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Explicit Form of the Fundamental Units of Certain Real Quadratic Fields

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Abstract. In this paper, for all real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ such that d is a positive square free integer congruent to 2 or 3 modulo 4 and the period k_d of the continued fraction expansion of the quadratic irrational number $\omega_d = \sqrt{d}$ is equal to 7, we describe T_d, U_d explicitly in the fundamental unit $\varepsilon_d = (\frac{T_d+U_d\sqrt{d}}{2})(>1)$ of $\mathbb{Q}(\sqrt{d})$ and d itself by using five parameters appearing in the continued fraction expansion of ω_d.

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1. Introduction

Explicit form of the fundamental units of real quadratic fields $\mathbb{Q}(\sqrt{d})$ where d is congruent to 1 modulo 4 and the period k_d in the continued fraction expansion of the quadratic irrational number ω_d in $\mathbb{Q}(\sqrt{d})$ is equal to 3 and 4, 5 was described in [5, 6] respectively. Later in [3], explicit form of the fundamental units of all real quadratic fields $\mathbb{Q}(\sqrt{d})$ such that the period in the continued fraction expansion of the quadratic irrational number ω_d in $\mathbb{Q}(\sqrt{d})$ is equal to 6 was obtained.

In this paper, for all real quadratic fields $\mathbb{Q}(\sqrt{d})$ such that d is congruent to 1 modulo 4 and the period k_d in the continued fraction expansion of the quadratic irrational number $\omega_d = \frac{1+\sqrt{d}}{2}$ is equal to 7, we described T_d, U_d explicitly in the fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$ and d itself by using five parameters appearing in the continued fraction expansion of ω_d.

In this paper, we consider all real quadratic fields $\mathbb{Q}(\sqrt{d})$ where $d \equiv 2, 3(mod 4)$ and the period k_d of the continued fraction expansion of $\omega_d = \sqrt{d}$ is equal to 7 and describe explicitly coefficients T_d and U_d in the fundamental unit $\varepsilon_d = (\frac{T_d+U_d\sqrt{d}}{2})(>1)$ of $\mathbb{Q}(\sqrt{d})$ and d itself by using five parameters appearing in the continued fraction expansion of ω_d.

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Let $I(d)$ be the set of all quadratic irrational numbers in $\mathbb{Q}(\sqrt{d})$. For an element ξ of $I(d)$ if $\xi > 1$, $-1 < \xi' < 0$ then ξ is called reduced, where ξ' is the conjugate of ξ with respect to $\mathbb{Q}$. More information on reduced irrational numbers may be found in [2, 7]. We denote by $R(d)$ the set of all reduced quadratic irrational numbers in $I(d)$. It is well known that if an element ξ of $I(d)$ is in $R(d)$ then the continued fractional expansion of ξ is purely periodic. Moreover, the denominator of its modular automorphism is equal to fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$ and the norm of ε_d is $(-1)^{k_d}$ [4]. In this paper $[x]$ means the greatest integer less than or equal to x and continued fraction with period k is generally denoted by $[a_0, \overline{a_1, a_2, \ldots, a_k}]$.

2. Preliminaries

In this section some of the important required preliminaries and lemmas are given.

For any square-free positive integer d, we can put $d = a^2 + b$ with $a, b \in \mathbb{Z}$, $0 < b \leq 2a$. Here, since $\sqrt{d} - 1 < a < \sqrt{d}$ the integers a and b are uniquely determined by d. In this paper we will concern with all real quadratic fields $\mathbb{Q}(\sqrt{d})$ such that d is congruent to 2 or 3 modulo 4 and the period k_d is equal to 7.

Let $d = a^2 + b \equiv 2, 3(mod4)$, then we consider the following three cases:

Case 1. If b is congruent to 1 modulo 4, then d can only be congruent to 2 modulo 4. And for this case it is obvious that a is odd.

Case 2. If b is congruent to 2 modulo 4, then d can be congruent to 2 or 3 modulo 4. In this case, a is even when d is congruent to 2 modulo 4 and a is odd when d is congruent to 3 modulo 4.

Case 3. If b is congruent to 3 modulo 4, then d can only be congruent to 3 modulo 4. And for this case it is obvious that a is even.

