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On Hybrid Caputo Fractional Differential Equations with Variable Moments of Impulse

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Abstract. In this paper existence and continuation results for hybrid Caputo fractional differential equations of order $q \in (0,1)$ with variable moments of impulse are established under the weakened hypothesis of C^q continuity.

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1. Introduction

The concept of a fractional derivative, as is well known, has its inception in a question posed during a communication between Leibnitz and L'Hospital. The five century old question has become a major area of research both in the realm of applications and in the theoretical set up. The potential it offers in both these branches has attracted the attention of both theoretical and applied scientists as well as engineers and other technologists. The major contributions in this field are given in [6, 8, 9, 11–14] and the references therein. In [6] Diethelm gave a simple example which naturally introduces the fractional derivative. We briefly introduce it here, so as to connect it later, to the problem considered in this paper.

Consider the stress $\sigma(t)$ and strain $\epsilon(t)$ of a viscous liquid. It is known that the Newton's law

$$\sigma(t) = \eta D^1 \epsilon(t), \tag{1}$$

describes the relation between stress and strain for a viscous liquid, where η is the viscosity of the material.

The Hooke's law states the stress-strain relationship for elastic solid and is given by

$$\sigma(t) = E D^0 \epsilon(t), \tag{2}$$

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where E is the modulus of elasticity of the material. Now it can be naturally concluded that the behaviour of viscoelastic material must have a behaviour that is modeled by an equation having derivative of order $k \in (0,1)$ that lies in betweeen the equations of (1) and (2) and is given by

$$\sigma(t) = \nu D^k \epsilon(t), \tag{3}$$

where ν is the constant of the material and $k \in (0,1)$. The relation (3) in a slightly different set up is called as Nutting's law. It has been observed in [9] that viscoelastic materials like polymers, some biological tissue etc. may follow the relation (3). The operator D^k is called as the fractional derivative and is described in Section 2.

It has been observed that the theory of ordinary differential equations is being systematically extended to the set up of fractional differential equations. Further an effort is being put to obtain better results by using fractional derivatives in place of ordinary derivatives, where there is nonlocality or memory involved.

It is known that many evolutionary processes experience a change of state abruptly. These undergo short term perturbations, where the time span is negligible with respect to the duration of the process. Thus, it is natural to assume that the perturbations act instantaneously and hence can be modelled as impulses. These perturbations or impulses can be considered as of two types. The moments of impulse can be decided in advance or can depend on the solution of the model described by the physical phenomenon. Some examples that can be modelled in this set up are biological phenomena involving thresholds, bursting rhythmic models in biology and medicine, optimal control in economics, to name a few.

In some of the afore mentioned models, the occurrence of a change of state or perturbation depends on the solution. Also, the constraints of any physical model can be considered as a barrier or a surface. If a solution of the model encounters the surface, it must be given an impulse or a perturbation to avoid it or to get out of it. Thus a mathematical model involving differential equation with variable moments of impulse is a system worth studying. This model exhibits many interesting phenomena which are discussed in [9].

The momentum that the research on fractional differential equation is gaining had prompted us to take up the study of fractional differential equations with variable moments of impulse, in this paper. We proceeded along the lines of the theory established in [9] and have constructed examples in the present set up. The fact that there are many physical phenomena that can be modeled using fractional derivatives had encouraged us to obtain conditions using fractional derivatives. The remaining part of the paper is organized as follows. In Section 2, we deal with the preliminaries of fractional differential equations. Adapting the description of solution of an evolutionary process in [9] we described the solution of an impulsive or a hybrid fractional differential equation in Section 3. An example illustrating the proposed system is given in Section 4. In Section 5 we deal with existence and continuation of solutions. Section 6 concludes the work done in the paper.

2. Preliminaries

In this section, we introduce notations, definitions, results and preliminary facts from [10], [4] that are required in the remainder of this paper.

Definition 1. *The Riemann-Liouville fractional integral of order q, where q is a positive real number, of a function x given on the interval* $[t_0, T]$, $t_0 \geq 0$ *is defined as*

$$D^{-q}x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} x(s) ds, \quad t_0 \leq t \leq T,$$

where Γ *is the Gamma function.*

Definition 2. *The Riemann-Liouville fractional derivative of order q, where q is a positive real number, of a function x given on the interval* $[t_0, T]$, $t_0 \geq 0$ *is defined as*

$$D^{q}x(t) = \frac{1}{\Gamma(p)} \frac{d^m}{dt^m} \left\{ \int_{t_0}^{t} (t-s)^{p-1} x(s) ds \right\}, \quad t_0 \leq t \leq T,$$

where $m - p = q$ *and m is the least positive integer greater than q so that* $0 < p \leq 1$.

Definition 3. *The Caputo's fractional derivative of order q, where q is a positive real number, of a function x given on the interval* $[t_0, T]$, $t_0 \geq 0$ *is defined as*

$${}^{c}D^{q}x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s) ds, \quad t_0 \leq t \leq T,$$

where n is a positive integer such that $n - 1 < q < n$.

