EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 4, 2014, 472-485
ISSN 1307-5543 – www.ejpam.com

The Linear Span of Four Points in the Plücker's Quadric in $\mathbb{P}^5$

Jacqueline Rojas[1,*], Ramón Mendoza[2]

[1] *CCEN - Departamento de Matemática - UFPB Cidade Universitária, 58051-900, João Pessoa - PB - Brasil*
[2] *CCEN - Departamento de Matemática - UFPE Cidade Universitária, 50740-540, Recife - PE - Brasil*

Abstract. Given four (distinct) lines ℓ_1, ℓ_2, ℓ_3, ℓ_4 in $\mathbb{P}^3$. Let P_i $(i = 1, \ldots, 4)$ be the image of ℓ_i in the Plücker's quadric $\mathscr{Q} \subset \mathbb{P}^5$ under the Plücker embedding $\mathscr{P}$ (in (1)). Set $\Lambda = \langle P_1, \ldots, P_4 \rangle$ be the linear span of those four points in $\mathbb{P}^5$. The purpose of this article is to write specifically what kind of quadric $\Lambda \cap \mathscr{Q}$ can be, taking under considerations all possible configurations of these four lines in $\mathbb{P}^3$. In particular, having in mind the classical problem in Schubert Calculus: *How many lines in 3-space meet four given lines in general position*? whose answer is 2 (see p. 272 in [3] or p. 746 in [4]). We verified that four lines in $\mathbb{P}^3$ are in general position if and only if Λ is a 3-plane and $\Lambda \cap \mathscr{Q}$ is an irreducible quadric surface. In fact, we prove that there are exactly two solutions if and only if Λ is a 3-plane and $\Lambda \cap \mathscr{Q}$ is a nonsingular quadric.

2010 Mathematics Subject Classifications: AMS 14N05, 14N20

Key Words and Phrases: Plücker's quadric, linear span, 4-line problem.

1. Introduction

Plücker's coordinates were introduced by the German geometer Julius Plücker (1801-1868) in the 19th century, as a way to assign six homogenous coordinates to each line in the complex projective 3-space $\mathbb{P}^3$. Since they satisfy a homogeneous quadratic equation, it follows an embedding of the 4-dimensional space of lines in $\mathbb{P}^3$ (denoted by $\mathbb{G}_1(\mathbb{P}^3)$ and called *grassmannian of lines in* $\mathbb{P}^3$) onto a nonsingular quadric hypersurface $\mathscr{Q}$ in $\mathbb{P}^5$ (see Proposition 3). Thus, reminding the Schubert's classical enumerative problem: *How many lines in 3-space meet four given lines in general position*? Whose answer can be found in many texts and is given by: "*there are two lines in* $\mathbb{P}^3$ *which meet 4 given lines in general position*" (see Example 14.7.2 at p. 272 in [3]). In fact, if you want to know an algorithm to determine explicit solutions, then see [7]. On the other hand, any beginner in the art of solving enumerative problems will ask: what does *general position* means? In Algebraic Geometry, general position is a notion of genericity for a set of points, or other geometric objects. It means the

*Corresponding author.

Email addresses: `jacq@mat.ufpb.br` (J. Rojas), `ramon@dmat.ufpe.br` (R. Mendoza),

general case situation, as opposite to some more special or coincident cases that are possible. Its precise meaning differs in different settings. For example, in [8] the authors imposed the condition $\ell_i \cap \ell_j = \emptyset$ $(1 \le i < j \le 4)$ to the four given lines ℓ_1, ℓ_2, ℓ_3, ℓ_4 in $\mathbb{P}^3$ and, even under this assumption they found (in one case) infinitely many solutions for the 4-lines problem (cf. **3.1** in the last subsection). So, this condition is not enough for the four given lines to be in general position.

In this article, we use the identification between lines in $\mathbb{P}^3$ and points in the quadric hypersurface $\mathcal{Q}$ to explain what is the precise meaning of *general position* for that problem. Nevertheless, the emphasis in our work lies on to take under considerations all possible configurations of these four lines in $\mathbb{P}^3$ and write specifically what kind of quadric $\Lambda \cap \mathcal{Q}$ can be. One key ingredient in this work lies on the well known description of all linear subspaces contained in a quadric hypersurface in $\mathbb{P}^3$ and $\mathbb{P}^5$ (see subsection 2.1 and 3.1).

Finally, we note that Plücker was a geometer that firmly believed in the importance of the applications of Mathematics to the physical sciences. So, in 1847 he turned to Physics, accepting the chair of Physics at Bonn and working on magnetism, electronics and atomic physics. He anticipated Gustav Kirchhoff and Robert Wilhelm Bunsen in announcing that the lines of the spectrum were characteristic of the chemical substance which emitted them, and in indicating the value of this discovery in chemical analysis. According to Johann Hittorf he was the first who saw the three lines of the hydrogen spectrum, which a few months after his death were recognized in the spectrum of the solar protuberances.

2. Notations and Preliminary Results

We denote by $\mathbb{C}$ the field of complex numbers. Let V be an n-dimensional vector space over $\mathbb{C}$. Denote by $[\mathrm{v}_1, \ldots, \mathrm{v}_k]$ the subspace of V generated by the vectors $\mathrm{v}_1, \ldots, \mathrm{v}_k \in V$.

The k-grassmannian associated to the vector space V. For each integer k, $0 \le k \le n = \dim V$, we denote by $G_k(V)$ the set of all k-dimensional linear subspaces of V and call it the *k-grassmannian* associated to V. In the particular case $k = 1$, the 1-grassmannian associated to V it is also called *projective space associated to* V and it is denoted by $\mathbb{P}(V)$ (i.e. $\mathbb{P}(V) := G_1(V)$). We use the notation $\mathbb{P}^n$ instead of $\mathbb{P}(\mathbb{C}^{n+1})$ and $p = [a_0 : \ldots : a_n]$ for $p = [(a_0, \ldots, a_n)] \in \mathbb{P}^n$, just for the sake of simplicity.

If $W \in G_{k+1}(V)$ then $\mathbb{P}(W) \subseteq \mathbb{P}(V)$ will be called *k-linear subspace* of $\mathbb{P}(V)$. The set of all k-linear subspaces of $\mathbb{P}(V)$ will be denoted by $\mathbb{G}_k(\mathbb{P}(V))$, the *grassmannian of k-linear subspaces* of $\mathbb{P}(V)$. Moreover, we shall call $\mathbb{G}_1(\mathbb{P}(V))$, $\mathbb{G}_2(\mathbb{P}(V))$ and $\mathbb{G}_{n-1}(\mathbb{P}(V))$ the grassmannian of lines, planes and hyperplanes in $\mathbb{P}(V)$, respectively. So, since a line ℓ in $\mathbb{P}^3$ is equal to $\mathbb{P}(W)$ for some $W \in G_2(\mathbb{C}^4)$, we have the correspondence

$$\begin{array}{ccc} G_2(\mathbb{C}^4) & \longrightarrow & \mathbb{G}_1(\mathbb{P}^3) \\ W & \longmapsto & \mathbb{P}(W), \end{array}$$

between the 2-grassmannian associated to $\mathbb{C}^4$ and the grassmannian of lines in $\mathbb{P}^3$. Thus, all assertions involving $G_2(\mathbb{C}^4)$ can be translated into $\mathbb{G}_1(\mathbb{P}^3)$.

