for his homotopy connected 3-types "quadratic module". Atik has constructed pullback and induced nil(2)- modules in his thesis [2]. In this work, by using a similar way given in these cited works, we have shown that the category of nil(n)-modules is bifibred over groups in the sense of A. Grothendieck [9].

2. Nil(n)-Modules

A pre-crossed module is a group homomorphism $\partial : M \to Q$ together with an action of Q on M, written m^q for $q \in Q$ and $m \in M$, satisfying the condition $\partial(m^q) = q^{-1}\partial(m)q$ for all $m \in M$ and $q \in Q$. This is a crossed module if in addition

$$x^{-1}y^{-1}x = (y)^{\partial x}$$

Email addresses: hasan.atik@medeniyet.edu.tr, hasanatik@yahoo.com

We define Peiffer commutator in a pre-crossed module

$$\langle x, y \rangle = x^{-1} y^{-1} x(y)^{\partial_1 x}$$

Thus ∂ is a crossed module if and only if $\langle x, y\rangle = 1$ for all $x, y \in M$.

In a group G we have the lower central series

$$\Gamma_{n+1} \subset \Gamma_n \subset \ldots \subset \Gamma_1 = G$$

where $\Gamma_n = \Gamma_n(G)$ is the subgroup of G generated by all iterated commutators $(x_1, x_2, \ldots, x_n)$ of length n. Here $\Gamma_2(G)$ is the commutator subgroup of G. Similarly we obtain the lower Peiffer central series

$$P_{n+1} \subset P_n \subset \ldots \subset P_1 = M$$

in a pre-crossed module $\partial : M \to N$. Where $P_n = P_n(\partial)$ is the subgroup of M generated by all iterated Peiffer commutators $\langle x_1, x_2, \ldots, x_n\rangle$ of length n. The group $P_n(\partial)$ is the Peiffer subgroup of M, this generalizes the commutator subgroup in a group. The following definition is given by Baues [3].

Definition 1. *A pre-crossed module $\partial : M \to N$ is a Peiffer nilpotent of class n if $P_{n+1}(\partial) = 1$, in this case we call ∂ is a nil(n)-module. That is*

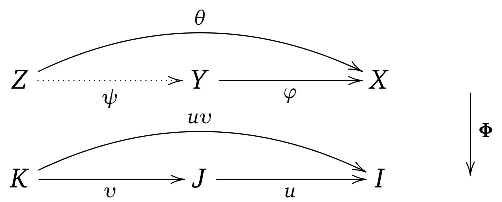
$$\langle x_1, x_2, x_3 \cdots, x_n\rangle = 1,$$

A morphism between two nil(n)-modules $\partial : M \to Q$ and $\partial' : M' \to Q'$ is a pair (g, f) of homomorphisms of groups $g : M \to M'$ and $f : Q \to Q'$ such that $f\partial = \partial' g$ and the actions preserved, i.e. $g(m^q) = g(m)^{f(q)}$ for any $m \in M, q \in Q$. We shall denote the category of *nil(n)-modules* by $\mathbf{Nil}(n)$.

3. Bifibration of Categories

We recall the definition of fibration of categories from [7].

Definition 2. *Let $\Phi : \mathbf{X} \to \mathbf{B}$ be a functor. A morphism $\varphi : Y \to X$ in $\mathbf{X}$ over $u := \Phi(\varphi)$ is called Cartesian if and only if for all $\upsilon : K \to J$ in $\mathbf{B}$ and $\theta : Z \to X$ with $\Phi(\theta) = u\upsilon$ there is a unique morphism $\psi : Z \to Y$ with $\Phi(\psi) = \upsilon$ and $\theta = \varphi\psi$.*

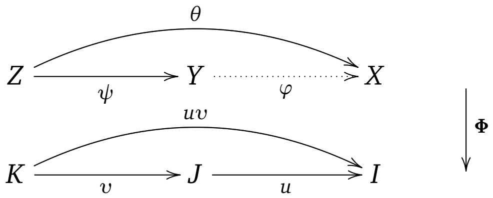


A morphism $\alpha : Z \to Y$ is called vertical (with respect to Φ) if and only if $\Phi(\alpha)$ is an identity isomorphism in $\mathbf{B}$. In particular, for $I \in \mathbf{B}$ we write $\mathbf{X_I}$, called the fibre over I, for the subcategory of $\mathbf{X}$ consisting of those morphisms α with $\Phi(\alpha) = id_I$,

Definition 3. *The functor $\Phi : \mathbf{X} \to \mathbf{B}$ is fibration or category fibred over $\mathbf{B}$ if and only if for all $u : J \to I$ in $\mathbf{B}$ and $X \in \mathbf{X_I}$ there is a Cartesian morphism $\varphi : Y \to X$ over u : such a φ is called a Cartesian lifting of X along u.*

We now give the duals of the above definition.

