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Skew-Laurent rings over $\sigma(*)$-rings

V. K. Bhat

School of Mathematics, SMVD University, P/o SMVD University, Katra, J and K, India- 182320

Abstract. Let R be an associative ring with identity $1 \neq 0$, and σ an endomorphism of R. We recall $\sigma(*)$ property on R (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R). Also recall that a ring R is said to be 2-primal if and only if $P(R)$ and the set of nilpotent elements of R coincide, if and only if the prime radical is a completely semiprime ideal. It can be seen that a $\sigma(*)$-ring is a 2-primal ring.

Let R be a ring and σ an automorphism of R. Then we know that σ can be extended to an automorphism (say $\overline{\sigma}$) of the skew-Laurent ring $R[x, x^{-1}; \sigma]$. In this paper we show that if R is a Noetherian ring and σ is an automorphism of R such that R is a $\sigma(*)$-ring, then $R[x, x^{-1}; \sigma]$ is a $\overline{\sigma}(*)$-ring. We also prove a similar result for the general Ore extension $R[x; \sigma, \delta]$, where σ is an automorphism of R and δ a σ-derivation of R.

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1. Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The set of prime ideals of R is denoted by $Spec(R)$. The sets of minimal prime ideals of R is denoted by $Min.Spec(R)$. Prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. Let R be a ring and σ an automorphism of R. Let I be an ideal of R such that $\sigma^m(I) = I$ for some $m \in \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers). We denote $\cap_{i=1}^{m}\sigma^i(I)$ by I^0. The field of rational numbers is denoted by $\mathbb{Q}$ and the field of real numbers is denoted by $\mathbb{R}$ unless otherwise stated.

This article concerns the study of skew-Laurent rings over $\sigma(*)$-rings, where σ is an automorphism of R.

Email address: vijaykumarbhat2000@yahoo.com

$\sigma(*)$-rings

Recall that in Krempa [8], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [9], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1. *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$*, where F is a field. Then* $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$*. Let $\sigma : R \to R$ be defined by* $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$*. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.*

2-primal Rings

We do not want to talk about 2-primal rings, but because of a close relation between a $\sigma(*)$-ring and a 2-primal ring, we have the following:

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i.e. if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a commutative ring is 2-primal and so is a reduced ring.

2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [10], Greg Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R. He has proved that when R is a local ring with a nilpotent maximal ideal, the Ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R.

In [9], Kwak establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. It has been proved that if R is a ring and σ an endomorphism of R such that $\sigma(P(R)) \subseteq P(R)$, then R is a $\sigma(*)$-ring implies that R is 2-primal. Therefore, we see that if R is a Noetherian ring and σ an automorphism of R, then R is a $\sigma(*)$-ring implies that R is 2-primal.

The following example shows that if R is a Noetherian ring, then even $R[x]$ need not be 2-primal.

Example 2. *Let $R = M_2(\mathbb{Q})$, the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.*

Skew Polynomial Rings

Let R be a ring, σ be an endomorphism of R and δ a σ-derivation of R. Recall that δ is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$.

Example 3. *Let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ defined by* $\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}$*, for all $r \in R$ be a homomorphism. Then δ is a σ-derivation of R.*

Recall that the skew polynomial ring (Ore extension) $R[x;\sigma,\delta]$ is the usual ring of polynomials with coefficients in R, in which multiplication is subject to the relation $ax = x\sigma(a)+\delta(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that $\sigma(I) = I$ and $\delta(I) \subseteq I$, then $O(I)$ denotes $I[x;\sigma,\delta]$, which is an ideal of $O(R)$.

Skew-Laurent Rings

Recall that $R[x,x^{-1};\sigma]$ is the usual ring of Laurent polynomials with coefficients in R, in which multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We take any $f(x) \in R[x,x^{-1};\sigma]$ to be of the form $f(x) = \sum_{i=-m}^{n} x^i a_i$. We denote $R[x,x^{-1};\sigma]$ by $L(R)$. If an ideal I of a ring R is σ-stable (i.e. $\sigma(I) = I$), then we denote as usual $I[x,x^{-1};\sigma]$ by $L(I)$.

