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Intersections of Rational Parametrized Plane Curves

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Abstract. In this paper, we introduce and compare three different methods of computing the intersections of rational parametrized plane curves. The common approach of these methods is to apply the μ-basis of the plane curves, and avoid of computing the implicit equations of the curves, which increase the computation efficiency.

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1. Introduction

Through this paper, we shall consider two rational plane curves $\mathscr{C}_1$ and $\mathscr{C}_2$ in the complex projective two-space given as the image of generic one-to-one rational parametrizations:

$$\mathbf{F}(s,t) = (f_0(s,t),\ f_1(s,t),\ f_2(s,t)),\ (s,t) \neq (0,0), \tag{1}$$

and

$$\mathbf{G}(u,v) = (g_0(u,v),\ g_1(u,v),\ g_2(u,v)),\ (u,v) \neq (0,0), \tag{2}$$

where f_0, f_1, f_2 (respectively, g_0, g_1, g_2) are linearly independent homogeneous polynomials of the same degree $d \geq 2$ (respectively, $d' \geq 2$), and $\gcd(f_0, f_1, f_2) = 1$ (respectively, $\gcd(g_0, g_1, g_2) = 1$).

The implicit equation of a parametric curve is a polynomial f in the polynomial ring $\mathbb{C}[x_0, x_1, x_2]$ such that $f(a_0, a_1, a_2) = 0$ whenever $[a_0, a_1, a_2]$ is a point on the parametrized curve, and f is irreducible. Without loss of generality, we let $F(x,y,z) = 0$ and $G(x,y,z) = 0$ be the implicit equation of the parametrized curves $\mathbf{F}(s,t)$ and $\mathbf{G}(u,v)$ respectively.

It is known that the intersection number of $\mathscr{C}_1$ and $\mathscr{C}_2$ is dd' counting the intersection multiplicity. In general, one can obtain this information by computing the implicit equation $G(x,y,z) = 0$ of the curve $\mathscr{C}_2$, then solve the equation $G(f_0(s,t), f_1(s,t), f_2(s,t)) = 0$ for (s,t).

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(Or computing the implicit equation $F(x,y,z)=0$ of the curve $\mathscr{C}_1$, then solve the equation $F(g_0(u,v),g_1(u,v),g_2(u,v))=0$ for (u,v).)

The aim of this paper is to find the intersection of the curve $\mathscr{C}_1$ and the curve $\mathscr{C}_2$ only use the parametrizations of the curves without computing the implicit equation of the curves. We will accomplish this by using the μ-basis of the syzygies of the curve.

Moving lines and moving planes (syzygies) were introduced into Computer Aided Geometric Design by Sederberg, Cox and their collaborators in order to develop robust, efficient algorithms for implicitizing rational curves and surfaces [5, 7–9]. Their success motivated people to develop fast algorithms for computing special bases, called μ-bases, for moving lines and moving planes [1–3, 10].

We begin in Section 2 with a brief review of moving lines and μ-basis of the moving lines or rational parametrized plane curves. In Sections 3, we describe three different different methods of computing the intersections of two parametric curves. For each method, we first provide the theoretical background behind the algorithm, then we give the specific algorithm, and then we illustrate the algorithm via an example. In Section 4, we provide a brief summary of the paper, and a open questions for future study.

2. Moving Lines

A *moving line* is a family of lines with each pair of parameters (s,t) corresponding to a line:

$$L(x_0,x_1,x_2;s,t)=\sum_{i=0}^{2}A_i(s,t)x_i=A_0x_0+A_1x_1+A_2x_2, \tag{3}$$

where A_0,A_1,A_2 are homogeneous polynomials in s,t of the same degree. For simplicity, sometimes we write a moving plane as $\mathbf{L}=(A_0,A_1,A_2)$. The moving plane (3) follows the parametrization (1) if

$$\mathbf{L}\cdot\mathbf{F}=\sum_{i=0}^{2}A_i(s,t)f_i(s,t)=A_0f_0+A_1f_1+A_2f_2\equiv 0, \tag{4}$$

that is, if the point on the curve at the parameter (s,t) lies on the plane at the parameter (s,t).