In particular, the Caputo's fractional derivative of order q, where $0 < q < 1$ is defined as

$${}^{c}D^{q}x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x'(s) ds, \quad t_0 \leq t \leq T.$$

Definition 4. *A function u is said to be* C_p *continuous i.e.,* $u \in C_p\left([t_0, t_0 + a], \mathbb{R}\right)$ *if and only if* $u \in C((t_0, t_0 + a], \mathbb{R})$ *and* $(t - t_0)^p u(t) \in C([t_0, t_0 + a], \mathbb{R})$ *with* $p + q = 1$, $0 < q < 1$.

Definition 5. *A function u is said to be* C^q *continuous i.e.,* $u \in C^q\left([t_0, T], \mathbb{R}\right)$ *if and only if the Caputo derivative* ${}^{c}D^{q}u(t)$ *exists and satisfies*

$${}^{c}D^{q}u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds, \quad t_0 \leq t \leq T.$$

We observe that $u \in C^q\left([t_0, t_0 + a], \mathbb{R}\right)$, implies that u is continuous and differentiable. Next we state the following result from [10], which is used in Section 5.

Lemma 1. *$x(t) \in C^q([t_0, t_0 + a], \mathbb{R})$ is solution of the initial value problem*

$$^cD^q x = f(t,x), \quad x(t_0) = x_0, \ 0 < q < 1$$

if and only if it satisfies corresponding Volterra fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \quad t_0 \le t \le t_0 + a.$$

Finally, we state the following Lemma [1], which is used in Section 5 and Section 6 to prove our main results.

Lemma 2. *Let $0 < q < 1$. Consider the Caputo fractional differential equation*

$$^cD^q_{t_0} u(t) = g(t, u(t)), \ t \ge t_0,$$

where $g(t,u) \ge 0$ and $t_0 \in \mathbb{R}$. If the solutions exist and $u(t_0) \ge 0$, then they are nonnegative. Furthermore, if $g(t,u) = \lambda u$ for $\lambda \ge 0$, then the solutions are nondecreasing in t.

3. Hybrid Caputo Fractional Differential Equations

As observed in the introduction it is natural to expect viscoelastic materials to be modeled after fractional differential equations. Let the natural constraints that arise in the model be construed as a barrier or a threshold. When the solution of the model, come into contact with the barrier, some additional input or impulse must be given for the solution to get out of the barrier. This general behaviour can be understood through an evolutionary process of a physical phenomenon given below. We adapt the process given in [9] to the set up of fractional differential equations. Consider an evolutionary process or a physical phenomenon that exhibits behaviour which can be described by

(i) a Caputo fractional differential equation of order $q \in (0,1)$

$$^cD^q x = f(t,x), \tag{4}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}$ is an open set, and $\mathbb{R}$ is the space of real numbers and $\mathbb{R}_+$ is the nonnegative real line;

(ii) the sets $M(t), N(t) \subset \Omega$ for each $t \in \mathbb{R}_+$; and

(iii) the operator $A(t) : M(t) \to N(t)$ for each $t \in \mathbb{R}_+$.

Let $x(t) = x(t, t_0, x_0)$ be any solution of (4) starting at (t_0, x_0). The behaviour of the evolutionary process or the physical phenomenon can be described by the point $P_t = (t, x(t))$ starting at $P_{t_0} = (t_0, x_0)$, it moves along the curve $\{(t, x) : t \geq t_0, x = x(t)\}$ until the point P_t meets the set $M(t)$ at $t = t_1$. Then at the point $t = t_1$, the operator $A(t)$ transfers the point $P_{t_1} = (t_1, x(t_1)) \in M(t_1)$ to the position $P_{t_1^+} = (t_1, x_1^+) \in N(t_1)$ where $x_1^+ = A(t_1)x(t_1)$. Now the motion of the point P_t continues forward from $P_{t_1^+}$ along the the curve $\{(t, x) : t \geq t_1, x = x(t, t_1, x_1^+)\}$ as the solution of (4) with starting point (t_1, x_1^+) until it again meets the set $M(t)$ at $t = t_2$. This again yields, by the effect of operator $A(t)$, that the point $P_{t_2} = (t_2, x(t_2))$ is transferred to the position $P_{t_2^+} = (t_2, x_2^+)$ where $x_2^+ = A(t_2)x(t_2) \in N(t_2)$. From then on, the motion of the evolutionary process moves the point P_t forward following the equation ${}^cD^q x = f(t, x)$ till it touches $M(t)$ at $t = t_3$. This process can be continued forward as long as the solution of (4) exists.

The set up given by (i), (ii), (iii) characterizes the considered evolutionary process as a hybrid Caputo fractional differential system. The motion of the point P_t is a curve represents the solution curve of the hybrid Caputo fractional differential equation (HCFDE for short).

Thus a solution of a hybrid Caputo fractional differential equation may exhibit a variety of behaviour as given below.

(i) The solution may be a continuous function, if the integral curve does not intersect $M(t)$ or hits it at the fixed points of the operator $A(t)$.

(ii) The solution can be a piecewise continuous function having a finite number of discontinuities of first kind which are are not fixed points of the operator $A(t)$.