Lemma 1. *For a square-free positive integer d congruent to* 2 *or* 3 *modulo* 4*, we put* $\omega_d = \sqrt{d}$, $q_0 = [\omega_d]$, $\omega_R = q_0 + \omega_d$. *Then* $\omega_d \notin R(d)$, *but* $\omega_R \in R(d)$ *holds. Moreover, for the period k of* ω_R, *we get* $\omega_R = [\overline{2q_0, q_1, \ldots, q_{k-1}}]$ *and* $\omega_d = [q_0, \overline{q_1, \ldots, q_{k-1}, 2q_0}]$. *Furthermore, let* $\omega_R = \frac{(P_{k-1}\omega_R + P_{k-2})}{(Q_{k-1}\omega_R + Q_{k-2})} = [2q_0, q_1, \ldots, q_{k-1}, \omega_R]$ *be a modular automorphism of* ω_R, *then the fundamental unit* ε_d *of* $Q(\sqrt{d})$ *is given by the following formula:*

$$\varepsilon_d = (\frac{T_d + U_d\sqrt{d}}{2}) > 1, \quad T_d = 2q_0Q_{k-1} + 2Q_{k-2}, \quad U_d = 2Q_{k-1}$$

where Q_i *is determined by* $Q_{-1} = 0$, $Q_0 = 1$, $Q_{i+1} = q_{i+1}Q_i + Q_{i-1}$, $(i \geq 0)$.

Proof. See [5, Lemma 1].

Lemma 2. *For a square-free positive integer d, we put* $d = a^2 + b$ $(0 < b \leq 2a)$, $a, b \in \mathbb{Z}$. *Moreover let* $\omega_i = \ell_i + \frac{1}{\omega_{i+1}}$ $(\ell_i = [\omega_i],\ i \geq 0)$ *be the continued fraction expansion of* $\omega = \omega_0$

in $R(d)$. Then each ω_i is expressed in the form $\omega_i = \frac{a-r_i+\sqrt{d}}{c_i}$ $(c_i, r_i \in \mathbb{Z})$, and ℓ_i, c_i, r_i can be obtained from the following recurrence formula:

$$\begin{aligned} \omega_0 =& \frac{a-r_0+\sqrt{d}}{c_0}, \\ 2a-r_i =& c_i\ell_i + r_{i+1}, \\ c_{i+1} =& c_{i-1} + (r_{i+1}-r_i)\ell_i \quad (i \geq 0), \text{ where } 0 \leq r_{i+1} < c_i, c_{-1} = \frac{(b+2ar_0-r_0{}^2)}{c_0}. \end{aligned}$$

Moreover for the period $k \geq 1$ of ω_0, we get

$$\begin{aligned} \ell_i =& \ell_{k-i} \quad (1 \leq i \leq k-1), \\ r_i =& r_{k-i+1}, c_i = c_{k-i} \quad (1 \leq i \leq k). \end{aligned}$$

Proof. See [1, Proposition 1].

Lemma 3. *For a square-free positive integer d congruent to 2 or 3 modulo 4, we put $\omega_d = \sqrt{d}$, $q_0 = [\omega_d]$ and $\omega_R = q_0 + \omega_d$.*

If we put $\omega = \omega_R$ in Lemma 2 , then we have the following recurrence formula:

$$\begin{aligned} r_0 =& r_1 = 0, \\ c_0 =& 1, c_1 = b, \\ \ell_0 =& 2q_0, \ell_i = q_i \quad (1 \leq i \leq k-1). \end{aligned}$$

Proof. The proof follows easily from Lemma 2.

3. Main Results

Theorem 1. *For a positive square-free integer d congruent to 2 modulo 4, we assume $k_d = 7$. Then, if b is congruent to 1 modulo 4, we get*

$$\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$$

for three positive integers ℓ_1, ℓ_2, ℓ_3 such that $\ell_i \geq 1$ $(i = 1, 2, 3)$ and then

$$(T_d, U_d) = (2[a(A^2+B^2)+BC+A\ell_2], 2(A^2+B^2))$$

and

$$d = A^2r^2 + 2rD + E$$

hold. Moreover r and s are positive integers determined uniquely by

$$\begin{aligned} a =& Ar + \ell_1 s \\ A^2 + & B^2 - C^2 - \ell_2{}^2 = 2rB - 2s(A + B\ell_3) \end{aligned}$$

where A, B, C, D and E are determined uniquely as follows:

$$\begin{aligned} A =& \ell_1 \ell_2 + 1 \\ B =& \ell_1 + A\ell_3 \\ C =& \ell_2 \ell_3 + 1 \\ D =& A\ell_1 s + \ell_2 \\ E =& {\ell_1}^2 s^2 + 2s + 1 \end{aligned}$$