Definition 1. *Two moving lines* $\mathbf{p}(s,t),\mathbf{q}(s,t)$ *are called a μ-basis of the rational plane curve* $\mathbf{F}(s,t)$ *if* $\mathbf{p},\mathbf{q}$ *are moving lines that follow* $\mathbf{F}(s,t)$ *and satisfy the following two conditions:*

1. $[\mathbf{p},\mathbf{q}]=\kappa\mathbf{F}(s,t)$,

2. $\deg(\mathbf{p})+\deg(\mathbf{q}))=\deg(\mathbf{F})$,

where κ is some nonzero constant and $[\mathbf{p},\mathbf{q}]$ *is the outer product of* $\mathbf{p},\mathbf{q}$.

The notation of a μ-basis for rational space curves can be generalized in an obvious way to rational curves of arbitrary dimension. The existence of a μ-basis for a rational curve in any dimension follows directly from the Hilbert-Burch Theorem [Theorem 20.15, 6]. In particular, the proof of the existence of a μ-basis for a rational curve in an affine n-space is given in [Exercise 17, Page 286, 4]. An alternative existence proof as well as a simple algorithm to compute a μ-basis based solely on Gaussian elimination is presented in [10].

The μ-basis elements $\mathbf{p}, \mathbf{q}$ for a rational plane curve are not unique. But the degrees of the μ-basis elements $\mu_1 = \deg(\mathbf{p})$, $\mu_2 = \deg(\mathbf{q})$ for a rational plane curve are unique. Song-Goldman [10] proved this result and the following proposition for rational space curves. For the convenience of the reader, we will focus our attention to plane curves, and below is the statement of their result for plane curves.

Proposition 1. *Suppose* $\mathbf{p}(s,t)$, $\mathbf{q}(s,t)$ *are a μ-basis of degrees* μ_1, μ_2 *for the rational plane curve* $\mathbf{F}(s,t)$ *and let* $\mathbf{L}(s,t)$ *be a moving curve of degree m that follows the curve. Then there exist polynomials* $\alpha(s,t)$, $\beta(s,t)$ *such that* $\mathbf{L} = \alpha\mathbf{p} + \beta\mathbf{q}$, *where* $\deg(\alpha) = m - \mu_1$, $\deg(\beta) = m - \mu_2$.

3. Comparison of Computational Methods

In this section, we will introduce three different methods, GCD method, Resultant matrix method, and Smith normal form method, to compute the intersection of two parametrized curves $\mathscr{C}_1$ and $\mathscr{C}_2$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are given as the image of generic one-to-one rational parametrizations:

$$\mathbf{F}(s,t) = (f_0(s,t),\ f_1(s,t),\ f_2(s,t)),\ (s,t) \neq (0,0),$$

and

$$\mathbf{G}(u,v) = (g_0(u,v),\ g_1(u,v),\ g_2(u,v)),\ (u,v) \neq (0,0),$$

where f_0, f_1, f_2 (respectively, g_0, g_1, g_2) are linearly independent homogeneous polynomials of the same degree $d \geq 2$ (respectively, $d' \geq 2$), and $\gcd(f_0, f_1, f_2) = 1$ (respectively, $\gcd(g_0, g_1, g_2) = 1$).

These three methods only use the parametrizations of the curves without computing the implicit equation of the curves. We will accomplish this by using the μ-basis of the syzygies of the curve. For each of the method, we will first prove the validity of the method, and then provide the computational algorithm for each of the method.

3.1. GCD Method

First, we will study the GCD method.