(iii) The solution may also have a countable number of discontinuities of first kind. Then the solution is a piecewise continuous function with a countable number of discontinuities. Clearly all these points are not fixed points of the operator $A(t)$. The question of uncountable number of discontinuities does not arise as we have assumed $x(t)$ is a solution of (4).

The moments of time at which the integral curve hits the set $M(t)$ are called as the moments of the impulse and are denoted by $t = t_k$. Following the standard notation in impulsive differential equations [9], we assume that the solution of the HCFDE is left continuous at $t = t_k$, $k = 1, 2, 3, \ldots$ that is $x(t_k^-) = \lim_{h \to 0^+} x(t_k - h) = x(t_k)$.

The generality in the above set up, gives rise to two types of HCFDE known as HCFDE with fixed moments of impulse and the other by HCFDE with variable moments of impulse. The former is quite popular and runs parallel to ordinary differential equations with fixed moments of impulse. There are many papers on fractional differential equations with fixed moments of impulse, of which some are [2, 3, 5, 7].

The latter gives rise to very interesting phenomenon and exhibits complex behaviour. We concentrate on the latter types of equations in this paper.

4. HCFDE with Variable Moments of Impulse

Consider a sequence of surfaces $\{S_k\}$ given by $S_k : t = \tau_k(x)$, $k = 1, 2, 3, \ldots$, $\tau_k : \mathbb{R} \to \mathbb{R}$ such that $\tau_k(x) < \tau_{k+1}(x)$ and $\lim_{k\to\infty} \tau_k(x) = \infty$. Then the HCFDE with variable moments of impulse is given by

$$\begin{aligned} {}^cD^q x =& f(t,x), \quad t \neq \tau_k(x), \\ x(t^+) =& x(t) + I_k(x(t)), \quad t = \tau_k(x), \end{aligned} \tag{5}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}$ is an open set, $\tau_k \in C[\Omega, (0, \infty)]$, $k = 1, 2, 3, \ldots,$
$I_k(x(t)) = \Delta(x(t)) = x(t^+) - x(t^-)$, and $I_k \in C[\Omega, \mathbb{R}]$.

In this case, the moments of the impulsive effect for the system (5) depend on the solutions satisfying $t_k = \tau_k(x(t_k))$, for each k. Thus, the solutions starting at different points will have different points of discontinuity. Also a solution may hit the same surface $t = \tau_k(x)$ several times and we shall call such a behaviour as "*pulse phenomena*". In addition, different solutions may coincide after some time and behaves as a single solution there after. This phenomena is called "*confluence*". In the following example, the different solutions that arise in this context are described and the graphs are drawn.

Consider the hybrid Caputo fractional differential equation with variable moments of impulse

$$\begin{aligned} {}^cD^q x =& 0, t \neq \tau_k(x), t \geq 0, \\ x(t^+) =& x^2 sgn\, x, \quad t = \tau_k(x), k = 1, 2, 3, \ldots \end{aligned} \tag{6}$$

where $\tau_k(x) = x^2 + 20(k-1)$ for $|x| < 6$ describe the surfaces $S_k : t = \tau_k(x)$. Here $I_k(x) = \Delta(x) = x^2 sgn\, x - x$.

If ${}^cD^q x = 0$ then $x(t) = x_0$, a constant.

Case (1) The solutions $x(t)$ with initial condition $x(0) = x_0, |x_0| \geq 6$ are free from impulsive effect since they do not intersect the surfaces S_k for any k. For example, consider the solution $x(t)$ of (6) starting at the point $(0, 8.5)$. It does not hit any surface, see Figure 1a.

Case (2) The solutions starting at the points $(0, x_0)$ where $1 < x_0 < 6$ or $-6 < x_0 < -1$, undergo impulsive effect a finite number of times. For example, consider the solution $x(t)$ with initial condition $x(0) = \sqrt{2}$. The point $p_t = (t, x(t))$ starts its motion from $(t_0, x_0) = (0, \sqrt{2})$ and moves along the curve