Proof. In the case of $b \equiv 1(mod 4)$, it can be easily seen that a is an odd integer since d is congruent to 2 modulo 4. We can put $b = 4m + 1$ for a non-negative integer m satisfying $0 \le 4m < 2a$. Since $q_0 = [\omega_d] = [\sqrt{d}] = a$ and $\omega_R = a + \sqrt{d}$, it follows from Lemma 3 that $r_0 = r_1 = 0$, $c_0 = 1$, $c_1 = 4m + 1$ and $\ell_0 = 2a$. Since $k_d = 7$, we get $\ell_1 = \ell_6$, $\ell_2 = \ell_5$ and $\ell_3 = \ell_4$ from Lemma 2. Then we have

$$\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$$

for three positive integers ℓ_1, ℓ_2, ℓ_3 such that $\ell_i \ge 1$ $(i = 1, 2, 3)$. From Lemma 2 we get

$$2a = (4m + 1)\ell_1 + r_2 \tag{1}$$

since $r_1 = 0$ and $c_1 = 4m + 1$. From (1), we obtain $(4m + 1)\ell_1 + r_2 \equiv 0 \pmod 2$. So there exists a positive integer r such that $r_2 = 2r - \ell_1$. By substitution of r_2 in (1) we get

$$a = 2m\ell_1 + r. \tag{2}$$

Here since a is odd, r must be an odd integer, too. It follows from Lemma 2 that $c_2 = 1 + r_2\ell_1$ and $2a = c_2\ell_2 + r_3 + r_2$. Thus,

$$2a = (1 + r_2\ell_1)\ell_2 + r_3 + r_2 \tag{3}$$

is obtained. Then

$$(4m + 1)\ell_1 = (1 + r_2\ell_1)\ell_2 + r_3 \tag{4}$$

holds from (1) and (3). Thus we get $\ell_2 + \ell_3 \equiv 0 \pmod{\ell_1}$. There exists a positive integer t such that $r_3 = \ell_1 t - \ell_2$. By substitution of r_3 in (4), we get $4m = t + 2r\ell_2 - \ell_1\ell_2 - 1$. Thus if we put $A = \ell_1\ell_2 + 1$, then we get $t - A = 4m - 2r\ell_2$. Since $t - A$ is even, we can put $t - A = 2s$ for a positive integer s. Hence, $4m = 2s + 2r\ell_2$ is obtained. Therefore we get $a = Ar + \ell_1 s$ from (2). On the other hand,

$$c_3 = 4m + 1 + (r_3 - r_2)\ell_2 \tag{5}$$

is obtained from Lemma 2. By substitution of $r_2 = 2r - \ell_1$ and $r_3 = \ell_1 t - \ell_2$ in (5), we get $c_3 = At - {\ell_2}^2$. Moreover from Lemma 2, we get $2a = c_3\ell_3 + r_3 + r_4$. Thus

$$r_4 = (2r - \ell_1 - t\ell_3)A + \ell_2(\ell_2\ell_3 + 1) \tag{6}$$

is obtained because of (3) and $c_3 = At - \ell_2{}^2$. Furthermore, $c_3 = At - \ell_2{}^2$ and $c_4 = (1 + r_2\ell_1) + (r_4 - r_3)\ell_3$ imply $At - \ell_2{}^2 = 1 + r_2\ell_1 + r_4\ell_3 - r_3\ell_3$ since $c_3 = c_4$. Thus,

$$At - \ell_2{}^2 = (1 + \ell_2\ell_3)^2 + 2r(\ell_1 + A\ell_3) - t\ell_3(\ell_1 + A\ell_3) - \ell_1(\ell_1 + A\ell_3)$$