Theorem 1. *With about notation, if* $\mathbf{p}(s,t)$ *and* $\mathbf{p}(s,t)$ *are the μ-basis for the rational plan curve* $\mathbf{F}(s,t)$, *then*

$$\begin{aligned}
&\{(u_0,v_0) \neq (0,0) \mid \mathbf{p}(s,t)\cdot\mathbf{G}(u_0,v_0) = \mathbf{q}(s,t)\cdot\mathbf{G}(u_0,v_0) = 0\}\\
=&\{(u_0,v_0) \neq (0,0) \mid \gcd(\mathrm{Res}_s(\mathbf{p}(s,t)\cdot\mathbf{G}(u_0,v_0), \mathbf{q}(s,t)\cdot\mathbf{G}(u_0,v_0)),\\
&\mathrm{Res}_t(\mathbf{p}(s,t)\cdot\mathbf{G}(u_0,v_0), \mathbf{q}(s,t)\cdot\mathbf{G}(u_0,v_0))) = 0\}\\
\Leftrightarrow &\mathbf{G}(u_0,v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2.
\end{aligned}$$

Proof. First, we will show that

$$\{(u_0, v_0) \neq (0,0) \mid \mathbf{p}(s,t) \cdot \mathbf{G}(u_0, v_0) = \mathbf{q}(s,t) \cdot \mathbf{G}(u_0, v_0) = 0\} \iff \mathbf{G}(u_0, v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2.$$

To do so, we let $(u_0, v_0) \neq (0,0)$ be such that $\mathbf{p}(s,t) \cdot \mathbf{G}(u_0, v_0) = \mathbf{q}(s,t) \cdot \mathbf{G}(u_0, v_0) = 0$. This means that $\mathbf{G}(u_0, v_0) \parallel [\mathbf{p}(s,t)\ \mathbf{q}(s,t)]$. Hence, $\mathbf{G}(u_0, v_0) = k\mathbf{F}(s,t)$, and $\mathbf{G}(u_0, v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2$. On the other hand, if $\mathbf{G}(u_0, v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2$, then $\mathbf{G}(u_0, v_0) = \mathbf{F}(s_0, t_0)$ for some $(s_0, t_0) \neq (0,0)$. Thus, Definition 1 yields that $\mathbf{p}(s,t) \cdot \mathbf{G}(u_0, v_0) = \mathbf{q}(s,t) \cdot \mathbf{G}(u_0, v_0) = 0$. Therefore, the claim is proved.

Now, we will show that

$$\begin{aligned} &\{(u_0, v_0) \neq (0,0) \mid \mathbf{p}(s,t) \cdot \mathbf{G}(u_0, v_0) = \mathbf{q}(s,t) \cdot \mathbf{G}(u_0, v_0) = 0\} \\ = &\{(u_0, v_0) \neq (0,0) \mid \gcd(\mathrm{Res}_s(\mathbf{p}(s,t) \cdot \mathbf{G}(u_0, v_0), \mathbf{q}(s,t) \cdot \mathbf{G}(u_0, v_0)), \\ &\mathrm{Res}_t(\mathbf{p}(s,t) \cdot \mathbf{G}(u_0, v_0), \mathbf{q}(s,t) \cdot \mathbf{G}(u_0, v_0))) = 0\}. \end{aligned}$$

Let $G_p = \mathbf{p}(s,t) \cdot \mathbf{G}(u,v)$ and $G_q = \mathbf{q}(s,t) \cdot \mathbf{G}(u,v)$ be polynomials in variables s,t,u,v. Then $f = \mathrm{Res}_s(G_p, G_q)$ is a polynomial in t,u,v and $g = \mathrm{Res}_t(G_p, G_q)$ is a polynomial in s,u,v. Moreover, $\gcd(f,g)$ is a polynomial in u,v, and the solution set

$$\{(u_0, v_0) \neq (0,0) \mid \gcd(f,g) = 0\}$$

is exactly the set of $\{(u_0, v_0) \neq (0,0) \mid \mathbf{p}(s,t) \cdot \mathbf{G}(u_0, v_0) = \mathbf{q}(s,t) \cdot \mathbf{G}(u_0, v_0) = 0$. □

Hence, the natural algorithm to compute the intersection of two plane curves via GCD are stated below:

GCD Algorithm

Input: Parametrized curves $\mathbf{F}(s,t)$ and $\mathbf{G}(u,v)$.