Next we introduce the notion of algebraic projective set in $\mathbb{P}^n$. We will see in Proposition 1 that k-linear subspaces of $\mathbb{P}^n$ are examples of algebraic sets.

Algebraic projective sets in $\mathbb{P}^n$. Let $\mathbb{C}[\underline{X}] = \mathbb{C}[X_0, \ldots, X_n]$ be the polynomial ring over $\mathbb{C}$ in the variables $X_0, \ldots, X_n$. Now, for each integer $d \geq 0$ consider the vector subspace $\mathbb{C}[\underline{X}]_d$, generated by all monomials in $X_0, \ldots, X_n$ of degree d. Each element in $\mathbb{C}[\underline{X}]_d$ will be called an homogeneous polynomial of degree d. If $F \in \mathbb{C}[\underline{X}]_d$, then we define, $\mathscr{Z}(F)$, *the zero set* of F in $\mathbb{P}^n$ by

$$\mathscr{Z}(F) = \left\{ [\mathrm{v}] \in \mathbb{P}^n \mid F(\mathrm{v}) = 0 \right\}.$$

For example, if $L \in \mathbb{C}[\underline{X}]_1$, then $\mathscr{Z}(L) = \mathbb{P}(W)$ with $W = \{\mathrm{v} \in \mathbb{C}^{n+1} \mid L(\mathrm{v}) = 0\}$. Therefore, $\mathscr{Z}(L)$ is a hyperplane of $\mathbb{P}^n$, if $L \neq 0$ else $\mathscr{Z}(L) = \mathbb{P}^n$.

If $d \geq 1$, then an element $[F]$ in the projectivization of $\mathbb{C}[\underline{X}]_d$ will be called *hypersurface of degree d* in $\mathbb{P}^n$. From here on, when we say: Let $X \subset \mathbb{P}^n$ be the reduced hypersurface defined by $F \in \mathbb{C}[\underline{X}]_d$, that means that $X = \mathscr{Z}(F) \subset \mathbb{P}^n$ and F is square-free.

A subset X of $\mathbb{P}^n$ will be called *algebraic projective set*, if there exist homogeneous polynomials $F_1, \ldots, F_k$ in $\mathbb{C}[\underline{X}]$ such that $X = \mathscr{Z}(F_1) \cap \cdots \cap \mathscr{Z}(F_k)$.

Next we show that any r-linear subspace in $\mathbb{P}^n$ is the intersection of exactly $n - r$ hyperplanes.

Proposition 1. *Let Λ be an r-linear subspace in $\mathbb{P}^n$ with $n > r$. Then, there are exactly $n - r$ linearly independent linear forms $L_1, \ldots, L_{n-r}$ in $\mathbb{C}[\underline{X}]$ such that*

$$\Lambda = \mathscr{Z}(L_1) \cap \cdots \cap \mathscr{Z}(L_{n-r}).$$

Proof. Assume that $\Lambda = \mathbb{P}(W)$ with $W \in G_{r+1}(\mathbb{C}^{n+1})$. Let $\alpha = \{e_1, \ldots, e_{n+1}\}$ be the canonical base of $\mathbb{C}^{n+1}$ and let $\beta = \{e_1^*, \ldots, e_{n+1}^*\}$ be the associated dual base of $(\mathbb{C}^{n+1})^*$. Next we consider the linear isomorphism

$$\begin{array}{rccc} \varphi: & (\mathbb{C}^{n+1})^* & \to & \mathbb{C}[\underline{X}]_1 \\ & e_i^* & \mapsto & X_{i-1}. \end{array}$$

Now, let W^0 be the annihilator of W, then $\varphi(W^0)$ is an $(n-r)$-dimensional linear subspace of $\mathbb{C}[\underline{X}]_1$. Finally, an easy verification show that any base $\{L_1, \ldots, L_{n-r}\}$ of $\varphi(W^0)$ verified that $\Lambda = \mathscr{Z}(L_1) \cap \cdots \cap \mathscr{Z}(L_{n-r})$. □

Incidence of r-linear subspaces with reduced hypersurfaces in $\mathbb{P}^n$. The following proposition will play an important role in our investigations.

Proposition 2. *Let $Z \subset \mathbb{P}^n$ be the reduced hypersurface defined by the homogeneous polynomial F of degree d and Λ be an r-linear subspace of $\mathbb{P}^n$ with $r \geq 1$. Then the following conditions are verified.*

(1) $Z \cap \Lambda \neq \emptyset$;

(2) $Z \cap \Lambda$ *consists of infinitely many points, if* $\Lambda \subset Z$ *or* $r \geq 2$*, else* $Z \cap \Lambda$ *consists of at most* d *points.*

Proof. To arrive at statements (1) and first part of (2) have in mind that $\dim Z = n-1$, $\dim \Lambda = r$ and apply Theorem 7.2 at p. 48 in [5]. Already, the last part of statement (2) follows easily from the fundamental theorem of algebra. □

Projective Tangent Space and Nonsingular Reduced hypersurface. Let $Z \subset \mathbb{P}^n$ be the reduced hypersurface defined by $F \in \mathbb{C}[\underline{X}]_d$ and $p = [\mathrm{v}] \in \mathrm{Z}$. Let $F'_{\mathrm{v}} : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be the differential of F at v. We define the *projective tangent space*, $\mathbb{T}_p Z$, to the hypersurface Z at p by

$$\mathbb{T}_p Z = \mathbb{P}(\ker(F'_{\mathrm{v}})) = \Big\{[u_0 : \ldots : u_n] \in \mathbb{P}^n \mid \sum_{i=0}^{n} \frac{\partial F}{\partial X_i}(\mathrm{v}) \cdot u_i = 0\Big\}.$$

Now, note that:

- follows from the Euler relation $\sum_{i=0}^{n} \frac{\partial F}{\partial X_i} \cdot X_i = dF$ that $p \in \mathbb{T}_p Z$.
- $\mathbb{T}_p Z$ is a hyperplane in $\mathbb{P}^n$ if and only if $p \notin \cap_{i=0}^{n} \mathscr{Z}(\frac{\partial F}{\partial X_i})$.

A point $p \in Z$ satisfying the last condition above will be called a *nonsingular point* of Z, else p will be called a *singular point* of Z. If all the points in Z are nonsingular, then Z will be called a *nonsingular* hypersurface in $\mathbb{P}^n$.

For example, after a linear change of coordinates (see Theorem 4 at p. 411 in [2]) we concluded that $\mathscr{Z}(X_0^2+X_1^2+X_2^2+X_3^2)$ is the unique nonsingular quadric surface in $\mathbb{P}^3$, whereas the singular reduced quadric surfaces in $\mathbb{P}^3$ correspond either to the union of two planes or a quadric cone. Moreover, it is straightforward to show that the vertex of a quadric cone is its unique singular point. In the case of the union of two (distinct) planes, it is verified that the points in the line where the two planes meet are their singular points.

2.1. Lines on Reduced Quadrics Surfaces in $\mathbb{P}^3$

Next, we present some observations about lines contained on reduced quadrics surfaces in $\mathbb{P}^3$.

- **Union of two planes**. Of course any line in this surface will be contained in one of the planes. Thus this surface could have at most two disjoint lines.
- **Quadric cone**. Any line in this surface passes through its vertex. Thus this surface does not contain disjoint lines.
- **The nonsingular quadric surface**. As we shall describe in the next Lemma, a nonsingular quadric surface in $\mathbb{P}^3$ contains exactly two families of lines parametrize by $\mathbb{P}^1$. In fact, this Lemma is part of exercise 2.15 in Hartshorne's book [5]. See the proof at p. 478-479 in [4] or [8].