We also note that if σ is an automorphism of R, then it can be extended to an automorphism (say $\overline{\sigma}$) of $R[x,x^{-1};\sigma]$ such that $\overline{\sigma}(x) = x$; i.e. $\overline{\sigma}(\Sigma_{i=-m}^{n} x^i a_i) = \Sigma_{i=-m}^{n} x^i \sigma(a_i)$. The study of skew polynomial rings and skew-Laurent rings has been of interest to many authors. For example [1, 6, 7, 9].

In this paper we prove the following results:

Theorem 2: Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if $R[x,x^{-1};\sigma]$ is a $\overline{\sigma}(*)$-ring.

Theorem 3: Let R be a Noetherian ring which is also an algebra over Q. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $R[x;\sigma,\delta]$ is a $\overline{\sigma}(*)$-ring.

2. Preliminaries

We begin this section with the following Proposition:

Proposition 1. *Let R be a ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies R is 2-primal.*

Proof. Let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. □

The following example shows that there exists an endomorphism σ of a ring R such that the converse of the above Proposition does not hold.

Example 4. *Let $R = F[x]$, F a field. Then R is a commutative domain, and therefore is 2-primal with $P(R) = 0$. Let $\sigma : R \to R$ be defined by $\sigma(f(x)) = f(0)$. Let $f(x) = xa$, $0 \neq a \in F$. Then $f(x)\sigma(f(x)) \in P(R)$, but $f(x) \notin P(R)$. Therefore R is not a $\sigma(*)$-ring.*

Before we give a characterization of a Noetherian $\sigma(*)$-ring, we require the following:

Recall that an ideal P of a ring R is completely prime if R/P is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$ (McCoy [11]).

Note that a completely prime ideal is a prime ideal, but the converse need not be true.

For example, let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R, but is not strongly prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

Proposition 2 (Proposition 2.1 of Bhat [6]). *Let R be a Noetherian ring, and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. To make the article self contained, we give a proof (a modified one):

Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. Now for each i, $a \in U_i$ or $\sigma(a) \in U_i$ as U_i are completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring and let $U = U_1$ be a minimal prime ideal of R. Now by Proposition 1, $P(R)$ is completely semiprime. Now $Min.Spec(R)$ is finite by Theorem (2.4) of Goodearl and Warfield [7]. Let $U_2, U_3, \ldots, U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now since a $\sigma(*)$-ring is 2-primal, minimal prime ideals are completely prime. Hence U is completely prime. □

Note that in above Theorem the condition of completely primeness of minimal prime ideals can not be deleted. Towards this we have the following:

Remark 1. *Let R be a Noetherian ring and σ an automorphism of R such that $\sigma(U) = U$ for each minimal prime ideal U of R. Then R need not be a $\sigma(*)$-ring (Example 4).*

3. Skew-Laurent Rings Over $\sigma(*)$-rings

Goodearl and Warfield proved in (2ZA) of [7] that if R is a commutative Noetherian ring, and if σ is an automorphism of R, then an ideal I of R is of the form $P \cap R$ for some prime ideal P of $R[x, x^{-1}; \sigma]$ if and only if there is a prime ideal S of R and a positive integer m with $\sigma^m(S) = S$, such that $I = \cap \sigma^i(S)$, $i = 1, 2, \ldots, m$.

We note that if R is a Noetherian ring, then as mentioned above, $Min.Spec(R)$ is finite. Now if σ is an automorphism of R, then $\sigma^j(U) \in Min.Spec(R)$ for any $U \in Min.Spec(R)$ for all $j \in \mathbb{N}$. Therefore, there exists some $m \in \mathbb{N}$ such that $\sigma^m(U) = U$ for all $U \in Min.Spec(R)$. We denote $\cap_{i=1}^{m} \sigma^i(U)$ by U^0.

We now have the following:

Theorem 1. *Let R be a Noetherian ring and σ an automorphism of R. Then $P \in Min.Spec(L(R))$ if and only if there exists $U \in Min.Spec(R)$ such that $L(P \cap R) = (P \cap R)[x, x^{-1}; \sigma] = P$ and $P \cap R = U^0$.*

Proof. See Theorem (2.4) of Bhat [1]. □

As mentioned in the introduction, we note that if σ is an automorphism of R, then it can be extended to an automorphism (say $\overline{\sigma}$) of $R[x, x^{-1}; \sigma]$ such that $\overline{\sigma}(x) = x$; i.e. $\overline{\sigma}(\Sigma_{i=-m}^{n} x^i a_i) = \Sigma_{i=-m}^{n} x^i \sigma(a_i)$.