Output: The set of parameters and their corresponding points of the intersection of two curves given by parametrization $\mathbf{F}(s,t)$ and $\mathbf{G}(u,v)$.

Procedure:

1. Compute $\mathbf{p}(s,t), \mathbf{q}(s,t)$, the μ-basis for $\mathbf{F}(s,t)$.
2. Compute $G_p = \mathbf{p}(s,t) \cdot \mathbf{G}(u,v)$, and $G_q = \mathbf{q}(s,t) \cdot \mathbf{G}(u,v)$.
3. Compute $f = \mathrm{Res}_s(G_p, G_q)$ and $g = \mathrm{Res}_t(G_p, G_q)$, where $\mathrm{Res}_s(f,g)$ stands for the resultant of polynomial f,g with respect to s.
4. The solution to $\gcd(f,g) = 0$ corresponding the the parameters (u_0, v_0) such that $\mathbf{G}(u_0, v_0)$ are the intersection points of the two curves with correct multiplicity.

Example 1. *Below are two rational paramatrized plane curves given by Wang-Goldman [11]: Lemniscate of Bernoulli, a rational quartic curve* $\mathbf{F}$ *and a rational cubic curve* $\mathbf{G}(u,v)$.

$$\begin{aligned} \mathbf{F}(s,t) =&(s^4 - t^4,\ -2st(t^2 - s^2),\ t^4 + 6s^2t^2 + s^4), \\ \mathbf{G}(u,v) =&(-3v^3 - 2uv^2 + 4u^2v + 2u^3,\ -v^3 + 3uv^2 - u^3 + 2u^2v,\ 2uv^2 + 2u^2v + u^3). \end{aligned}$$

GCD Method:

First, compute a μ-basis for $\mathbf{F}(s,t)$:

$$\mathbf{p}(s,t) = (-s^2 - t^2,\ -2st,\ s^2 - t^2), \quad \mathbf{q}(s,t) = (2st,\ -s^2 - t^2, 0).$$

Compute

$$\begin{aligned} G_p :=& \mathbf{p}(s,t)\cdot \mathbf{G}(u,v) \\ =& -s^2(u^3 + 2u^2v - 4uv^2 - 3v^3) - 3t^2(u^3 + 2u^2v - v^3) + 2st(u^3 - 2u^2v - 3uv^2 + v^3), \\ G_q :=& \mathbf{q}(s,t)\cdot \mathbf{G}(u,v) \\ =& 2st(2u^3 + 4u^2v - 2uv^2 - 3v^3) + s^2(u^3 - 2u^2v - 3uv^2 + v^3) + t^2(u^3 - 2u^2v - 3uv^2 + v^3). \end{aligned}$$

And

$$\begin{aligned} f :=& \mathrm{Res}_s(G_p, G_q) \\ =& 4t^4(u^2 + 2uv + 2v^2)^2(22u^8 - 86u^6v^2 - 10u^5v^3 + 131u^4v^4 + 34u^3v^5 - 84u^2v^6 - 20uv^7 + 25v^8), \\ g :=& \mathrm{Res}_t(G_p, G_q) \\ =& 4s^4(u^2 + 2uv + 2v^2)^2(22u^8 - 86u^6v^2 - 10u^5v^3 + 131u^4v^4 + 34u^3v^5 - 84u^2v^6 - 20uv^7 + 25v^8). \end{aligned}$$