is obtained since $r_2 = 2r - \ell_1$, $r_3 = \ell_1 t - \ell_2$ and $r_4 = (2r - \ell_1 - t\ell_3)A + \ell_2(\ell_2\ell_3 + 1)$. Thus if we put $B = \ell_1 + A\ell_3$ and $C = \ell_2\ell_3 + 1$, we get $A^2 + B^2 - C^2 - \ell_2{}^2 = 2rB - 2s(A + B\ell_3)$ since $t - A = 2s$. If we assume that the integers r and s are not uniquely determined, we get $A^2 + B^2 = 0$ which is a contradiction. Therefore, the integers r and s are uniquely determined by $a = Ar + \ell_1 s$ and $A^2 + B^2 - C^2 - \ell_2{}^2 = 2rB - 2s(A + B\ell_3)$.

Now, since $\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$ implies $Q_5 = BC + A\ell_2$ and $Q_6 = A^2 + B^2$ by Lemma 1, we obtain

$$(T_d, U_d) = (2[a(A^2 + B^2) + BC + A\ell_2], 2(A^2 + B^2)).$$

Furthermore, if we put $D = A\ell_1 s + \ell_2$ and $E = \ell_1{}^2 s^2 + 2s + 1$, then we get $d = A^2r^2 + 2rD + E$ because $b = 2s + 2r\ell_2 + 1$. Thus, the theorem is proved. □

As an application of this theorem, we can practically determine ω_d where $d = 314 = 17^2 + 25$. Since $q_0 = a$ and $\ell_0 = 2a$, it follows that $q_0 = 17$ and $\ell_0 = 34$. On the other hand, we get $m = 6$, since $b = 4m + 1$. From $a = 2m\ell_1 + r$, we obtain $\ell_1 = 1$ and $r = 5$. Thus we get $r_2 = 9$ immediately. Since $2a = (1 + r_2\ell_1)\ell_2 + r_2 + r_3$, $c_3 = 4m + 1 + (r_3 - r_2)\ell_2$ and $2a = c_3\ell_3 + r_3 + r_4$, we obtain $\ell_2 = 2$, $r_3 = 5$, $c_3 = 17$, $\ell_3 = 1$ and $r_4 = 12$. Hence ω_d can be determined as follows:

$$\omega_d = [17, \overline{1, 2, 1, 1, 2, 1, 34}]$$

Moreover fundamental unit of $\mathbb{Q}(\sqrt{314})$ can be easily determined as

$$\varepsilon_d = \frac{886 + 50\sqrt{314}}{2}$$

since $A = 3$, $B = 4$, $C = 3$. Furthermore by using $r_3 = \ell_1 t - \ell_2$ and $t - A = 2s$, we get $t = 7$ and $s = 2$. Thus $D = 8$ and $E = 9$ is obtained easily.

Theorem 2. *Let $d = a^2 + b \equiv 2, 3 \pmod 4$ be a positive square-free integer with $b \equiv 2 \pmod 4$. If $k_d = 7$, then we get*

$$\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$$

for the three positive integers ℓ_1, ℓ_2, ℓ_3 such that $\ell_i \geq$ $(i = 1, 2, 3)$ and then

$$(T_d, U_d) = (2[a(A^2 + B^2) + BC + A\ell_2], 2(A^2 + B^2))$$

and

$$d = A^2r^2 + 2rD + E$$

hold. Moreover, r and s are positive integers determined uniquely by

$$a = Ar + \ell_1 s$$

$$-\ell_2[\ell_2+\ell_3(C+1)]-1=2r(\ell_2+A\ell_3)-2s(B\ell_3+A)$$

where A, B, C, D and E are determined uniquely as follows:

$$\begin{aligned}A=&\ell_1\ell_2+1\\ B=&A\ell_3+\ell_1\\ C=&\ell_2\ell_3+1\\ D=&A\ell_1 s+\ell_2\\ E=&{\ell_1}^2s^2+2s.\end{aligned}$$