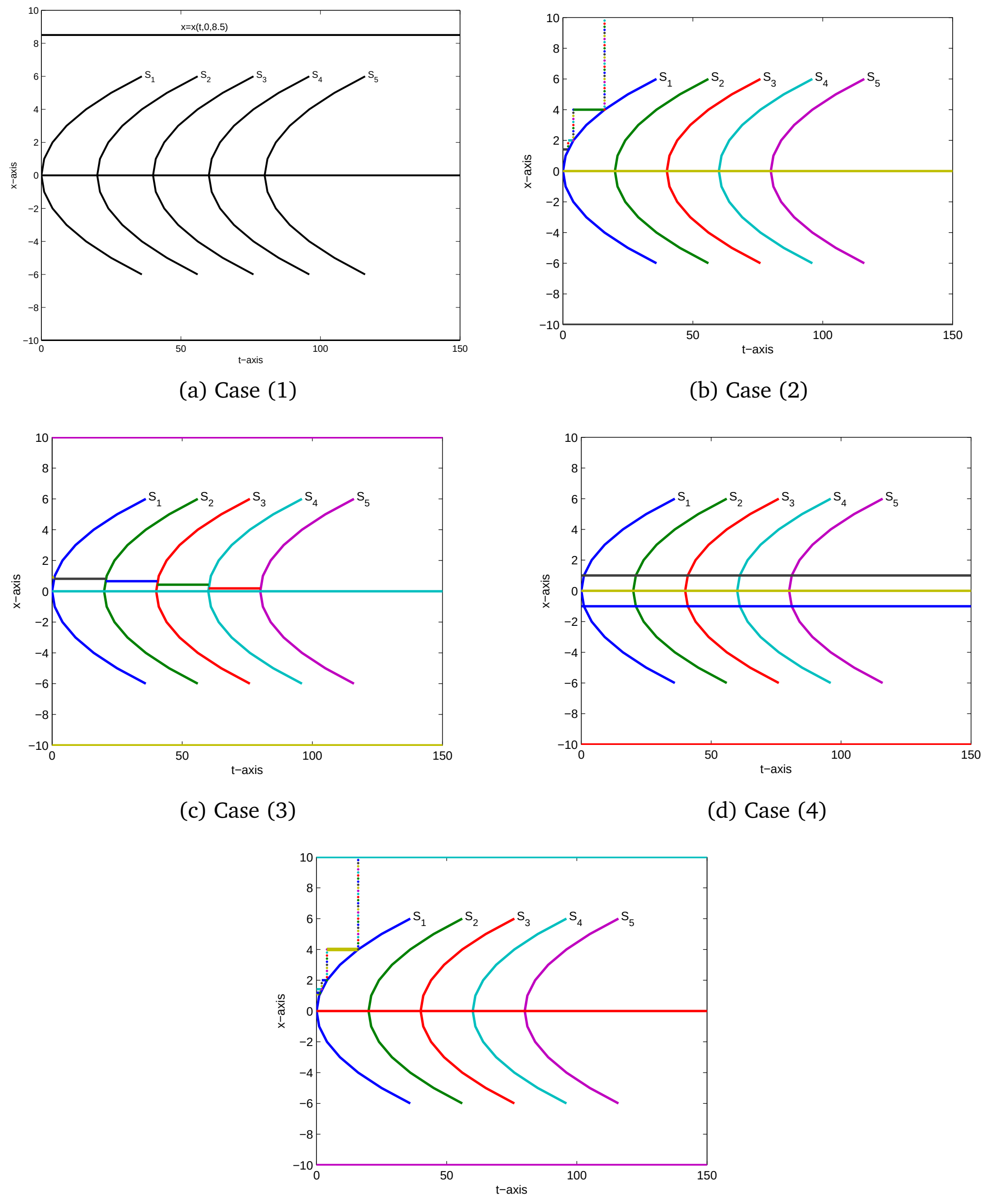
$$\{(t,x) : t \geq 0, x = x(t)\} = \{(t,x) t \geq 0, x = \sqrt{2}\}$$

until the time $t_1 = 2 > t_0 = 0$ at which the point P_t meets the surface $S_1 : t = x^2$. The point $p_{t_1} = (t_1, x_1) = (2, \sqrt{2})$ which lies on the surface S_1 is transferred to the point $P_{t_1^+} = (t_1, x_1^+) = (2, 2)$ where $x_1^+ = x_1^2 sgn\, x_1 = 2$. Then the point P_t continues to move further along the curve with $x(t) = x(t; t_1, x_1^+)$ as the solution of (6) starting at (t_1, x_1^+) until it hits the same surface S_1 at the next moment $t_2 = 4 > t_1 = 2$. Then once again the point $P_{t_2} = (t_2, x_2) = (4,2)$ is transferred to the point $P_{t_2^+} = (t_2, x_2^+) = (4,4)$ where $x_2^+ = x_2^2 \, sgn \, x_2$. As before the point P_t continuous to move forward with $x(t) = x(t, t_2, x_2^+) = x(t,4,4)$ as the solution of (6) starting at $(t_2, x_2^+) = (4,4)$ until it hits the same surface S_1 at the moment $t_3 = 16 > t_2 = 4$. Then the point $(t_3, x_3) = (16,4)$ is transferred to the point $P_{t_3^+} = (t_3, x_3^+) = (16,16)$, where $x_3^+ = x_3^2 \, sgn \, x_3$ and it does not encounter any surface, beyond time $t_3 = 16$. In this case the solution $x(t)$ is a piecewise continuous function having finite number of discontinuities of the first kind since the integral curve meets the surfaces at a finite number of times which are not the fixed points of the operator $A(t)$ given by $A(t)x = x^2 \, sgn \, x$. In this case the solution hit the same surface S_1 three times exhibiting pulse phenomenon, see Figure 1b.