Hence, the solutions to

$$\gcd(f,g) = (u^2+2uv+2v^2)^2(22u^8-86u^6v^2-10u^5v^3+131u^4v^4+34u^3v^5-84u^2v^6-20uv^7+25v^8) = 0$$

are the parameters (u_0, t_0) such that $\mathbf{G}(u_0, v_0)$ are the intersections of the two curves. The intersections are listed below:

$$\begin{aligned} \mathbf{G}(-1-1i,1) =& (3+6i, -6+3i, 0) \text{ double points,} \\ \mathbf{G}(-1+1i,1) =& (3-6i, -6-3i, 0) \text{ double points,} \\ \mathbf{G}(-1.163-0.135i,1) =& (1.645+0.436i, -0.311+0.771i, -1.166-0.187i), \\ \mathbf{G}(-1.163+0.135i,1) =& (1.645-0.436i, -0.311-0.771i, -1.166+0.187i), \\ \mathbf{G}(-0.840-0.483i,1) =& (0.566+2.396i, -2.572+1.089i, -0.739-0.253i), \\ \mathbf{G}(-0.840+0.483i,1) =& (0.566-2.396i, -2.572-1.089i, -0.739+0.253i), \\ \mathbf{G}(0.661-0.145i,1) =& (-2.166-0.855i, 1.567-0.635i, 2.399-0.865i), \\ \mathbf{G}(0.661+0.145i,1) =& (-2.166+0.855i, 1.567+0.635i, 2.399+0.865i), \\ \mathbf{G}(1.343-0.343i,1) =& (4.954-6.630i, 4.452-1.056i, 8.006-4.344i), \\ \mathbf{G}(1.343+0.3430i,1) =& (4.954+6.630i, 4.452+1.056i, 8.006+4.344i). \end{aligned}$$

We may check our solution by a conventional computation via implicit equation of $\mathbf{F}(s,t)$. We note that the implicit equation of $\mathbf{F}(s,t)$ is:

$$F(x,y,z) = 4(x^2 + y^2)^2 + 4w^2(y^2 - x^2) = 0,$$

and

$$F(\mathbf{G}(u,v)) = 4(u^2+2uv+2v^2)^2(22u^8-86u^6v^2-10u^5v^3+131u^4v^4+34u^3v^5-84u^2v^6-20uv^7+25v^8).$$

Obviously, $\gcd(f,g) = F(\mathbf{G}(u,v))$*, hence, the solutions to* $\gcd(f,g) = 0$ *are indeed the parameters of* $\mathbf{G}(u,v)$ *corresponding to the intersections of the two curves.*

3.2. Resultant Matrix Method

Now, we will use resultant matrix to find the intersection of two parametrized curves. First, we let $p(x, y, z; s, t) = \mathbf{p}(s, t) \cdot (x, y, z)$ and $q(x, y, z; s, t) = \mathbf{q}(s, t) \cdot (x, y, z)$ be a μ-basis of a rational parametric plane curve $\mathscr{C}_1$ of degree d given by

$$\mathbf{F}(s, t) = (f_0(s, t),\ f_1(s, t),\ f_2(s, t)),\ (s, t) \neq (0, 0).$$

It is know that the implicit equation of the curve is $F(x, y, z) = \mathrm{Res}_s(p(x, y, z; s, 1), q(x, y, z; s, 1))$. Let the resultant matrix of the curve $\mathscr{C}_1$ be denoted by $M(x, y, z)$. We note that $M(x, y, z)$ is a square matrix of size $d \times d$, and the rank of the matrix $\mathrm{rank} M(x, y, z) = d$.

Let the rational parametric plane curve $\mathscr{C}_2$ of degree d' be given by

$$\mathbf{G}(u, v) = (g_0(u, v),\ g_1(u, v),\ g_2(u, v)),\ (u, v) \neq (0, 0).$$

By substituting the variables x, y, z in matrix $M(x, y, z)$ with $(g_0(u, v),\ g_1(u, v),\ g_2(u, v))$, we obtain a matrix

$$\mathbb{M}(u, v) = M(\mathbf{G}(u, v)) = M(g_0(u, v),\ g_1(u, v),\ g_2(u, v)).$$