Lemma 1. *Let Q be a nonsingular quadric surface in $\mathbb{P}^3$. Then there exist two families of lines $\mathscr{L}=\{L_p\}_{p\in\mathbb{P}^1}$ and $\mathscr{M}=\{M_p\}_{p\in\mathbb{P}^1}$ in Q such that*

(1) $L_p\cap L_q=\emptyset$ and $M_p\cap M_q=\emptyset$ for all $L_p,L_q\in\mathscr{L}$, $M_p,M_q\in\mathscr{M}$ and $p\neq q\in\mathbb{P}^1$.

(2) $L_p\cap M_q\neq\emptyset$ for all $L_p\in\mathscr{L}$, $M_q\in\mathscr{M}$ and $p,q\in\mathbb{P}^1$.

(3) If ℓ is a line contained in Q then $\ell\in\mathscr{L}$ or $\ell\in\mathscr{M}$.

(4) Given $x\in Q$ there exist unique lines $L_{p(x)}\in\mathscr{L}$ and $M_{q(x)}\in\mathscr{M}$ such that $\{x\}=L_{p(x)}\cap M_{q(x)}$.

One other simple but important fact which will help us to prove our main result (Theorem 1) is the next Lemma.

Lemma 2. *Given the lines ℓ_1, ℓ_2 and ℓ_3 in $\mathbb{P}^3$ such that $\ell_i\cap\ell_j=\emptyset$ for $1\le i<j\le 3$, there exists a nonsingular quadric surface Q in $\mathbb{P}^3$ containing ℓ_1, ℓ_2 and ℓ_3.*

Proof. Take 3 points p_{1i},p_{2i},p_{3i} on each line ℓ_i $(i=1,2,3)$ and note that

$$W=\{G\in\mathbb{C}[\underline{X}]_2 \mid G(p_{ij})=0 \text{ for all } 1\le i,j\le 3\} \text{ (here } n=3)$$

is a subspace of $\mathbb{C}[\underline{X}]_2$ of dimension greater than or equal to 1. So we can choose $G\neq 0$ in W and take $Q=\mathscr{Z}(G)$. Now follows from Proposition 2 that each line ℓ_i is contained in Q. Finally, since Q contains three pairwise disjoint lines, then Q must be a nonsingular surface in $\mathbb{P}^3$. □

3. The Plücker's Quadric $\mathscr{Q}$ in $\mathbb{P}^5$

The Plücker embedding $\mathscr{P}:\mathbb{G}_k(\mathbb{P}(V))\longrightarrow\mathbb{P}(\Lambda^{k+1}V)$ is the map that enable us to identify the grassmannian $\mathbb{G}_k(\mathbb{P}(V))$ with a projective variety in $\mathbb{P}(\Lambda^{k+1}V)$ ($\Lambda^{k+1}V$ denotes the $(k+1)$-th exterior power of V). It is defined by $\mathbb{P}(W)\mapsto[\mathrm{u}_0\wedge\ldots\wedge\mathrm{u}_k]$, if $W=[\mathrm{u}_0,\ldots,\mathrm{u}_k]$.

The Plücker's quadric $\mathscr{Q}$ in $\mathbb{P}^5$. In what follows we will consider $V=\mathbb{C}^4$. Now, fix the base $\{e_i\wedge e_j\}_{1\le i<j\le 4}$ of $\Lambda^2\mathbb{C}^4$ where $\{e_i\}_{i=1}^4$ is the canonical basis of $\mathbb{C}^4$. Let us consider $W=[\mathrm{u},\mathrm{v}]\in G_2(\mathbb{C}^4)$ with $\mathrm{u}=(u_0,u_1,u_2,u_3)$ and $\mathrm{v}=(v_0,v_1,v_2,v_3)$, then we see that

$$\mathrm{u}\wedge\mathrm{v}=\sum_{1\le k<l\le 4} w_{k-1,l-1}e_k\wedge e_l \text{ where } w_{ij}=u_iv_j-u_jv_i \text{ for } 0\le i<j\le 3.$$

Thus the Plücker embedding $\mathscr{P}$ above in coordinates is given by

$$\begin{aligned}\mathscr{P}:\mathbb{G}_1(\mathbb{P}^3) &\longrightarrow\mathbb{P}^5\\ \mathbb{P}([\mathrm{u},\mathrm{v}]) &\longmapsto[w_{01}:w_{02}:w_{03}:w_{12}:w_{13}:w_{23}].\end{aligned} \tag{1}$$

Proposition 3. *The Plücker map $\mathscr{P}:\mathbb{G}_1(\mathbb{P}^3)\longrightarrow\mathbb{P}^5$ defined in (1) is an embedding of the grassmannian $\mathbb{G}_1(\mathbb{P}^3)$ over the nonsingular quadric hypersurface $\mathscr{Q}=\mathscr{Z}(F)\subset\mathbb{P}^5$ where*

$$F=X_0X_5-X_1X_4+X_2X_3\in\mathbb{C}[\underline{X}]\,(n=5).$$

Proof. See Theorem 11 at p. 409 in [2] or p. 209-211 in [4]. □

We will refer to $\mathscr{Q}$ in Proposition 3 as the Plücker's quadric $\mathscr{Q}$ (in $\mathbb{P}^5$).

Next, we will make a remark that tells us who is the image of the families of lines $\mathscr{L}$ and $\mathscr{M}$ of a nonsingular quadric surface in $\mathbb{P}^3$, under the Plücker embedding $\mathscr{P}$ (cf. Lemma 1).

Remark 1. *It can be shown that* $\mathscr{P}(\mathscr{L}) = \mathscr{Z}(X_0 + X_5, X_1 - X_4, X_2 + X_3, F)$ *and* $\mathscr{P}(\mathscr{M}) = \mathscr{Z}(X_0 - X_5, X_1 + X_4, X_2 - X_3, F)$*, so the image of the families of lines* $\mathscr{L}$ *and* $\mathscr{M}$*, under the Plücker embedding* $\mathscr{P}$ *(in* (1)*) are nonsingular conics lying in complementary planes (also see p. 196 in [6]).*

3.1. Linear Subspaces in the Plücker's quadric $\mathscr{Q}$

We begin by proving a simple result, which allows us to conclude that the Plücker's quadric $\mathscr{Q}$ does not contain 3-planes or hyperplanes. In fact, it is verified that $\mathscr{Q}$ only contains lines and planes, and this will be described in Propositions 5 and 6, respectively (see [1]).

Proposition 4. *Let* $Z \subset \mathbb{P}^n$ *be a nonsingular quadric hypersurface defined by* $G \in \mathbb{C}[\underline{X}]_2$ *and* Λ *a* r*-linear subspaces of* $\mathbb{P}^n$ *. If* $\Lambda \subset Z$*, then we have* $2r < n$*.*

Proof. Set $\Lambda = \mathbb{P}(W)$ where $W = [\mathrm{w}_0, \dots, \mathrm{w}_r]$. Assume that $G = \sum_{i=0}^n X_i^2$ (otherwise make a linear change of coordinates). So G induz the bilinear form $B : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ given by $B(\mathrm{v}, \mathrm{w}) = v_0 w_0 + \cdots + v_n w_n$, if $\mathrm{v} = (v_0, \dots, v_n)$ and $\mathrm{w} = (w_0, \dots, w_n)$. Let

$$W^{\perp} = \{\mathrm{v} \in \mathbb{C}^{n+1} \mid B(\mathrm{v}, \mathrm{w}) = 0 \text{ for all } \mathrm{w} \in W\}$$

be the the orthogonal subspace associated to W.