Proof. In the case of $b\equiv 2 \pmod 4$, we put $b=4m+2$ for a positive integer m satisfying $0<2m+1\leq a$. Since $q_0=[\omega_d]=[\sqrt{d}]=a$, it follows from Lemma 3 that $r_0=r_1=0$, $c_0=1$, $c_1=4m+2$, $\ell_0=2a$. Since $k_d=7$, we get $\ell_1=\ell_6$, $\ell_2=\ell_5$ and $\ell_3=\ell_4$ by Lemma 2. Then we have

$$\omega_d=[a,\overline{\ell_1,\ell_2,\ell_3,\ell_3,\ell_2,\ell_1,2a}]$$

for three integers ℓ_1, ℓ_2, ℓ_3 such that $\ell_i\geq 1$ holds. From Lemma 2 we get

$$2a=(4m+2)\ell_1+r_2 \tag{7}$$

since $r_1=0$, $c_1=4m+2$. From (7), we have $(4m+2)\ell_1+r_2\equiv 0 \pmod 2$ and we can put $r_2=2r$ for an integer r such that $r\geq 0$. Hence, it follows from (7)

$$a=(2m+1)\ell_1+r. \tag{8}$$

It follows from Lemma 2 that

$$c_2=1+\ell_1 r_2 \tag{9}$$

and

$$2a=c_2\ell_2+r_2+r_3. \tag{10}$$

Then from (7), (9) and (10) we have

$$(4m+2)\ell_1=c_2\ell_2+r_3. \tag{11}$$

Moreover, we can write $\ell_2+r_3\equiv 0 \pmod{\ell_1}$ from (9) and (11). So there exists a positive even integer t such that $r_3=\ell_1 t-\ell_2$. Since t is even, we can put $t=2s$ for a positive integer s. Thus $r_3=2s\ell_1-\ell_2$ is obtained. By substitution of r_3 in (11), we get

$$4m=2s+2r\ell_2-2. \tag{12}$$

It follows from (8) and (12) that $a=r(\ell_1\ell_2+1)+s\ell_1$ is written. Thus if we put $A=\ell_1\ell_2+1$, then we get $a=Ar+s\ell_1$. On the other hand, we get $2a=c_3\ell_3+r_3+r_4$ and $c_3=4m+2+(r_3-r_2)\ell_2$ from Lemma 2. It follows from $a=Ar+\ell_1 s$, $2a=c_3\ell_3+r_3+r_4$,

$c_3 = 4m + 2 + (r_3 - r_2)\ell_2$ and (12) that $r_4 = 2Ar - 2s\ell_3 A + \ell_2(\ell_2\ell_3 + 1)$. Thus, if we put $C = \ell_2\ell_3 + 1$ then we get

$$r_4 = 2Ar - 2s\ell_3 A + C\ell_2. \tag{13}$$

Moreover, $c_3 = 4m + 2 + (r_3 - r_2)\ell_2$ and $c_4 = 1 + r_2\ell_2 + (r_4 - r_3)\ell_3$ imply

$$4m = 2r_2\ell_2 + r_4\ell_3 - r_3\ell_3 - r_3\ell_2 - 1 \tag{14}$$

since $c_3 = c_4$. Then by substitution $r_3 = 2s\ell_1 - \ell_2$, (12) and (13) in (14) we obtain

$$-\ell_2{}^2 - \ell_2\ell_3(C+1) - 1 = 2r(\ell_2 + A\ell_3) - 2s[(\ell_1 + A\ell_3)\ell_3 + A].$$

Then if we put $B = \ell_1 + A\ell_3$ we get $-\ell_2{}^2 - \ell_2\ell_3(C+1) - 1 = 2r(\ell_2 + A\ell_3) - 2s(B\ell_3 + A)$. If we assume that the integers r and s are not determined uniquely from $a = Ar + \ell_1 s$ and $-\ell_2{}^2 - \ell_2\ell_3(C+1) - 1 = 2r(\ell_2 + A\ell_3) - 2s(B\ell_3 + A)$ we get $A(A + B\ell_3) + \ell_1(A\ell_3 + \ell_2) = 0$ which is a contradiction. Therefore, the integers r and s are uniquely determined.