We can derive the following relationship between the rank of the matrix $\mathbb{M}(u, v)$ and the intersection of the two curves:

Theorem 2.

$$\{(u_0, v_0) \neq (0, 0) \mid \mathrm{rank}\mathbb{M}(u_0, v_0) < d\} \iff \mathbf{G}(u_0, v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2.$$

Proof. $\mathbf{G}(u_0, v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2$ if and only if $\mathbf{G}(u_0, v_0)$ is such that $\det M(\mathbf{G}(u_0, v_0)) = 0$, which is equivalent to the condition that $\mathrm{rank}\mathbb{M}(u_0, v_0) < d$. □

Hence, the computational algorithm follows directly as below:

Resultant Matrix Algorithm

Input: Parametrized curves $\mathbf{F}(s, t)$ and $\mathbf{G}(u, v)$.

Output: The set of parameters and their corresponding points of the intersection of two curves given by parametrization $\mathbf{F}(s, t)$ and $\mathbf{G}(u, v)$.

Procedure:

1. Compute $\mathbf{p}(s, t), \mathbf{q}(s, t)$, the μ-basis for $\mathbf{F}(s, t)$.
2. Construct the resultant matrix of the curve $\mathbf{F}(s, t)$ denoted by $M(x, y, z)$ using information of the μ-basis.
3. Substituting the variables x, y, z in matrix $M(x, y, z)$ with $(g_0(u, v),\ g_1(u, v),\ g_2(u, v))$.
4. The solution to $\det(M(g_0(u, v),\ g_1(u, v),\ g_2(u, v)) = 0$ corresponding the the parameters (u_0, v_0) such that $\mathbf{G}(u_0, v_0)$ are the intersection points of the two curves with correct multiplicity.

Example 2. *To compare the algorithm, we will use the same example as before, two rational paramatrized plane curves given by Wang-Goldman [11]: Lemniscate of Bernoulli, a rational quartic curve* $\mathbf{F}$ *and a rational cubic curve* $\mathbf{G}(u,v)$*.*

$$\begin{aligned}\mathbf{F}(s,t) =&(s^4-t^4,\ -2st(t^2-s^2),\ t^4+6s^2t^2+s^4),\\ bG(u,v) =&(-3v^3-2uv^2+4u^2v+2u^3,\ -v^3+3uv^2-u^3+2u^2v,\ 2uv^2+2u^2v+u^3).\end{aligned}$$

Resultant Matrix Method:
Compute the μ*-basis of the parametrized curve* $\mathbf{F}(s,t)$ *in terms of the moving lines as below:*

$$\mu\text{-basis: } \mathbf{p}(s,t)=(-s^2-t^2,\ -2st,\ s^2-t^2),\ \mathbf{q}(s,t)=(2st,\ -s^2-t^2,0)$$

moving line form: $p(x,y,z;s,1)=s^2(-x+z)-2sy-(x+z),\ q(x,y,z;s,1)=-s^2y+2sx-y.$

Then, construct matrix $M(x,y,z)$ *as*

$$M(x,y,z)=\begin{bmatrix}-x+z & 0 & -y & 0\\ -2y & -x+z & 2x & -y\\ -(x+z) & -2y & -y & 2x\\ 0 & -(x+z) & 0 & -y\end{bmatrix}.$$

And

$$\begin{aligned}\mathbb{M}(u,v) =&M(\mathbf{G}(u,v))\\ =&4(u^2+2uv+2v^2)^2(22u^8-86u^6v^2-10u^5v^3+131u^4v^4+34u^3v^5-84u^2v^6-20uv^7+25v^8).\end{aligned}$$

Since $\mathbb{M}(u,v)$ *is exactly the same as* $F(\mathbf{G}(u,v))$*, the solutions to* $\mathbb{M}(u,v)=0$ *are indeed the parameters of* $\mathbf{G}(u,v)$ *corresponding to the intersections of the two curves.*