Definition 4. *Let $\Phi : \mathbf{X} \to \mathbf{B}$ be a functor. A morphism $\psi : Z \to Y$ in $\mathbf{X}$ over $\upsilon := \Phi(\psi)$ is called cocartesian if and only if for all $u : J \to I$ in $\mathbf{B}$ and $\theta : Z \to X$ with $\Phi(\theta) = u\upsilon$ there is a unique morphism $\varphi : Y \to X$ with $\Phi(\varphi) = u$ and $\theta = \varphi\psi$.*



Definition 5. *The functor $\Phi : \mathbf{X} \to \mathbf{B}$ is cofibration or category cofibred over $\mathbf{B}$ if and only if for all $\upsilon : K \to J$ in $\mathbf{B}$ and $Z \in \mathbf{X_K}$ there is a Cartesian morphism $\psi : Z \to Z'$ over υ : such a ψ is called a cocartesian lifting of X along υ.*

Proposition 1. *Let $\Phi : \mathbf{X} \to \mathbf{B}$ be a fibration of categories. Then $\psi : Z \to Y$ in $\mathbf{X}$ over $\upsilon : K \to J$ in $\mathbf{B}$ is cocartesian if and only if for all $\theta' : Z \to X'$ over υ there is a unique morphism $\psi' : Y \to X'$ in X_J with $\theta' = \psi'\psi$.*

Corollary 1. *Let $\Phi : \mathbf{X} \to \mathbf{B}$ be a fibration of categories which has a left adjoint and suppose that X admits pushouts. Then Φ is also a cofibration.*

For detailed information about bifibration categories we advise carefull reading of T.Streicher [11].

4. Nil(n)-Modules Bifibred over Groups

Proposition 2. *We have a forgetful functor $\Phi_N : \mathbf{Nil(n)} \to \mathbf{Grp}$ in which $(M \to N) \longrightarrow N$. This forgetful functor is fibred.*

Suppose that $\partial : M \to Q$ is a nil(n)-module and $\sigma : P \to Q$ is a homomorphism of groups. Take $\sigma^*(M) = \{(p, m) : \partial(m) = \sigma(p)\}$ as the fiber product of ∂ and σ. Thus we have the following pullback diagram

$$\begin{array}{ccc} \sigma^*(M) & \xrightarrow{\sigma_1} & M \\ {\scriptstyle \beta_1}\downarrow & & \downarrow{\scriptstyle \partial} \\ P & \xrightarrow[\sigma]{} & Q \end{array} \qquad (1)$$

where $\sigma_1 : \sigma^*(M) \to P$ is given by $\sigma_1(p, m) = m$ and $\beta_1 : \sigma^*(M) \to P$ is given by $\beta_1(p, m) = p$ for all $(p, m) \in \sigma^*(M)$. The action of $p' \in P$ on $(p, m) \in \sigma^*(M)$ can be given by

$$(p, m)^{p'} = (p'^{-1}pp', m^{\sigma(p')}).$$

This action obviously is a group action of P on $\sigma^*(M)$ and according to this action, β_1 becomes a nil(n)-module. Indeed, β_1 is a pre-crossed module since for all $(p,m)\in\sigma^*(M)$,

$$\beta_1((p,m)^{p'})=\beta_1(p'^{-1}pp',m^{\sigma(p')})=p'^{-1}pp'=p'^{-1}\beta_1(p,m)p'.$$