Note that $W^{\perp} = \ker(T_0) \cap \cdots \cap \ker(T_r)$ where T_i is the linear functional over $\mathbb{C}^{n+1}$ given by $T_i(\mathrm{v}) = B(\mathrm{v}, \mathrm{w}_i)$, $i \in \{0, \dots, r\}$. Since these linear functionals are linearly independent we conclude that $\dim W^{\perp} = n + 1 - \dim W = n - r$.

On the other hand, $\Lambda \subset Z$ if and only if $G(\mathrm{w}) = B(\mathrm{w}, \mathrm{w}) = 0$ for all $\mathrm{w} \in W$. Thus $\Lambda \subset Z$ if and only if $W \subseteq W^{\perp}$. So, if $\Lambda \subset Z$ then $\dim W \leq \dim W^{\perp} = n + 1 - \dim W$. Therefore, $2 \dim W \leq n + 1$ which implies $2r < n$. □

In what follows, we denoted by $\langle \ell_1, \dots, \ell_k \rangle$ the smallest linear subspace of $\mathbb{P}^n$ containing the lines $\ell_1, \dots, \ell_k$ of $\mathbb{P}^n$. For example, if ℓ_1 and ℓ_2 are two distinct lines in $\mathbb{P}^n$ having a common point, then $\langle \ell_1, \ell_2 \rangle$ is a plane, else $\langle \ell_1, \ell_2 \rangle$ is a 3-plane.

Next we give the description of lines and planes in $\mathscr{Q}$.

Lines in the Plücker's quadric $\mathscr{Q}$. The lines in the Plücker's quadric $\mathscr{Q}$ are parametrized by the incidence variety $\Gamma = \{(p, \Pi) \mid p \in \Pi\} \subset \mathbb{P}^3 \times \mathbb{G}_2(\mathbb{P}^3)$. In fact, let $p \in \mathbb{P}^3$ and $\Pi \subset \mathbb{P}^3$ be a plane through p. Set

$$\Omega_p(\Pi) = \left\{ \ell \in \mathbb{G}_1(\mathbb{P}^3) \mid p \in \ell \subset \Pi \right\}.$$

Proposition 5. *Let $\mathscr{P} : \mathbb{G}_1(\mathbb{P}^3) \to \mathbb{P}^5$ be the Plücker embedding in* (1). *If L is a line in $\mathbb{P}^5$ contained in $\mathscr{Q}$, then we have:*

(i) $\mathscr{P}(\Omega_p(\Pi))$ *is a line in $\mathbb{P}^5$ contained in $\mathscr{Q}$ for all $(p,\Pi) \in \Gamma$.*

(ii) If $P_0, P \in L$ are two different points such that $P_0 = \mathscr{P}(\ell_0)$ and $P = \mathscr{P}(\ell)$, then $\ell_0 \cap \ell = \{p\}$ and they determine the plane $\Pi = \left\langle \ell_0, \ell \right\rangle$. In fact, for every point $Q = \mathscr{P}(m) \in L$ holds that $m \in \Omega_p(\Pi)$.

Planes in the Plücker's quadric $\mathscr{Q}$. There are two families of planes in the Plücker's quadric $\mathscr{Q}$ parametrized by $\mathbb{P}^3$ and its dual $\overset{\vee}{\mathbb{P}^3} = \mathbb{G}_2(\mathbb{P}^3)$, respectively. In fact, let $p \in \mathbb{P}^3$ and $\Pi \subset \mathbb{P}^3$ be a plane. Set

$$\Omega_p = \left\{ \ell \in \mathbb{G}_1(\mathbb{P}^3) \mid p \in \ell \right\} \text{ and } \Omega(\Pi) = \left\{ \ell \in \mathbb{G}_1(\mathbb{P}^3) \mid \ell \subset \Pi \right\}.$$

Proposition 6. *Let $\mathscr{P} : \mathbb{G}_1(\mathbb{P}^3) \to \mathbb{P}^5$ be the Plücker embedding in* (1). *If Λ is a plane in $\mathbb{P}^5$ contained in $\mathscr{Q}$, then we have:*

(i) $\mathscr{P}(\Omega_p)$ *and* $\mathscr{P}(\Omega(\Pi))$ *are planes in $\mathbb{P}^5$ contained in $\mathscr{Q}$.*

(ii) The pre-image of Λ under the Plücker embedding $\mathscr{P}$ is either: Ω_p for some $p \in \mathbb{P}^3$ or $\Omega(\Pi)$ for some $\Pi \in \mathbb{G}_2(\mathbb{P}^3)$. In fact, if $P_i = \mathscr{P}(\ell_i)$, $i = 0,1,2$ are three points in $\mathscr{Q}$ whose linear span it is equal to Λ, then $\Lambda = \Omega_p$ if $p \in \cap_{i=0}^{2} \ell_i$ (and these lines are non-coplanar) else $\Lambda = \Omega(\Pi)$ with $\Pi = \langle \ell_0, \ell_1, \ell_2 \rangle$.

4. Description of $\Lambda \cap \mathscr{Q}$ According to the Relative Position of the Four Given Lines in $\mathbb{P}^3$

We begin this section by introducing some more notation. As we did for lines we denote by $\langle p_1, \ldots, p_k \rangle$ *the linear span* of $p_1, \ldots, p_k \in \mathbb{P}^n$ (i.e. the smallest linear subspace of $\mathbb{P}^n$ containing these points). Moreover, if p is a point and ℓ is a line in $\mathbb{P}^n$, then $\langle p, \ell \rangle$ also denote the linear span of p and ℓ in $\mathbb{P}^n$. So $\langle p, \ell \rangle = \ell$, if $p \in \ell$ else $\langle p, \ell \rangle$ is a 3-plane.

Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be four distinct lines in $\mathbb{P}^3$. Let $P_i \in \mathscr{Q}$ $(i = 1, \ldots, 4)$ be the image of ℓ_i under the Plücker embedding $\mathscr{P}$ (in (1)) and $\Lambda = \langle P_1, P_2, P_3, P_4 \rangle \subset \mathbb{P}^5$. Set $\mathscr{S}$ be the set of solutions of the 4-lines problem in Schubert calculus, i.e.

$$\mathscr{S} = \left\{ \ell \in \mathbb{G}_1(\mathbb{P}^3) \mid \ell \cap \ell_i \neq \emptyset \text{ for } 1 \leq i \leq 4 \right\}.$$

We have three major cases to be considered (cf. subsections 4.1, 4.2 and 4.3). In each of these cases, the linear space Λ and $\Lambda \cap \mathscr{Q}$ are reported in tables, whose rows are labeled according to the position that the lines can have in $\mathbb{P}^3$. Next, we determine Λ and $\Lambda \cap \mathscr{Q}$ in some of these cases, which we believe are sufficient to illustrate the technique used in this computations, we left to the reader the other cases

In what follows, for simplicity, we will use the notation $\Omega_p(\Pi)$, Ω_p and $\Omega(\Pi)$ for lines and planes in $\mathscr{Q}$, according to the identification given by the Plücker embedding in Propositions 5 and 6, respectively.