Now, since $\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$ implies $Q_5 = BC + A\ell_2$ and $Q_6 = A^2 + B^2$ by Lemma 1, we obtain $T_d = 2[a(A^2 + B^2) + BC + A\ell_2]$ and $U_d = 2(A^2 + B^2)$, respectively. Moreover if we put $D = A\ell_1 s + \ell_2$ and $E = \ell_1{}^2 s^2 + 2s$, then we get $d = A^2r^2 + 2rD + E$ since $b = 2s + r\ell_2$. Thus, the theorem is proved completely. □

As an application of this theorem, we can easily determine ω_d where $d = 202 = 14^2 + 6$. Since $q_0 = a$ and $\ell_0 = 2a$, we get $q_0 = 14$ and $\ell_0 = 28$. On the other hand, we get $m = 1$ since $b = 4m + 2$. Then we get $\ell_1 = 4$ and $r = 2$ from $a = (2m+1)\ell_1 + r$ and $r < 2m + 1$. Hence $r_2 = 4$ is obtained. It follows from $4m = 2s + 2r\ell_2 - 2$ that $\ell_2 = 1$ and $s = 1$. Moreover, we get $r_3 = 7$ since $r_3 = 2s\ell_1 - \ell_2$. Since $c_3 = 4m + 2 + (r_3 - r_2)\ell_2$ we obtain $c_3 = 9$. By using $2a = c_3\ell_3 + r_3 + r_4$ and $r_4 < c_3$, we get $\ell_3 = 2$ and $r_4 = 3$. Hence ω_d can be determined as follows:

$$\omega_d = [14, \overline{4, 1, 2, 2, 1, 4, 28}]$$

Furthermore the fundamental unit of $\mathbb{Q}(\sqrt{202})$ can be easily determined as

$$\varepsilon_d = \frac{6282 + 442\sqrt{202}}{2}$$

since $A = 5$, $B = 14$, $C = 3$. Moreover it is easily seen that $D = 21$ and $E = 18$.

Theorem 3. *For a positive square-free integer d congruent to 3 modulo 4, we assume $k_d = 7$. Then, if b is congruent to 3 modulo 4, we get*

$$\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$$

for three positive integers ℓ_1, ℓ_2, ℓ_3 such that $\ell_i \geq 1$ $(i = 1, 2, 3)$, and then

$$(T_d, U_d) = (2[a(A^2 + B^2) + BC + A\ell_2], 2(A^2 + B^2))$$

and

$$d = A^2r^2 + 2rD + E$$

hold where A, B, C, D and E are determined uniquely as follows:

$$\begin{aligned} A =&\ell_1\ell_2 + 1\\ B =&\ell_1 + A\ell_3\\ C =&\ell_2\ell_3 + 1\\ D =&A\ell_1 s + \ell_2\\ E =&{\ell_1}^2 s^2 + 2s + 3. \end{aligned}$$

Moreover, r is an odd integer and s is a positive integer determined uniquely by

$$\begin{aligned} a =&Ar + \ell_1 s\\ &3(A^2 + B^2) - C^2 - {\ell_2}^2 = 2rB - 2s(A + B\ell_3). \end{aligned}$$

Proof. In the case of $b \equiv 3 \pmod 4$, it can be easily seen that a is an even integer since d is congruent to 3 modulo 4. We can put $b = 4m + 3$ for a non-negative integer m satisfying $0 \leq 4m < 2a - 2$. Since $q_0 = [\omega_d] = [\sqrt{d}] = a$ and $\omega_R = a + \sqrt{d}$, it follows from Lemma 3 that $r_0 = r_1 = 0$, $c_0 = 1$, $c_1 = 4m + 3$ and $\ell_0 = 2a$. Since $k_d = 7$, we get $\ell_1 = \ell_6$, $\ell_2 = \ell_5$ and $\ell_3 = \ell_4$ from Lemma 2. Then we have

$$\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$$

for three positive integers ℓ_1, ℓ_2, ℓ_3 such that $\ell_i \geq 1$ $(i = 1, 2, 3)$. From Lemma 2 we get

$$2a = (4m + 3)\ell_1 + r_2 \tag{15}$$

since $r_1 = 0$ and $c_1 = 4m+3$. From (15), we obtain $(4m+3)\ell_1 + r_2 \equiv 0(mod 2)$. So there exists a positive integer r such that $r_2 = 2r - 3\ell_1$. By substitution of r_2 in (15) we get $a = 2m\ell_1 + r$. Here r is an even integer, since a is even. It follows from Lemma 2 that $c_2 = 1 + r_2\ell_1$ and $2a = c_2\ell_2 + r_3 + r_2$. Thus, $2a = (1 + r_2\ell_1)\ell_2 + r_3 + r_2$ is obtained. Then