Moreover,for $(p_1,m_1),(p_2,m_2),\ldots,(p_n,m_n)\in\sigma^*(M)$, we have

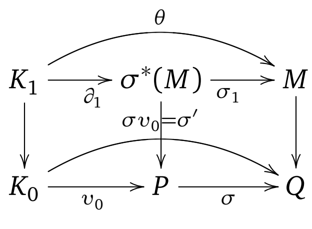
$$\begin{aligned}
&\langle\ldots\langle\langle(p_1,m_1),(p_2,m_2)\rangle,(p_3,m_3)\rangle,\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle\langle(p_1,m_1)^{-1}(p_2,m_2)^{-1}(p_1,m_1)(p_2,m_2)^{\beta_1(p_1,m_1)},(p_3,m_3)\rangle,(p_4,m_4)\rangle,\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle\langle(p_1{}^{-1},m_1{}^{-1})(p_2{}^{-1},m_2{}^{-1})(p_1,m_1)(p_2,m_2)^{p_1},(p_3,m_3)\rangle,(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle\langle(1,m_1{}^{-1}m_2{}^{-1}m_1m_2{}^{\sigma(p_1)}),(p_3,m_3)\rangle,(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle\langle(1,m_1{}^{-1}m_2{}^{-1}m_1m_2{}^{\partial(m_1)}),(p_3,m_3)\rangle,(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle(1,m_1{}^{-1}m_2{}^{-1}m_1m_2{}^{\partial(m_1)})^{-1}(p_3{}^{-1},m_3{}^{-1}),\\
&\quad(1,m_1{}^{-1}m_2{}^{-1}m_1m_2{}^{\partial(m_1)})(p_3,m_3)^{\beta_1(1,m_1{}^{-1}m_2{}^{-1}m_1m_2{}^{\partial(m_1)})},(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle(1,m_2{}^{\partial(m_1)^{-1}}m_1{}^{-1}m_2m_1)(p_3{}^{-1},m_3{}^{-1})(1,m_1{}^{-1}m_2{}^{-1}m_1m_2{}^{\partial(m_1)})\\
&\quad(p_3,m_3),(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle(1,\langle m_1,m_2\rangle^{-1})(p_3{}^{-1},m_3{}^{-1})(1,\langle m_1,m_2\rangle)(p_3,m_3),(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle(1,\langle m_1,m_2\rangle^{-1}m_3{}^{-1}\langle m_1,m_2\rangle m_3),(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle(1,\langle m_1,m_2\rangle^{-1}m_3{}^{-1}\langle m_1,m_2\rangle m_3{}^{\partial_1(\langle m_1,m_2\rangle)}),(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle\\
&=\langle\ldots\langle(1,\langle\langle m_1,m_2\rangle,m_3\rangle),(p_4,m_4)\rangle\ldots\rangle,(p_n,m_n)\rangle.
\end{aligned}$$

if we continue calculations in this way, we obtain; $(1,\langle m_1,m_2,m_3,\ldots m_n\rangle)$. Since ∂_1 is a nil(n)-module then $(\langle m_1,m_2,m_3,\ldots m_n\rangle)=1$, it gives the following result: β_1 is nil(n)-module.

Thus $\beta_1:\sigma^*(M)\to P$ is a nil(n)-module. In the diagram (1), the pair of homomorphisms (σ_1,σ) is a nil(n)-module morphism. This diagram is commutative since $\partial\sigma_1(p,m)=\partial(m)=\sigma(p)=\sigma\beta_1(p,m)$ for $p\in P$ and $m\in M$. We have

$$\sigma_1((p,m)^{p'})=\sigma_1((p')^{-1}pp',m^{\sigma(p')})=m^{\sigma(p')}=\sigma_1(p,m)^{\sigma(p')}$$

for all $(p,m)\in\sigma^*(M)$ and $p\in P$. Therefore we have a pullback nil(n)-module. Further we will show that σ_1 is a Cartesian morphism over σ. Let $(\upsilon_1,\upsilon_0):(K_1\to K_0)\to(\sigma^*(M)\to P)$ be homomorphism of nil(n)-modules and $\theta:K_1\to P$ be a unique nil(n)-module morphism. Then we have the following commutative diagram



where $\upsilon_1:K_1\to\sigma^*(M)$ is given by $\upsilon_1(k_1)=(\upsilon_0\partial_1k_1,\theta k_1)$. Since $\sigma(\upsilon_0\partial_1k_1)=\partial\theta k_1=\partial m$ so ψ is a well defined homomorphism.

Proposition 3. *The functor* Φ_N : **Nil(n)** $\to$ **Grp** *is cofibred.*

Let $\mu : M \to P$ be a nil(2)-module and $f : P \to Q$ be a homomorphism of groups. Let $f_*(M) = F(M \times Q)$ be a free group generated by the set $M \times Q$. Let S be a subgroup of $f_*(M)$ generated by the following relations: $(m, m' \in M,\ q \in Q)$

1. $(m,q)(m',q)(mm',q)^{-1} \in S$
2. $(m^p,q)(m,f(p)q)^{-1} \in S$

Now, consider the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \theta\ } & f_*(M)/S \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \overline{\mu}} \\ P & \xrightarrow[\ f\ ]{} & Q \end{array}$$

in which $\overline{\mu} : f_*(M)/S \to Q$ is given by $\overline{\mu}((m,q)S) = q^{-1} f\mu(m) q$ and $\theta : M \to f_*(M)/S$ is given by $\theta(m) = (m,1)S$ for $m \in M$ and $q \in Q$. This diagram is commutative, since $\overline{\mu}\theta(m) = \overline{\mu}((m,1)S) = f\mu(m)$ for all $m \in M$. The action of Q on $f^*(M)/S$ can be given by $((m,q)S)^{q'} = (m,qq')S$ for $m \in M$ and $q,q' \in Q$. By using this action, we have the following result.