4.1. The Four Lines are Concurrent, Say $p \in \cap_{i=1}^{4} \ell_i$.

Table 1: Four Lines are Concurrent.

Subcase	Position of the lines	$\mathscr{S}$	Λ	$\Lambda \cap \mathscr{Q}$
1.1	ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are contained in the plane Π	$\Omega_p \cup \Omega(\Pi)$	Line	$\Omega_p(\Pi)$
1.2	ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are non coplanar lines	Ω_p	Plane	Ω_p

1.1 Naturally any line passing trough p is a solution. On the other hand, if $\ell \in \mathscr{S}$ is a solution such that $p \notin \ell$, it certainly meets the lines ℓ_1 and ℓ_2 at two different points, which implies that $\ell \subset \Pi$. Therefore $\mathscr{S} = \Omega_p \cup \Omega(\Pi)$. On the other hand, since $P_i \in \Omega_p(\Pi)$ for $i = 1, \ldots, 4$, then $\Lambda \equiv \Omega_p(\Pi)$ is a line contained in $\mathscr{Q}$.

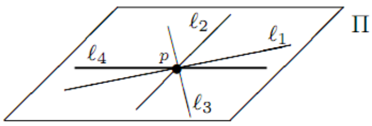

Figure 1: Case 1.1

4.2. The Four Lines have no Common Point and at Least Two are Coplanar.

Of course in all the subcases listed below we consider an index reordering if necessary.

Table 2: Four Lines have no Common Point and at Least Two are Coplanar.

Subcase	Position of the lines	$\mathscr{S}$	Λ	$\Lambda \cap \mathscr{Q}$
2.1	ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are contained in the plane Π	$\Omega(\Pi)$	Plane	$\Omega(\Pi)$

2.1 Of course any line contained in the plane Π is a solution. Now, since the four given lines have not common point, then any solution will meets at least two of these at different points. Therefore, $\mathscr{S} = \Omega(\Pi)$. Now, since the four lines are not concurrent, then three of these four points (the P_i's) determine the plane $\Omega(\Pi)$. In fact, $\Lambda \equiv \Omega(\Pi)$, so it is a plane contained in $\mathscr{Q}$.

Table 3: **2.2 Exactly Three Lines are Coplanar**. Assume that ℓ_1, ℓ_2 and ℓ_3 are contained in the plane Π and $\Pi \cap \ell_4 = \{q\}$.

Subcase	Position of the lines	$\mathscr{S}$	Λ	$\Lambda \cap \mathscr{Q}$
2.2.1	$\bigcap_{i=1}^{3} \ell_i = \{p\}$	$\Omega_p(\langle p, \ell_4\rangle) \cup \Omega_q(\Pi)$	Plane	Union of two lines
2.2.2	$\bigcap_{i=1}^{3} \ell_i = \emptyset$	$\Omega_q(\Pi)$	3-plane	Union of two planes

2.2.1 Let $\ell \in \mathscr{S}$ be a solution which meets the lines ℓ_1 and ℓ_2 at two different points, then $\ell \in \Omega_q(\Pi)$ (since $\Pi \cap \ell_4 = \{q\}$). Else ℓ meets ℓ_1 and ℓ_2 at p and ℓ_4 at some point different of p, then $\ell \subset \langle p, \ell_4\rangle$. So $\ell \in \Omega_p(\langle p, \ell_4\rangle)$. Therefore, $\mathscr{S} = \Omega_q(\Pi) \cup \Omega_p(\langle p, \ell_4\rangle)$ and it can be identified with two projective lines having a common point (just a cross!). On the other hand, since P_1, P_2 and P_i lies on the line $\Omega_p(\Pi)$ and $P_4 \notin \Omega_p(\Pi)$ ($p \notin \ell_4$) we conclude that Λ is a plane in $\mathbb{P}^5$ not contained in $\mathscr{Q}$. So $\Lambda \cap \mathscr{Q}$ is a conic. Note that the line $L_{1,2} = \langle P_1, P_2\rangle \subset \Lambda \cap \mathscr{Q}$ and $P_4 \notin L_{1,2}$. Therefore, $\Lambda \cap \mathscr{Q}$ is the union two lines. In fact, $\Lambda \cap \mathscr{Q} = L_{1,2} \cup L$ where $L = \langle P_4, M\rangle$ with $M = \mathscr{P}(\langle p, q\rangle)$.

Table 4: **2.3 Any Three Lines are non Coplanar and at Least Two Pair of Lines are Coplanar with** $\ell_i \neq \Pi_1 \cap \Pi_2 \, \forall i$. Assume that $\ell_1 \cap \ell_2 = \{p\}$ and $\ell_3 \cap \ell_4 = \{q\}$. Set $\Pi_1 = \langle \ell_1, \ell_2\rangle$ and $\Pi_2 = \langle \ell_3, \ell_4\rangle$.

Subcase	Position of the lines	$\mathscr{S}$	Λ	$\Lambda \cap \mathscr{Q}$
2.3.1	$p, q \in \Pi_1 \cap \Pi_2$	$\Omega_p(\Pi_2) \cup \Omega_q(\Pi_1)$	Plane	Union of two lines
2.3.2	$p \in \Pi_1 \cap \Pi_2 \not\ni q$	$\Omega_p(\Pi_2)$	3-plane	Union of two planes
2.3.3	$p \notin \Pi_1 \cap \Pi_2 \not\ni q$	$\left\{\Pi_1 \cap \Pi_2, \ell_{p,q}\right\}$	3-plane	Nonsingular quadric

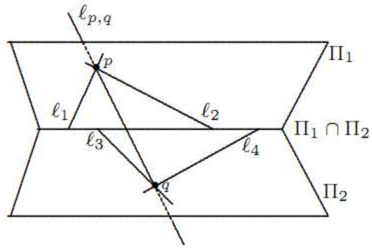

Figure 2: Case 2.3.3

2.3.3 Let ℓ be a solution. Next we consider the following two cases.

(i) Assume that $p \in \ell$. Suppose that ℓ meets the lines ℓ_3 and ℓ_4 at two different points, then $\ell \subset \Pi_2$ which implies that $p \in \Pi_1 \cap \Pi_2$. But this is impossible, so $q \in \ell$. Therefore $\ell = \ell_{p,q}$. Similarly, we conclude that $\ell = \ell_{p,q}$, if $q \in \ell$.

(ii) Assume that $p \notin \ell$ and $q \notin \ell$. Then ℓ meets the lines ℓ_1 and ℓ_2 (respectively, ℓ_3 and ℓ_4) at two different points which implies that $\ell \subset \Pi_1$ (respectively, $\ell \subset \Pi_2$). Therefore, $\ell = \Pi_1 \cap \Pi_2$.