Case (3) The solutions $x(t)$ starting at the points $(0, x_0), 0 < x_0 < 1$, meets the surfaces S_k at an infinite number of times t_k and we have $t_k \to \infty$ as $k \to \infty$ as well as $\lim_{k\to\infty} x(t_k) = 0$. For example, let us take $x_0 = 0.9$. The solution $x(t)$ begins its motion at $(0, 0.9)$ and continuous to move along the curve $x = 0.9$ until it hits the surface $S_1 : t = x^2$ at $(0.81, 0.9)$. This point $(0.81, 0.9)$ is transferred to $(0.81, 0.81)$. Then the solution starts at $(0.81, 0.81)$ and continuous to move along the line $x = 0.81$ until it hits the surface $S_2 : t = x^2 + 20$ at $(20.6561, 0.81)$. It is then transferred to $(20.6561, 0.6561)$. The solution then starts at $(20.6561, 0.6561)$ and continuous to move until it hits the surface $S_3 : t = x^2 + 40$ at $(40.43047, 0.6561)$ and so on. As $k \to \infty$ we have $\lim_{k\to\infty} x(t_k) = 0$. In this case the solution undergo an impulsive effect an infinite number of times, see Figure 1c.

Case (4) The solutions starting at $(0,0), (0,1)$ and $(0,-1)$ hit the surface S_k at times t_k, $(k = 0,1,2,3,\dots)$ which are the fixed points of the operator $A(t)\,x = x^2 \, sgn \, x$, and there is no impulsive effect, see Figure 1d.

Case (5) The solutions starting at $\left(0, 2^{\frac{1}{8}}\right)$, $\left(0, 2^{\frac{1}{4}}\right)$, $\left(0, 2^{\frac{1}{2}}\right)$ unite for $t \geq \sqrt{2}$ and thus exhibit the phenomenon of confluence, see Figure 1e.



(a) Case (1)

(b) Case (2)

(c) Case (3)

(d) Case (4)

(e) Case (5)

Figure 1: Solutions to (6)

5. Existence and Continuation of Solutions

The example in Section 3, points to the rich potential that Caputo fractional differential equation with variable moments of impulse offers. The first question that one comes across is to discuss the meaning of a solution of this equation and obtain an existence result. In this section we answer this question. We define the solution of Caputo fractional differential equation with variable moments of impulse and proceed to prove an existence result.

Consider an open set $\Omega \subset \mathbb{R}$ and set $D = \mathbb{R}_+ \times \Omega$. Suppose that for each $k = 1, 2, 3, \ldots$, $\tau_k \in C[\Omega, (0, \infty)]$, $\tau_k(x) < \tau_{k+1}(x)$ and $\lim_{k\to\infty} \tau_k(x) = \infty$ for $x \in \Omega$. Suppose that k varies from 1 to ∞. Also assume that $S_k : t = \tau_k(x)$ are the surfaces.
Consider the initial value problem (IVP) for the hybrid Caputo fractional differential equation with variable moments of impulse

$$\left.\begin{array}{rcl} {}^cD^q x &=& f(t,x), \quad t \neq \tau_k(x) \\ x(t^+) &=& x(t) + I_k(x(t)), \quad t = \tau_k(x) \\ x(t_0^+) &=& x_0, \quad t_0 \geq 0 \end{array}\right\} \tag{7}$$

where $f : D \to \mathbb{R}$ and $I_k : \Omega \to \mathbb{R}$. A function $x : [t_0, t_0 + a) \to \mathbb{R}$, $t_0 \geq 0$, $a > 0$ is said to be a solution of (7) if

(i) $x(t_0^+) = x_0$ and $(t, x(t)) \in D$ for $t \in [t_0, t_0 + a)$,

(ii) $x(t) \in C^q([t_0, t_0 + a), \mathbb{R})$, ${}^cD^q x(t)$ is continuous, and $x(t)$ satisfies ${}^cD^q x = f(t, x)$ for $t \in [t_0, t_0 + a)$ and $t \neq \tau_k(x(t))$,

(iii) if $t \in [t_0, t_0 + a)$ and $t = \tau_k(x(t))$, then $x(t^+) = x(t) + I_k(x(t))$, and at such t's we always assume that $x(t)$ is left continuous and $s \neq \tau_j(x(s))$ for any j, $t < s < t + \delta$, for some $\delta > 0$.

Whenever $t_0 \neq \tau_k(x_0)$ for any k, we mean the initial condition $x(t_0^+) = x_0$ in the usual sense, that is, $x(t_0) = x_0$. If $t_0 = \tau_k(x_0)$ for some k then $x(t_0^+) = x_0$, which, in general, is natural for the system (7), since (t_0, x_0) may be such that $t_0 = \tau_k(x_0)$ Unlike ordinary fractional differential equations, the system (7) may not possess any solution at all, even if, f is continuous (or continuously differentiable) since the only solution $x(t)$ of the problem ${}^cD^q x = f(t, x), x(t_0) = x_0$, may totally lie on a surface and hence by the definition, we conclude that the Caputo fractional differential equation with variable moments of impulse does not have any solution for all $t \in [t_0, T]$.