$$(4m + 3)\ell_1 = (1 + r_2\ell_1)\ell_2 + r_3 \tag{16}$$

holds from (15). Thus we get $\ell_2 + \ell_3 \equiv 0 \pmod{\ell_1}$. So there exists a positive integer t such that $r_3 = \ell_1 t - \ell_2$. By substitution of r_3 in (16), we get $4m = t + 2r\ell_2 - 3(\ell_1\ell_2 + 1)$. Thus if we put $A = \ell_1\ell_2 + 1$, then we get $t - 3A = 4m - 2r\ell_2$. Since $t - 3A$ is even, we can put $t - 3A = 2s$ for a positive integer s. Hence, $4m = 2s + 2r\ell_2$ is obtained. Therefore we get $a = Ar + \ell_1 s$ since $a = 2m\ell_1 + r$. On the other hand,

$$c_3 = 4m + 3 + (r_3 - r_2)\ell_2 \tag{17}$$

is obtained from Lemma 2. Thus we get from (17) $c_3 = At - {\ell_2}^2$ since $r_2 = 2r - 3\ell_1$ and $r_3 = \ell_1 t - \ell_2$. Moreover, from Lemma 2 we get $2a = c_3\ell_3 + r_3 + r_4$. Thus
$r_4 = (2r - 3\ell_1 - t\ell_3)A + \ell_2(\ell_2\ell_3 + 1)$ is obtained because of $2a = (1 + r_2\ell_1)\ell_2 + r_2 + r_3$ and $c_3 = At - {\ell_2}^2$.

Furthermore, $c_3 = At - \ell_2{}^2$ and $c_4 = (1 + r_2\ell_1) + (r_4 - r_3)\ell_3$ imply $At - \ell_2{}^2 = 1 + r_2\ell_1 + r_4\ell_3 - r_3\ell_3$ since $c_3 = c_4$. Thus,

$$At - \ell_2{}^2 = (1 + \ell_2\ell_3)^2 + 2r(\ell_1 + A\ell_3) - t\ell_3(\ell_1 + A\ell_3) - 3\ell_1(\ell_1 + A\ell_3)$$

is obtained since $r_2 = 2r - \ell_1$, $r_3 = \ell_1 t - \ell_2$ and $r_4 = (2r - 3\ell_1 - t\ell_3)A + \ell_2(\ell_2\ell_3 + 1)$. Thus, if we put $B = \ell_1 + A\ell_3$ and $C = \ell_2\ell_3 + 1$, we get $3(A^2 + B^2) - C^2 - \ell_2{}^2 = 2rB - 2s(A + B\ell_3)$ since $t - 3A = 2s$. If we assume that the integers r and s are not uniquely determined, we get $A^2 + B^2 = 0$ which is a contradiction. Therefore, the integers r and s are uniquely determined by $a = Ar + \ell_1 s$ and $3(A^2 + B^2) - C^2 - \ell_2{}^2 = 2rB - 2s(A + B\ell_3)$.

Now, since $\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$ implies $Q_5 = BC + A\ell_2$ and $Q_6 = A^2 + B^2$ by Lemma 1, we obtain $T_d = 2[a(A^2 + B^2) + BC + A\ell_2]$ and $U_d = 2(A^2 + B^2)$. Moreover, if we put $D = A\ell_1 s + \ell_2$ and $E = \ell_1{}^2 s^2 + 2s + 3$, then we get $d = A^2r^2 + 2rD + E$ since $b = 2s + 2r\ell_2 + 3$. Thus, the proof is completed. □

4. Conclusion

In this paper, some results are presented in order to determine the fundamental units of certain quadratic fields $\mathbb{Q}(\sqrt{d})$ with the period k_d of the continued fraction expansion of the quadratic irrational number ω_d is equal to 7. These results provide us a practical method in order to determine both the continued fraction expansion of the quadratic irrational number ω_d and the fundamental units of certain real quadratic fields. By using these results both the continued fraction expansion of the quadratic irrational number ω_d and the fundamental units of certain real quadratic fields can be rapidly determined without using long algorithms.

The similar results can be proved for all real quadratic fields with the period k_d of the continued fraction expansion of the quadratic irrational number ω_d is higher than 7.

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