Now, note that the line $L_{1,2} = \langle P_1, P_2 \rangle \subset \Lambda \cap \mathscr{Q}$ and $P_i \notin L_{1,2}$, $i = 3,4$. So $\langle P_1, P_2, P_3 \rangle$ is a plane not contained in $\mathscr{Q}$ (since $p \notin \ell_i$ and $\ell_i \not\subset \Pi_1$ for $i = 3,4$). On the other hand, the line $L_{3,4} = \langle P_3, P_4 \rangle$ is also contained in $\Lambda \cap \mathscr{Q}$ and we observe that $L_{1,2}$ and $L_{3,4}$ are disjoint (since $L_{1,2} = \Omega_p(\Pi_1)$ and $L_{3,4} = \Omega_q(\Pi_2)$), so P_4 does not belong to the plane $\langle P_1, P_2, P_3 \rangle$. Therefore, Λ is a 3-plane. Keeping in mind that $\Lambda \cap \mathscr{Q}$ is a quadric surface containing four non coplanar points and two disjoint lines, we conclude that $\Lambda \cap \mathscr{Q}$ is a union of two planes or a nonsingular quadric. Suppose that $\Lambda \cap \mathscr{Q}$ is a union of two planes, say $\Lambda \cap \mathscr{Q} = \Lambda_1 \cup \Lambda_2$. Thus we can assume that $L_{1,2} \subset \Lambda_1$ and $L_{3,4} \subset \Lambda_2$. Then necessarily Λ_1 it is either Ω_p or $\Omega(\Pi_1)$, and in the same form Λ_2 it is either Ω_q or $\Omega(\Pi_2)$. But in any case, $\Lambda_1 \cap \Lambda_2$ will be empty or a point. Therefore, $\Lambda \cap \mathscr{Q}$ is a nonsingular quadric surface.

Table 5: **2.4 Any Three Lines are non Coplanar and at Least Two Pair of Lines are Coplanar with** $\Pi_1 \cap \Pi_2 = \ell_1$. Assume that $\ell_1 \cap \ell_2 = \{p\}$ and $\ell_1 \cap \ell_3 = \{q\}$. Set $\Pi_1 = \langle \ell_1, \ell_2 \rangle$ and $\Pi_2 = \langle \ell_1, \ell_3 \rangle$. Let $\{r_i\} = \Pi_i \cap \ell_4$ for $i = 1,2$.

Subcase	Position of the lines	$\mathscr{S}$	Λ	$\Lambda \cap \mathscr{Q}$
2.4.1	$p = q$	$\Omega_p(\langle p, \ell_4 \rangle)$	3-plane	Union of two planes
2.4.2	$p \neq q$ and $p \in \ell_4$ (or $p \neq q$ and $q \in \ell_4$)	$\Omega_p(\langle p, \ell_3 \rangle)$ (or $\Omega_q(\langle q, \ell_2 \rangle)$)	3-plane	Union of two planes
2.4.3	$r_1 = r_2$ and $\#\{p, q, r_1\} = 3$	$\{\ell_1\}$	3-plane	Quadric cone
2.4.4	$p \neq q$ and $r_1 \neq r_2$	$\{\ell_{q,r_1}, \ell_{p,r_2}\}$	3-plane	Nonsingular quadric

2.4.4 Let $\ell \in \mathscr{S}$. Here we have two possibilities:

(i) $p \notin \ell$. Then $\ell \cap \ell_1 \neq \ell \cap \ell_2$ and we have that $\ell \subset \Pi_1$. Thus $\ell \cap \ell_3 \subset \Pi_1 \cap \ell_3 = \{q\}$ and $\ell \cap \ell_4 \subset \Pi_1 \cap \ell_4 = \{r_1\}$. Therefore, $\ell = \ell_{q,r_1}$ ($q \neq r_1$ since $\ell_4 \not\subset \Pi_1$).

(ii) $p \in \ell$. Note that ℓ meets the line ℓ_3 at a point p_1 ($p_1 \in \Pi_2$) different from p. Thus $\ell \subset \Pi_2$. On the other hand $\ell \cap \ell_4 \subset \Pi_2 \cap \ell_4 = \{r_2\}$. Therefore, $\ell = \ell_{p,r_2}$.

As in case **2.3.3** we have that $\langle P_1, P_2, P_3 \rangle$ is a plane not contained in $\mathscr{Q}$. Since the lines $L_{1,2} = \langle P_1, P_2 \rangle$ and $L_{1,3} = \langle P_1, P_3 \rangle$ are contained in $\langle P_1, P_2, P_3 \rangle \cap \mathscr{Q}$ we conclude that $\langle P_1, P_2, P_3 \rangle \cap \mathscr{Q} = L_{1,2} \cup L_{1,3}$. So $P_4 \notin \langle P_1, P_2, P_3 \rangle$. Therefore, Λ is a 3-plane. Keeping in mind that $\Lambda \cap \mathscr{Q}$ is a quadric surface containing four non coplanar points, $\{P_1\} = L_{1,2} \cap L_{1,3}$ and $L_{1,4} = \langle P_1, P_4 \rangle \not\subset \Lambda \cap \mathscr{Q}$ (since $L_{1,4} \not\subset \mathscr{Q}$), we conclude that $\Lambda \cap \mathscr{Q}$ is a union of two planes or a nonsingular quadric. Suppose that $\Lambda \cap \mathscr{Q}$ is a union of two planes, say $\Lambda \cap \mathscr{Q} = \Lambda_1 \cup \Lambda_2$. Thus we can assume that $L_{1,2} \subset \Lambda_1$. Now, note that $L_{2,3} \not\subset \Lambda \cap \mathscr{Q}$ (since $L_{2,3} \not\subset \mathscr{Q}$). So $P_3 \notin \Lambda_1$, which implies $L_{1,3} \subset \Lambda_2$. Finally, observe that $L_{2,4}$ and $L_{3,4}$ are not contained in $\Lambda \cap \mathscr{Q}$. Thus $P_4 \notin \Lambda_1 \cup \Lambda_2$. Therefore, $\Lambda \cap \mathscr{Q}$ is a nonsingular quadric.

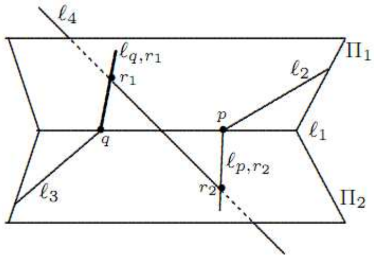

Figure 3: Case 2.4.4

Table 6: **2.5 Exactly Two Lines are Coplanar**. Assume that ℓ_1 and ℓ_2 are contained in the plane Π, $\ell_1 \cap \ell_2 = \{p\}$, $\Pi \cap \ell_i = \{p_i\}$ for $i = 3,4$ and $p_3 \neq p_4$. Let $C = \{p, p_3, p_4\}$.

Subcase	Position of the lines	$\mathscr{S}$	Λ	$\Lambda \cap \mathscr{Q}$
2.5.1	$p = p_3$ (or $p = p_4$)	$\Omega_p(\langle p, \ell_4 \rangle)$ (or $\Omega_p(\langle p, \ell_3 \rangle)$)	3-plane	Union of two planes
2.5.2	$\#C = 3$ and $p \in \ell_{p_3,p_4}$	$\left\{\ell_{p_3,p_4}\right\}$	3-plane	Quadric cone
2.5.3	$p \notin \ell_{p_3,p_4}$ (so $\#C = 3$)	$\left\{\langle p, \ell_3 \rangle \cap \langle p, \ell_4 \rangle, \ell_{p_3,p_4}\right\}$	3-plane	Nonsingular quadric

2.5.3 Let ℓ be a solution and consider the following two possibilities.

(i) $\ell \subset \Pi$. Since $\ell \cap \ell_i \subset \Pi \cap \ell_i = \{p_i\}$ then $p_i \in \ell$ for $i = 3,4$. Therefore $\ell = \ell_{p_3,p_4}$.