Example 1. *Consider the following IVP of Caputo fractional differential equation with $q = \frac{1}{2}$*

$$\begin{aligned} {}^cD^{\frac{1}{2}} x =& 1, t \neq \tau_k(x), \\ \Delta x =& I_k(x) = \frac{\pi}{4}(x-1)^2 + 1 - x, \quad t = \tau_k(x) \\ x(1^+) =& 1 \textit{ where } S_k : \tau_k(x) = \frac{\pi}{4}(x-1)^2 + k, \quad k = 1, 2, \ldots \end{aligned} \tag{8}$$

There is no solution to the above system 8 passing though $(1,1)$*, since*
${}^{c}D^{\frac{1}{2}}x = 1 \Leftrightarrow x(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}\frac{f(s,x(s))}{(t-s)^{1-q}}ds$. *Now*

$$
\begin{aligned}
x(t) =& x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}\frac{f(s,x(s))}{(t-s)^{1-q}}ds \\
=& 1 + \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{1}^{t}\frac{1}{(t-s)^{1-\left(\frac{1}{2}\right)}}ds \\
=& 1 + \frac{1}{\sqrt{\pi}}\int_{1}^{t}(t-s)^{-\left(\frac{1}{2}\right)}ds \\
=& 1 + \frac{1}{\sqrt{\pi}}\left[(-1)\frac{(t-s)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_{s=1}^{s=t} \\
=& 1 + \frac{1}{\sqrt{\pi}}\left(0 + \frac{(t-1)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right) \\
=& 1 + \frac{2}{\sqrt{\pi}}\sqrt{t-1} \\
\Rightarrow \quad x(t) =& 1 + \frac{2}{\sqrt{\pi}}\sqrt{t-1},
\end{aligned}
$$

which lies entirely on the surface S_1.

The above example clearly shows that we need to obtain some conditions that will guarantee that the solution will exist after it hits a surface or a barrier. As the solution satisfies the Caputo fractional differential equation, it is natural that the conditions must be in terms of the fractional derivatives. This is done in the following theorem, and the criteria obtained are not only new but are very interesting. Hence we need some extra conditions on τ_k and f, τ_k or f besides continuity in order to establish any general existence theory for the system (7). We now proceed to state and prove a result on existence of a solution for the considered IVP. The proof of the theorem is analogous to the proof of the corresponding Theorem 1.2.1 in [9].

Theorem 1. *Assume that*

(i) $f : D \to \mathbb{R}$ *is continuous at* $t \neq \tau_k(x)$, $k = 1,2,\ldots,$.

(ii) for each $(t,x) \in D$ *there exists a function* $\ell \in L^1_{loc}$ *such that* $\mid f(s,y) \mid \leq \ell(s)$ *in a neighbourhood of (t,x).*

(iii) $t_1 = \tau_k(x_1)$ *for any* $k \geq 1$ *implies that there exists* $\delta > 0$ *such that* $t \neq \tau_k(x)$ *for any* (t,x) *with* $0 < t - t_1 < \delta$ *and* $\mid x - x_1 \mid < \delta$.

Then for each $(t_0, x_0) \in D$, there exists a solution $x : [t_0, t_0 + \alpha) \to \mathbb{R}$ of the initial value problem (7) for some $\alpha > 0$.

Proof. If $t_0 \neq \tau_k(x_0)$ for all $k \geq 1$ then there exists $\delta_1 > 0$ such that $s \neq \tau_i(x(s))$ for all $i \geq 1$, $t_0 < s < t_0 + \delta_1$. The continuity of f imply the existence of a local solution $x(t)$ of ${}^cD^q x = f(t,x)$ and $x(t_0) = x_0$. Hence $x(t)$ is a local solution of the system (7).

If $t_0 = \tau_k(x_0)$ for some $k \geq 1$, then $x(t_0^+) = x(t_0) + I_k(x(t_0))$. The continuity of f and the condition (ii) imply the existence of a local solution $x(t)$ of ${}^cD^q x = f(t,x)$ and $x(t_0^+) = x(t_0) + I_k(x(t_0))$. Since $\tau_i(x) < \tau_j(x)$ for $i < j$ and, $t_0 = \tau_k(x_0)$ we have $t \neq \tau_j(x(t))$ for $j \neq k$ and t sufficiently close to t_0. Since $t_0 = \tau_k(x_0)$, by condition (iii) there exists $\delta > 0$ such that $t \neq \tau_k(x)$ for all (t,x) with $0 < t - t_0 < \delta$ and $| x - x_0 | < \delta$. Therefore, $s \neq \tau_i(x(s))$ for all i, $t_0 < s < t_0 + \delta$, for some $\delta > 0$. Hence $x(t)$ is a local solution of the system (7). □

Remark 1. *The condition (iii) in Theorem 1 is possible for only irregular functions $\tau_k(x)$ since the theory of implicit functions implies that if τ_k is differentiable at x_0 and $\tau_k'(x_0) \neq 0$, then the condition (iii) can never hold.*

We now proceed to state and prove a result on existence of a solution for the considered IVP with some regularity conditions on $\tau_k(x)$. The condition on $\tau_k(x)$ is given in terms of fractional derivative. This is natural as the solution follows the fractional differential equation model. This result is new and has been developed for hybrid Caputo fractional differential equation with variable moments of impulse.

Theorem 2. *Assume that*

(i) $f : D \to \mathbb{R}$ *is continuous.*

(ii) ${}^cD^q \tau_k(x)$ *exists,* $\tau_k : \Omega \to (0, \infty)$ *are differentiable and linear surfaces for all* $k \geq 1$.