3.3. Smith Normal Form Method

Recall that for every nonzero square univariate polynomial matrix A (that is the entries of matrix A are polynomials in one variable) with $\text{rank}(A)=r$, there exist invertible polynomial matrices P, Q (P is invertible if $\det(P)$ is a non-zero constant) such that

$$PAQ=\text{Diag}(f_1,f_2,\dots,f_r,0,0,\dots,0)$$

where $f_1,\dots,f_r$ are non-zero polynomials with $f_k \mid f_{k+1}$ for $1\le k\le r$, and $\text{Diag}(a_1,\dots,a_n)$ means the diagonal matrix with diagonal entries $a_1,\dots,a_n$. The diagonal matrix is called the *Smith normal form* of the polynomial matrix A is denoted by $S(A)$. The polynomials f_k for $k=1,\dots,r$ are called the *k-th invariant factors of A*, and $D_k=\prod_{i=1}^k f_i$ are called *k-th determinant factors of the matrix A.*

Since the implicit equation of the curve is $F(x,y,z)=\text{Res}_s(p(x,y,z;s,1),q(x,y,z;s,1))$ is obtained by $M(x,y,z)$, the determinant of the resultant matrix of the curve $\mathscr{C}_1$. In the previous subsection, substituting the variables x,y,z in matrix $M(x,y,z)$ with $(g_0(u,v),\ g_1(u,v),\ g_2(u,v))$, we obtain a matrix

$$\mathbb{M}(u,v)=M(\mathbf{G}(u,v))=M(g_0(u,v),\ g_1(u,v),\ g_2(u,v)).$$

Theorem 3. *$S(\mathbb{M}(u,1)) = Diag(f_1(u),\ldots,f_d(u))$ and $S(\mathbb{M}(1,v)) = Diag(g_1(v),\ldots,g_d(v))$. And*

$$\{(u_0,1) \text{ or } (1,v_0) \mid f_d(u_0)=0 \text{ or } g_d(v_0)=0\} \Longleftrightarrow \mathbf{G}(u_0,1) \text{ or } \mathbf{G}(1,v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2.$$

Proof. Since $\det(\mathbb{M}(u,v))$ is not identically zero, $\mathrm{rank}\mathbb{M}(u,v) = d$, therefore, the Smith normal forms $S(\mathbb{M}(u,1)) = \mathrm{Diag}(f_1(u),\ldots,f_d(u))$ and $S(\mathbb{M}(1,v)) = \mathrm{Diag}(g_1(v),\ldots,g_d(v))$.

By the property of Smith normal form, $\det S(\mathbb{M}) = \det(P\mathbb{M}Q) = c\det\mathbb{M} = 0$, and the solutions to the d-th invariant factor of $\mathbb{M}$ is exactly the solutions to $\det S(\mathbb{M}) = 0$. Therefore,

$$\begin{aligned}
&\mathbf{G}(u_0,v_0) \in \mathscr{C}_1 \cap \mathscr{C}_2\\
\Longleftrightarrow\ &\{(u_0,v_0) \neq (0,0) \mid \det S(\mathbb{M}(u_0,v_0)) = 0\}\\
\Longleftrightarrow\ &\{(u_0,1) \text{ or } (1,v_0) \mid f_d(u_0) = 0 \text{ or } g_d(v_0) = 0\}.
\end{aligned}$$

□

Hence, the algorithm via Smith normal form follows directly as below:

Smith Normal Form Algorithm

Input: Parametrized curves $\mathbf{F}(s,t)$ and $\mathbf{G}(u,v)$.

Output: The set of parameters and their corresponding points of the intersection of two curves given by parametrization $\mathbf{F}(s,t)$ and $\mathbf{G}(u,v)$.