(ii) $\ell \not\subset \Pi$. Now having in mind that $\ell \cap \ell_i \neq \emptyset$ for $i = 1,2$ and $\ell \not\subset \Pi$ we concluded that $p \in \ell$. On the other hand $\ell \cap \ell_i = \{q_i\} \subset \ell_i \subset \langle p, \ell_i \rangle$ for $i = 3,4$. Note that $q_i \neq p$ (since $p \notin \ell_i$ for $i = 3,4$) which implies $\ell = \ell_{p,q_i} \subset \langle p, \ell_i \rangle$ for $i = 3,4$. Therefore, $\ell = \langle p, \ell_3 \rangle \cap \langle p, \ell_4 \rangle$.

Note that $P_i \notin L_{1,2} = \langle P_1, P_2 \rangle \subset \mathscr{Q}$ for $i = 3,4$. Thus $\langle P_1, P_2, P_3 \rangle$ is a plane not contained in $\mathscr{Q}$ since $p \notin \ell_3 \not\subset \Pi$. Since $L_{1,2} \cup \{P_3\} \subset \langle P_1, P_2, P_3 \rangle \cap \mathscr{Q}$, if $P_4 \in \langle P_1, P_2, P_3 \rangle$ then $P_4 \in L_{1,2}$ or P_4 belong to the component L passing through P_3 in the conic $\langle P_1, P_2, P_3 \rangle \cap \mathscr{Q} = L_{1,2} \cup L$. But $P_4 \notin L_{1,2}$ because $\ell_4 \not\subset \Pi$. Moreover, $P_4 \notin L$ since $\ell_4 \cap \ell_3 = \emptyset$. Thus Λ is a 3-plane. Keeping in mind that $\Lambda \cap \mathscr{Q}$ is a quadric surface containing four non coplanar points and $L_{1,3}$ is not contained in $\Lambda \cap \mathscr{Q}$, we conclude that $\Lambda \cap \mathscr{Q}$ is a union of two planes or a nonsingular quadric. Suppose that $\Lambda \cap \mathscr{Q}$ is a union of two planes, say $\Lambda \cap \mathscr{Q} = \Lambda_1 \cup \Lambda_2$. Thus we can assume that $L_{1,2} \subset \Lambda_1$. Next, note that $L_{2,3} \not\subset \Lambda \cap \mathscr{Q}$ (since $L_{2,3} \not\subset \mathscr{Q}$). So $P_3 \notin \Lambda_1$, which implies $L_{1,3} \subset \Lambda_2$. Now, since $P_4 \in \Lambda_1 \cup \Lambda_2$ we conclude that $L_{2,4}$ and $L_{3,4}$ are contained in $\Lambda \cap \mathscr{Q}$ which it is impossible since $\ell_4 \cap \ell_i = \emptyset$, $i = 2,3$. Therefore, $\Lambda \cap \mathscr{Q}$ is a nonsingular quadric surface.

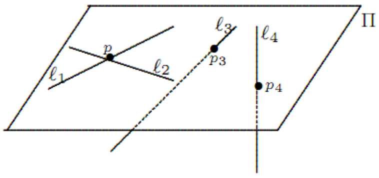

Figure 4: Case 2.5.3

4.3. No Pair of Lines is Coplanar

Let Q be a nonsingular quadric surface in $\mathbb{P}^3$ containing ℓ_1, ℓ_2 and ℓ_3 (as stated in Lemma 2). Since $\ell_i \cap \ell_j = \emptyset$ for $1 \le i < j \le 3$ then they belong to the same family of lines in Q (cf. Lemma 1). So we will assume that ℓ_1, ℓ_2 and ℓ_3 belong to the family $\mathscr{L}$.

Note that any solution $\ell \in \mathscr{S}$ meets the line ℓ_i at a point $p_i \in \ell_i \subset Q$ for $i = 1,2,3$. Since this three points p_1, p_2 and p_3 are different and belong to $\ell \cap Q$. Then follows from Proposition 2 that $\ell \subset Q$, which implies that $\ell \in \mathscr{M}$. Therefore any solution belong to the family $\mathscr{M}$.

In order to determine the set of solutions $\mathscr{S}$ and the linear space Λ, we will consider the following two subcases.

3.1 $\boxed{\ell_4 \subset Q}$. In this case ℓ_1, ℓ_2, ℓ_3 and ℓ_4 belong to the same family $\mathscr{L}$ (since ℓ_4 if disjoint to the others three). Therefore $\mathscr{S} = \mathscr{M}$.

Now, since the line $L_{i,j} = \langle P_i, P_j \rangle \not\subset \mathscr{Q}$ for $1 \le i < j \le 3$, we conclude from Proposition 6 that $\langle P_1, P_2, P_3 \rangle$ is a plane not contained in $\mathscr{Q}$. But, from Remark 1 we conclude that $\langle P_1, P_2, P_3 \rangle \cap \mathscr{Q} = \mathscr{P}(\mathscr{L})$. Consequently, $\Lambda = \langle P_1, P_2, P_3 \rangle$ (keep in mind that $P_4 \in \mathscr{P}(\mathscr{L})$ too) and $\Lambda \cap \mathscr{Q}$ is a nonsingular conic.

3.2 $\boxed{\ell_4 \not\subset Q}$. In this case follows from Proposition 2 that $\ell_4 \cap Q$ is non empty and $\ell_4 \cap Q = \{x, y\}$ (with x and y do not necessarily distinct). Let M_x and M_y be the unique lines in the family $\mathscr{M}$ passing through x and y, respectively (cf. (4) in Lemma 1). Indeed M_x and M_y are solutions (in fact, if $p \in \{x, y\}$ then $M_p \cap \ell_i$ is non empty for $i = 1,2,3$ because $\ell_i \in \mathscr{L}$ and $M_p \cap \ell_4 = \{p\}$). Now, we will show that M_x and M_y are the unique solutions. Let $\ell \in \mathscr{M}$ be a solution, then $\ell \cap \ell_4 \subset Q \cap \ell_4 = \{x, y\}$ which implies that $x \in \ell$ or $y \in \ell$. Therefore follows from (4) in the Lemma 1 that $\ell = M_x$ or $\ell = M_y$.

Again, since the line $L_{i,j} = \langle P_i, P_j \rangle \not\subset \mathscr{Q}$ for $1 \le i < j \le 3$, we conclude that $\langle P_1, P_2, P_3 \rangle$ is a plane not contained in $\mathscr{Q}$ and $\langle P_1, P_2, P_3 \rangle \cap \mathscr{Q}$ is a nonsingular conic. On the other hand, the condition $\ell_4 \not\subset Q$ implies that $\ell_4 \notin \mathscr{L}$. So, we are left to conclude that $P_4 \notin \langle P_1, P_2, P_3 \rangle$. Thus Λ is a 3-plane in $\mathbb{P}^5$. One more time, since $\Lambda \cap \mathscr{Q}$ is a quadric surface containing four non coplanar points and, the line $L_{i,j} \not\subset \mathscr{Q}$ for $1 \le i < j \le 3$, we conclude that $\Lambda \cap \mathscr{Q}$ is a quadric cone or a nonsingular quadric surface. In fact, since $\ell_4 \not\subset Q$, it must be either: tangent or secant to the surface Q.