(iii) *if* $t_1 = \tau_k(x_1)$ *for some* $(t_1, x_1) \in D$ *and* $k \geq 1$, *then there exists* $\delta > 0$ *such that* $\frac{\partial \tau_k(x)}{\partial x} . f(t,x) \neq \frac{(t-t_1)^{(1-q)}}{\Gamma(2-q)}$ *for* $(t,x) \in D$ *with* $| x - x_1 | < \delta$ *and* $0 < t - t_1 < \delta$.

Then for each $(t_0, x_0) \in D$, there exists a solution $x : [t_0, t_0 + \alpha) \to \mathbb{R}$ of the system (7) for some $\alpha > 0$.

Proof. If $t_0 \neq \tau_k(x_0)$ for all $k \geq 1$ then there exists $\delta_1 > 0$ such that $s \neq \tau_i(x(s))$ for all i, $t_0 < s < t_0 + \delta_1$. The continuity of f imply the existence of a local solution $x(t)$ of ${}^cD^q x = f(t,x)$ and $x(t_0) = x_0$. Hence $x(t)$ is a local solution of the system (7).

If $t_0 = \tau_k(x_0)$ for some $k \geq 1$, then $x(t_0^+) = x(t_0) + I_k(x(t_0))$. The continuity of f imply the existence of a local solution $x(t)$ of ${}^CD^q x = f(t,x)$ and $x(t_o^+) = x(t_0) + I_k(x(t_0))$. Set $\sigma(t) = t - \tau_k(x(t)$, then $\sigma(t_0) = 0$, since $t_0 = \tau_k(x_0)$. Then by using the fact that $\tau_k(x)$ are linear surfaces, and the hypothesis in a small right neighborhood of t_0 we obtain

$$ {}^cD^q \sigma(t) = {}^cD^q [t - \tau_k(x(t)] $$

$$
\begin{aligned}
&= {}^cD^q(t) - {}^cD^q[\tau_k(x(t))] \\
&= \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}ds - \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}\frac{d}{ds}[\tau_k(x(s)]ds \\
&= \frac{1}{\Gamma(1-q)}\left[-\frac{(t-s)^{1-q}}{1-q}\right]_{t_0}^{t} - \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}\frac{\partial}{\partial x}[\tau_k(x)]x'(s)ds \\
&= \frac{1}{\Gamma(1-q)}\frac{(t-t_0)^{1-q}}{1-q} - \frac{\partial}{\partial x}[\tau_k(x)]\,{}^cD^q x \\
&= \frac{(t-t_0)^{1-q}}{\Gamma(2-q)} - \frac{\partial}{\partial x}[\tau_k(x)]f(t,x) \neq 0
\end{aligned}
$$

Since ${}^cD^q\sigma \neq 0$ in a small right neighborhood of t_0 and $\sigma(t_0) = 0$, we have by Lemma 2, $\sigma(t)$ is either strictly increasing or decreasing in that neighbourhood and therefore, $t \neq \tau_k(x(t))$ for $0 < t - t_0 < \delta_2$ for some $\delta_2 > 0$.

Since $\tau_i(x) < \tau_j(x)$ for $i < j$ and $t_0 = \tau_k(x_0)$ we have $t \neq \tau_j(x(t))$ for $j \neq k$ and t sufficiently close to t_0. Therefore, $s \neq \tau_i(x(s))$ for any i, $t_0 < s < t_0 + \delta$, for some $\delta > 0$. Hence $x(t)$ is a local solution of the system (7). □

Regarding the initial value problem (7) we have the following two cases:

(i) If $t_0 \neq \tau_k(x_0)$, for all $k \geq 1$, then a solution of (7) is understood in the classical sense;

(ii) If $t_0 = \tau_k(x_0)$, for some $k \geq 1$, then a solution of (7) is understood in some extended sense depending on the smoothness of f.

A solution $x(t, t_0, x_0)$ of (7) existing on some interval $[t_0, t_0 + a)$ and experiencing impulses at the points $\{t_i\}$, $t_0 < t_i < t_0 + a$, $t_i < t_j$ for $i < j$, is described as follows:

$$
x(t,t_0,x_0) = \begin{cases} x(t,t_0,x_0), & t_0 \leq t \leq t_1, \\ x(t,t_1,x_1^+), & t_1 < t \leq t_2, \\ \cdots & \cdots \\ \cdots & \cdots \\ x(t,t_i,x_i^+), & t_i < t \leq t_{i+1}, \\ \cdots & \cdots \\ \cdots & \cdots \end{cases}.
$$

where $x_i^+ = x_i + I_k(x_i)$ and $x_i = x(t_i)$. Consequently, even when $t_0 \neq \tau_k(x_0)$, for any $k \geq 1$, it is possible that for some i, (t_i, x_i^+) lies on a surface S_j. In that case, that part of the solution $x(t, t_0, x_0)$ on the interval $(t_i,\ t_{i+1}]$ consists of $x(t, t_i, x_i^+)$ which is a solution of (7) on $[t_i,\ t_{i+1}]$ in the extended sense. Given a solution $x(t)$ of (7), defined on $[t_0, t_0 + a)$ with $a > 0$, we say that a solution $y(t)$ of (7) is a proper continuation to the right of $x(t)$ if $y(t)$ is defined on $[t_0, t_0 + b)$ for some $b > a$ and $x(t) = y(t)$ for $t \in [t_0, t_0 + a)$. The interval $[t_0, t_0 + a)$ is called the maximal interval of existence of a solution $x(t)$ of (7), if $x(t)$ is well defined on