With this we are now in a position to prove the following Theorem:

Theorem 2. *Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if $L(R) = R[x, x^{-1}; \sigma]$ is a Noetherian $\overline{\sigma}(*)$-ring.*

Proof. Let R be a Noetherian ring, σ an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. We shall prove that $O(R) = R[x; \sigma, \delta]$ is a Noetherian $\overline{\sigma}(*)$-ring. For this we will show that any minimal $P \in Min.Spec(O(R))$ is completely prime and $\overline{\sigma}(P) = P$.

Let $P \in Min.Spec(O(R))$. Then by Theorem 1, there exists $U \in Min.Spec(R)$ such that $P = U^0[x, x^{-1}; \sigma]$. Now R is a $\sigma(*)$-ring implies that $\sigma(U) = U$ by Proposition 2, and therefore $U^0 = U$. So $P = U[x, x^{-1}; \sigma]$ and thus $\overline{\sigma}(P) = P$.

We now show that $P = U[x, x^{-1}; \sigma]$ is completely prime. Now σ can be extended to an automorphism of R/U in a natural way. We note that $O(R)/P \cong (R/U)[x, x^{-1}; \sigma]$, and since U is completely prime, R/U is a domain and so $(R/U)[x, x^{-1}; \sigma]$ is also a domain. Hence $P = U[x, x^{-1}; \sigma]$ is completely prime.

Thus $\overline{\sigma}(P) = P$ and P is completely prime for all $P \in Min.Spec(L(R))$. Moreover $L(R) = R[x, x^{-1}; \sigma]$ is Noetherian by Theorem (1.17) of Goodearl and Warfield [7]. Hence by Proposition 2 $R[x, x^{-1}; \sigma]$ is a $\overline{\sigma}(*)$-ring.

Conversely let $L(R) = R[x, x^{-1}; \sigma]$ be a $\overline{\sigma}(*)$-ring. Let $U \in Min.Spec(R)$. Then Theorem 1 implies that $L(U^0) \in Min.Spec(L(R))$. Now $L(R)$ be a $\overline{\sigma}(*)$-ring implies that $\overline{\sigma}(L(U^0)) = L(U^0)$ and $L(U^0)$ is completely prime ideal of $L(R)$. Now there is an embedding $R/(L(U^0) \cap R) \to L(R)/L(U^0)$. Since $L(R)/L(U^0)$ is an integral domain, so is $R/(L(U^0) \cap R)$. Therefore, $U^0 = L(U^0) \cap R)$ is a completely prime ideal of R. Now $U^0 \subseteq U$ implies that $U^0 = U$. So $\sigma(U) = U$ and U is a completely prime ideal of R. Hence by Proposition 2 R is a $\sigma(*)$-ring. □

Remark 2.

i) *Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. Then $R[x, x^{-1}; \sigma]$ is a $\overline{\sigma}(*)$-ring. Therefore, Proposition 1 implies that $R[x, x^{-1}; \sigma]$ is 2-primal.*

ii) *If R is 2-primal Noetherian ring, then $R[x, x^{-1}; \sigma]$ need not be 2-primal. For example consider $\mathbb{Z}_2$ and let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then R is a commutative reduced ring with $P(R) = 0$, and therefore R is 2-primal. Define $\sigma : R \to R$ by $\sigma(a, b) = (b, a)$. Then it can be seen that*

$$P(R[x, x^{-1}; \sigma]) = 0, \text{ but } P(R[x, x^{-1}; \sigma])$$

is not completely semiprime as

$$((1, 0)x)^2 = 0 = P(R[x, x^{-1}; \sigma]), \text{ but } (1, 0)x \notin P(R[x, x^{-1}; \sigma]).$$

Thus $R[x, x^{-1}; \sigma]$ is not 2-primal.

4. Skew Polynomial Rings Over $\sigma(*)$-rings

Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x; \sigma, \delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x; \sigma, \delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.