Assume that ℓ_4 is tangent to Q. Let $x \in \ell_4 \cap Q$ be the tangent point. Let $M_x \in \mathscr{M}$ and $L_x \in \mathscr{L}$ be the (only) lines in Q passing through x. It is verified that $\ell_4 \subset \mathbb{T}_x Q = \langle M_x, L_x \rangle$. Therefore, $\ell_4 \in \Omega_x(\mathbb{T}_x Q)$. Set $M = \mathscr{P}(M_x), L = \mathscr{P}(L_x) \in \mathscr{Q}$. Thus L belong to the conic $\left\langle P_1, P_2, P_3 \right\rangle \cap \mathscr{Q} = \mathscr{P}(\mathscr{L})$. Moreover $M \in \langle L, P_4 \rangle \subset \Lambda \cap \mathscr{Q}$ (since, $x \in \ell_4 \subset \langle M_x, L_x \rangle$). On the other hand the lines $L_i = \langle M, P_i \rangle \subset \Lambda \cap \mathscr{Q}$, $i = 1, \ldots, 4$. Therefore, $\Lambda \cap \mathscr{Q}$ is a quadric cone.

Thus we have proved the following Theorem.

Theorem 1. *Given four lines ℓ_1, ℓ_2, ℓ_3, ℓ_4 in $\mathbb{P}^3$. Set*

$$\mathscr{S} = \left\{ \ell \in \mathbb{G}_1(\mathbb{P}^3) \mid \ell \cap \ell_i \neq \emptyset \text{ for } 1 \leq i \leq 4 \right\}.$$

Let P_i $(i = 1, \ldots, 4)$ be the image of ℓ_i in the Plücker's quadric $\mathscr{Q} \subset \mathbb{P}^5$ under the Plücker embedding $\mathscr{P}$ (in (1)). Set $\Lambda = \left\langle P_1, \ldots, P_4 \right\rangle$ be the linear span of those four points in $\mathbb{P}^5$. Then the cardinal of the set $\mathscr{S}$ is either 1, 2 or infinite. Moreover we have.

(i) $\#S = 2$ if and only if Λ is a 3-plane and $\Lambda \cap \mathscr{Q}$ is a smooth quadric.

(ii) $\#S = 1$ if and only if Λ is a 3-plane and $\Lambda \cap \mathscr{Q}$ is a quadric cone.

(iii) If $\#S = \infty$ then

- *(a) S is a line if and only if Λ is a 3-plane and $\Lambda \cap \mathscr{Q}$ is a union of two planes.*
- *(b) S is a conic if and only if Λ is a plane and $\Lambda \cap \mathscr{Q}$ is a nonsingular conic.*
- *(c) S is a reduced and reducible conic if and only if Λ is a plane and $\Lambda \cap \mathscr{Q}$ is a reduced and reducible conic.*
- *(d) S is a plane if and only if Λ is a plane in $\mathscr{Q}$.*
- *(e) S is a union of two distinct planes if and only if Λ is a line in $\mathscr{Q}$.*

In terms of the lines position Theorem 1 tell us:

$\#S = 2 \Longleftrightarrow$

- **$\cap_{i=1}^4 \ell_i = \emptyset$ and any three lines are non coplanar.**

(i) $\ell_1 \cap \ell_2 = \{p\}$, $\ell_3 \cap \ell_4 = \{q\}$ with $p \neq q$ and $p \notin \Pi_1 \cap \Pi_2 \not\ni q$, if $\Pi_1 = \langle \ell_1, \ell_2 \rangle$, $\Pi_2 = \langle \ell_3, \ell_4 \rangle$.

(ii) $\ell_1 \cap \ell_2 = \{p\}$, $\ell_1 \cap \ell_3 = \{q\}$ with $p \neq q$. If $\Pi_i = \langle \ell_1, \ell_{i+1} \rangle$ for $i = 1, 2$, then it is verified that $\Pi_i \cap \ell_4 = \{r_i\}$ $i = 1, 2$ and $r_1 \neq r_2$.

(iii) $\ell_1 \cap \ell_2 = \{p\}$. If $\Pi = \langle \ell_1, \ell_2 \rangle$ then it is verified for $i = 3, 4$ that $\Pi \cap \ell_i = \{r_i\}$, $r_3 \neq r_4$ and $p \notin \langle r_3, r_4 \rangle$.

- **No pair of lines is coplanar.**

(iv) The non singular quadric surface $Q \subset \mathbb{P}^3$ containing ℓ_1, ℓ_2 and ℓ_3, meets ℓ_4 exactly in two distinct points.

Note that (i), (ii) and (iii) above correspond to subcases **2.3.3**, **2.4.4** and **2.5.3** in subsection 4.2. Already (iv) corresponds to subcase **3.2** in subsection 4.3.

$$\#S = 1 \Longleftrightarrow \begin{cases} \bullet & \cap_{i=1}^{4} \ell_i = \emptyset \ \textbf{and any three lines are non coplanar.} \\ (\alpha) & \ell_1 \cap \ell_2 = \{p\},\ \ell_1 \cap \ell_3 = \{q\} \text{ with } p \neq q. \text{ If } \Pi_i = \langle \ell_1, \ell_{i+1} \rangle \\ & \text{for } i = 1,2, \text{ then it is verified that } \Pi_i \cap \ell_4 = \{r_i\}\ i = 1,2. \\ & \text{with } r_1 = r_2 \text{ and } \#\{p,q,r_1\} = 3. \\ (\beta) & \ell_1 \cap \ell_2 = \{p\}. \text{ If } \Pi = \langle \ell_1, \ell_2 \rangle \text{ then it is verified for } i = 3,4 \\ & \text{that } \Pi \cap \ell_i = \{r_i\},\ r_3 \neq r_4,\ p \in \langle r_3, r_4 \rangle \text{ and } p \neq r_i. \\ \bullet & \textbf{No pair of lines is coplanar.} \\ (\gamma) & \text{The non singular quadric surface } Q \subset \mathbb{P}^3 \text{ containing } \ell_1,\ \ell_2 \\ & \text{and } \ell_3, \text{ meets } \ell_4 \text{ exactly in one point.} \end{cases}$$

Note that (α) and (β) above correspond to subcases **2.4.3** and **2.5.2** in subsection 4.2. Already (γ) correspond to subcase **3.2** in subsection 4.3.

ACKNOWLEDGEMENTS The first author wishes to express her gratitude to I. Vainsencher (DM-UFMG) for a helpful conversation about the subject of this paper.

References

[1] D. Avritzer. Introdução à Geometria Enumerativa via Teoria de Deformações, 2^a *Bienal da Sociedade Brasileira de Matemática*, Mini Course, Salvador, Universidade Federal da Bahia. 2004.

[2] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms*, New York: Springer-Verlag. 1996.

[3] W. Fulton. *Intersection Theory*. Graduate Texts in Math, Springer-Verlag. 1977.

[4] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley-Interscience. 1994.

[5] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Math. **52**, Springer, New York. 1977.

[6] J. Harris and D. Eisenbud. *The Geometry of Schemes*. Graduate Texts in Math. **197**, Springer, New York. 1999.

[7] M. Hohmeyer and S. Teller. Determining the Lines Through Four Lines. *Journal of Graphics Tools*, 4(3):11–22. 1999.

[8] R. Mendoza and J. Rojas. Álgebra Linear e o Problema das quatro retas do Cálculo de Schubert, *Revista Matemática Universitária*, 45:55–69. 2009.