$[t_0, t_0 + a)$ and it does not have any proper continuation to the right. If $x(t)$ is a solution of the system (7) with maximal interval of existence $[t_0, t_0 + a)$ and if $a < \infty$, then either $x(t)$ approaches the boundary of Ω or $|x(t)|$ becomes unbounded as $t \to (t_0 + a)-$.

Definition 6 (Regular and Irregular Points). *A point (t_1, x_1) is said to be a regular point if $t_1 \neq \tau_k(x(t_1))$ for all $k \geq 1$, otherwise it is said to be an irregular point.*

We now proceed to state a result on continuation of solutions of the Caputo fractional differential equations with variable moments of impulse. The proof is parallel to the Theorem 1.2.3 in [9] and hence omitted.

Theorem 3. *Assume that*

(i) $f : D \to \mathbb{R}$ *is continuous.*

(ii) $I_k \in C[\mathbb{R}, \mathbb{R}]$, $\tau_k \in C[\mathbb{R}, \mathbb{R}_+]$ *for all* $k \geq 1$.

If $x(t)$ is any solution of the system (7) with a finite $[t_0, b)$ as its maximal interval of existence, with one of the following three conditions is satisfied,

a) If $t_1 = \tau_k(x_1)$ for some $k \geq 1$ then there exists $\delta > 0$ such that $t \neq \tau_k(x)$ for all (t, x) with $0 < t - t_1 < \delta$ and $|x - x_1| < \delta$.

b) If $t_1 = \tau_k(x_1)$ for some $k \geq 1$ then $t_1 \neq \tau_j(x_1 + I_k(x_1))$ for all $j \geq 1$.

c) If $t_1 = \tau_k(x_1)$ for some $k \geq 1$, ${}^cD^q\tau_k(x)$ exists, $\tau_i \in C^1[\mathbb{R}, \mathbb{R}_+]$ and $\tau_i(x)$ are linear surfaces for all $i \geq 1$ then $t_1 = \tau_j(x_1 + I_k(x_1))$ for some $j \geq 1$ and $\frac{\partial \tau_j(x)}{\partial x}.f(t, x) \neq \frac{(t-t_1)^{(1-q)}}{\Gamma(2-q)}$ at (t_1, x_1^+) where $x_1^+ = x_1 + I_k(x_1)$.

Then $\lim_{t \to b-} |x(t)| = \infty$.

6. Conclusion

In this paper we have introduced hybrid Caputo fractional differential equations of order $q \in (0, 1)$ with variable moments of impulse and have shown by examples the potential it has for further work. We have studied existence and continuation of solutions for initial value problems in this set up.

References

[1] S.K. Choi and N. Koo. *Monotone property and stability of solutions of fractional differential equations,* Non linear Analysis 74 (2011) 6530–6536.

[2] E.A. Dads, M. Benchohra, and S. Hamani. *Impulsive Fractional Differential Inclusions Involving the Caputo Fractional Derivative*, Fractional Calculus & Applied Analysis, Volume 12, Number 1 (2009).

[3] J.V. Devi and M.K. Sastry. *Monotone iterative technique for hybrid Caputo fractional differential equations*, DCDIS Series A: Mathematical Analysis, Volume 19, Number 3 (2012), 397-411.

[4] J.V. Devi, F.A. McRae, and Z. Drici. *Variational Lyapunov method for fractional differential equations*, Computers and Mathematics with Applications (2012), doi:10.1016/j.camwa.2012.01.070.

[5] J.V. Devi and V. Radhika. *Quasilinearization for hybrid Caputo fractional differential equations*, Dynamics of COntinuous, Discrete, and Impulsive Systems, Volume 19. 745–756. 2012.

[6] K. Diethelm. *The Analysis of Fractional Differential Equations, An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, New York,1993.

[7] Z. Drici, F.A. McRae, and J.V. Devi. *On the existence and stability of solutions of hybrid Caputo fractional differential equations*, Dynamics of Continuous, Discrete, and Impulsive Systems Series A: Mathematical Analysis, Volume 19. 501–512. 2012.

[8] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo. *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science, B.V., Amsterdam. 2006.

[9] V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov. *Theory of Impulsive Differential Equations*, Words Scientific, Singapore, 1989.

[10] V. Lakshmikantham, S. Leela, and J.V. Devi. *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers Ltd, 2009.

[11] K.S. Miller and B. Ross. *An introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, 1993.

[12] K.B. Oldham and J. Spanier. *The fractional calculus,* Academic Press, New York - London, 1974.

[13] I. Podlubny. *Fractional Differential Equations*, Academic Press, San Diego, 1999

[14] R. Herrmann. *Fractional calculus*, Gigahedron, Germany.