Procedure:

i) Compute $\mathbf{p}(s,t), \mathbf{q}(s,t)$, the μ-basis for $\mathbf{F}(s,t)$.

ii) Construct the resultant matrix of the curve $\mathbf{F}$ denoted by $M(x,y,z)$ using information of the μ-basis.

iii) Substituting the variables x,y,z in matrix $M(x,y,z)$ with $(g_0(u,v),\ g_1(u,v),\ g_2(u,v))$.

iv) Compute the Smith normal form $S(\mathbb{M}(u,1))$ and $S(\mathbb{M}(1,v))$. The solution for $f_d(u) = 0$ and $g_d(v) = 0$ corresponding the the parameters $(u_0,1)$ and $(1,v_0)$ such that $\mathbf{G}(u_0,1)$ and $\mathbf{G}(1,v_0)$ are the intersection points of the two curves with correct multiplicity.

Example 3. *To compare the algorithm, we will use the same example as before, two rational paramatrized plane curves given by Wang-Goldman [11]: Lemniscate of Bernoulli, a rational quartic curve* $\mathbf{F}$ *and a rational cubic curve* $\mathbf{G}(u,v)$.

$$\begin{aligned}
\mathbf{F}(s,t) =&(s^4 - t^4,\ -2st(t^2 - s^2),\ t^4 + 6s^2t^2 + s^4),\\
\mathbf{G}(u,v) =&(-3v^3 - 2uv^2 + 4u^2v + 2u^3,\ -v^3 + 3uv^2 - u^3 + 2u^2v,\ 2uv^2 + 2u^2v + u^3).
\end{aligned}$$

Smith Normal Form Method:

Compute the μ-basis of the parametrized curve $\mathbf{F}(s,t)$ *in terms of the moving lines as below:*

$$\mu\text{-basis: } \mathbf{p}(s,t) = (-s^2 - t^2,\ -2st,\ s^2 - t^2), \quad \mathbf{q}(s,t) = (2st,\ -s^2 - t^2, 0)$$

moving line form: $p(x, y, z; s, 1) = s^2(-x + z) - 2sy - (x + z)$, $q(x, y, z; s, 1) = -s^2 y + 2sx - y$.

Then, construct matrix $M(x, y, z)$ *as*

$$M(x,y,z) = \begin{bmatrix} -x+z & 0 & -y & 0 \\ -2y & -x+z & 2x & -y \\ -(x+z) & -2y & -y & 2x \\ 0 & -(x+z) & 0 & -y \end{bmatrix}.$$

And

$$\begin{aligned} \mathbb{M}(u,v) =& M(\mathbf{G}(u,v)) \\ =& 4(u^2+2uv+2v^2)^2(22u^8-86u^6v^2-10u^5v^3+131u^4v^4+34u^3v^5-84u^2v^6-20uv^7+25v^8). \end{aligned}$$

Then, we compute the Smith normal form of $\mathbb{M}(u, 1)$

$$S(\mathbb{M}(u,1)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2+2u+u^2 & 0 \\ 0 & 0 & 0 & \frac{1}{22}(2+2u+u^2)(25-20u-84u^2+34u^3+131u^4-10u^5-86u^6+22u^8) \end{bmatrix}.$$

The last diagonal entry is the polynomial

$$\frac{1}{22}(2+2u+u^2)(25-20u-84u^2+34u^3+131u^4-10u^5-86u^6+22u^8),$$

which is exactly $\mathbb{M}(u, 1)$ *and* $F(\mathbf{G}(u, 1))$. *Hence the solution to the equation*

$$\frac{1}{22}(2+2u+u^2)(25-20u-84u^2+34u^3+131u^4-10u^5-86u^6+22u^8) = 0$$

are the parameters of $\mathbf{G}(u, v)$ *corresponding to the intersection of the two curves.*

4. Conclusion

In this paper, we compared three different methods of finding the intersections of two rational parametrized plane curves. These methods only use the μ-basis of one curve without finding the implicit equations of the curves. Since finding implicit equation is not an easy task, these three algorithms increase the computation efficiency.

To our knowledge, no research has done to extend these three algorithms to the case of rational parametrized space curves. The authors are currently studying how to generalize these methods to higher dimensions.

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