Example 5 (Example 2.13 of Bhat [5]). *Let $R = \mathbb{R} \times \mathbb{R}$, $\sigma : R \to R$ defined by $\sigma((a, b)) = (b, a)$ for $a, b \in \mathbb{R}$. Then σ is an automorphism of R. Let now $r \in \mathbb{R}$. Define $\delta_r : R \to R$ by $\delta_r((a, b)) = (a, b)r - r\sigma((a, b))$ for $a, b \in R$. Then δ is a σ-derivation. Now for any $(u, v) \in R$,*

$$\begin{aligned}
\sigma(\delta_r((u, v))) =& \sigma((u, v)r - r\sigma((u, v)))\\
=& \sigma((u, v)r - r(v, u))\\
=& \sigma((ur, vr) - \sigma(vr, ur))\\
=& (vr, ur) - (ur, vr)).
\end{aligned}$$

Also

$$\begin{aligned}
\delta_r(\sigma((u, v))) =& \delta_r(v, u)\\
=& (v, u)r - r\sigma((v, u))\\
=& (v, u)r - r(u, v)\\
=& (vr, ur) - (ur, vr)).
\end{aligned}$$

Therefore $\sigma(\delta((u, v))) = \delta(\sigma((u, v)))$ for all $(u, v) \in R$.

Remark 3. *We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$ for all $a \in R$, then the above does not hold. For example let $f(x) = xl$ and $g(x) = xp$, $a, b \in R$. Then*

$$\overline{\delta}(f(x)g(x)) = x^2\{\delta(\sigma(l))\sigma(p) + \sigma(l)\delta(p)\} + x\{\delta^2(l)\sigma(p) + \delta(l)\sigma(p)\},$$

but

$$\overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x)) = x^2\{\sigma(\delta(l))\sigma(p) + \sigma(l)\delta(p)\} + x\{\delta^2(l)\sigma(p) + \delta(l)\sigma(p)\}.$$

So, $\overline{\delta}(f(x)g(x)) \neq \overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x))$, i.e. $\overline{\delta}$ is not a $\overline{\delta}$-derivation.

With this we now prove the following:

Theorem 3. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further let $P \in Min.Spec(O(R))$ implies that $P \cap R \in Min.Spec(R)$. Then R is a $\sigma(*)$-ring implies that $O(R) = R[x;\sigma,\delta]$ is a Noetherian $\overline{\sigma}(*)$-ring.*

Proof. Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. We shall prove that $O(R) = R[x;\sigma,\delta]$ is a Noetherian $\overline{\sigma}(*)$-ring. For this we will show that any minimal $P \in Min.Spec(O(R))$ is completely prime and $\overline{\sigma}(P) = P$.

Let $P \in Min.Spec(O(R))$. Now $P \cap R \in Min.Spec(R)$ and R is a $\sigma(*)$-ring implies that $\sigma(P \cap R) = P \cap R$ and $P \cap R$ is a completely prime ideal of R. Now Proposition (2.1) of Bhat [2] implies that $\delta(P \cap R) \subseteq P \cap R$. Now Theorem (2.4) of Bhat [4] implies that $O(P \cap R)$ is a completely prime ideal of $O(R)$. Now $O(P \cap R) \subseteq P$ implies that $O(P \cap R) = P$ as P is minimal. Now $\sigma(P \cap R) = P \cap R$ implies that $\overline{\sigma}(P) = P$.

Thus $\overline{\sigma}(P) = P$ and P is completely prime for all $P \in Min.Spec(O(R))$. Moreover $O(R) = R[x;\sigma,\delta]$ is Noetherian by Theorem (1.12) of Goodearl and Warfield [7]. Hence by Proposition 2 $R[x;\sigma,\delta]$ is a $\overline{\sigma}(*)$-ring. □

We note that the condition that $P \in Min.Spec(O(R))$ implies that $P \cap R \in Min.Spec(R)$ can not be ignored as follows:

Let $R = \mathbb{Q} \times \mathbb{Q}$. Let $\sigma : R \to R$ be defined by $\sigma((a,b)) = (b,a)$ and $\delta = 0$. Then $P = 0$ is a prime ideal of $O(R)$, but $P \cap R$ is not a prime ideal of R.

We have not been able to prove the converse part of the above result. The main reason being that a generalization of Theorem 1 in terms of $O(R)$ is not known. The known towards this is:

Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. Then $U \in Min.Spec(R)$ such that $\sigma(U) = U$ implies that $\delta(U) \subseteq U$ (Lemma 2.6 of Bhat [3]).

Question Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. If $O(R) = R[x;\sigma,\delta]$ is a Noetherian $\overline{\sigma}(*)$-ring. Is R is a $\sigma(*)$-ring?

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