Proposition 4. *The homomorphism* $\overline{\mu} : f_*(M)/S \to Q$ *given by* $\overline{\mu}((m,q)S) = q^{-1} f\mu(m) q$, *as defined above, is an induced nil(n)-module by the homomorphism of groups* $f : P \to Q$ *of the nil(n)-module* $\mu : M \to P$.

Proof. Since

$$\begin{aligned} \overline{\mu}(((m,q)S)^{q'}) =& \overline{\mu}((m,qq')S) \\ =& (qq')^{-1} f\mu(m) qq' = (q')^{-1}(q^{-1} f\mu(m) q) q' \\ =& (q')^{-1}\overline{\mu}((m,q)S) q', \end{aligned}$$

for all $m \in M$ and $q,q' \in Q$, $\overline{\mu}$ is a pre-crossed module.

Further, for all $(m,q)S, (m',q)S, \ldots, (m^{(n)},q)S \in f_*(M)/S$,

$$\begin{aligned} &\langle \ldots \langle \langle (m,q)S, (m',q)S \rangle, (m'',q)S \rangle, \ldots \rangle (m^{(n)},q)S \rangle \\ &= \langle \ldots \langle (m,q)S(m',q)S(m,q)S^{-1}((m',q)S^{-1})^{\overline{\mu}(m,q)S}, (m'',q)S \rangle, \ldots \rangle (m^{(n)},q)S \rangle \\ &= \langle \ldots \langle \langle (m,q)S(m',q)S(m^{-1},q)S((m'^{-1},q)S)^{q^{-1} f\mu(m) q}, (m'',q)S \rangle, \ldots \rangle (m^{(n)},q)S \rangle \\ &= \langle \ldots \langle \langle (mm'm^{-1},q)S((m'^{-1}, qq^{-1} f\mu(m) q)S, (m'',q)S \rangle, \ldots \rangle (m^{(n)},q)S \rangle \\ &= \langle \ldots \langle \langle (mm'm^{-1},q)S((m'^{-1})^{\mu(m)},q)S, (m'',q)S \rangle, \ldots \rangle (m^{(n)},q)S \rangle \\ &= \langle \ldots \langle \langle (mm'm^{-1}(m'^{-1})^{\mu(m)},q)S, (m'',q)S \rangle, \ldots \rangle (m^{(n)},q)S \rangle \end{aligned}$$

$$
\begin{aligned}
&=\langle \dots \langle \langle (\langle m,m'\rangle ,q)S,(m'',q)S\rangle \dots \rangle (m^{(n)},q)S\rangle \\
&=\langle \dots \langle (\langle m,m'\rangle ,q)S(m'',q)S(\langle m,m'\rangle ,q)^{-1}S((m'',q)S^{-1})^{\overline{\mu}(\langle m,m'\rangle ,q)S},\dots \rangle (m^{(n)},q)S\rangle \\
&=\langle \dots \langle (\langle m,m'\rangle ,q)S(m'',q)S(\langle m,m'\rangle ,q)^{-1}S((m'',q)S^{-1})^{q'^{-1}f\mu(\langle m,m'\rangle)q},\dots \rangle (m^{(n)},q)S\rangle \\
&=\langle \dots \langle (\langle m,m'\rangle ,q)S(m'',q)S(\langle m,m'\rangle ^{-1},q)S((m''^{-1},q)S),\dots \rangle (m^{(n)},q)S\rangle \\
&=\langle \dots \langle (\langle m,m'\rangle m''\langle m,m'\rangle ^{-1}(m''^{-1},q)S),\dots \rangle (m^{(n)},q)S\rangle \\
&=\langle \dots \langle (\langle m,m'\rangle m''\langle m,m'\rangle ^{-1}(m''^{-1})^{\mu(\langle m,m'\rangle)},q)S,\dots \rangle (m^{(n)},q)S\rangle \\
&=\langle \dots \langle (\langle \langle m,m'\rangle ,m''\rangle ,q)S,\dots \rangle (m^{(n)},q)S\rangle \\
&=\langle \dots \langle (1,q)S,\dots \rangle (m^{(n)},q)S\rangle \\
&\ \ \vdots \\
&=(\langle m_1,m_2,m_3,\dots m_n\rangle ,q)S\cong S
\end{aligned}
$$

Thus we have that $\overline{\mu}$ is a nil(n)-module. Now, we will show that (θ, f) is a nil(n)-module morphism. We have

$$\theta(m^p) = (m^p, 1)S = m, f(p)1)S = ((m, 1)S)^{f(p)} = \theta(m)^{f(p)}$$

and $\overline{\mu}\theta(m) = \overline{\mu}((m,1)S) = f\mu(m)$ for all $m \in M$ and $p \in P$. Then one can easily show that θ is a cocartesian morphism over f. □

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