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Exponentiated Transmuted Modified Weibull Distribution

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Abstract. The paper introduces the exponentiated transmuted modified Weibull distribution, which contains a number of distributions as special cases. The properties of the distribution are discussed and explicit expressions for the quantiles, mean deviations and the reliability are derived. The distribution and moments of order statistics are also studied. Estimation of the model parameters by the methods of least squares and maximum likelihood are discussed. Finally, the usefulness of the distribution for modeling data is illustrated using real data.

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1. Introduction

Modelling and analysis of lifetime data have become very crucial in different areas of research, like engineering, medicine, reliability, etc. In this regard, it is observed that the Weibull distribution is extensively used as it is found to provide reasonable fit in many practical situations. Attempts at generalization of the distribution have led to the exponentiated Weibull distribution, the modified Weibull distribution and the exponentiated modified Weibull distribution. An interesting idea of generalizing a distribution, which is known in the literature as transmutation has been used to develop further distributions. A random variable T is said to have a transmuted distribution if its distribution function is given by

$$Z(t) = (1+\lambda)G(t) - \lambda[G(t)]^2, \quad (1)$$

where $G(t)$ denotes the base distribution, and $\lambda \in$ [-1, 1] denotes the transmuting parameter. Aryall and Tsokos [2] introduced the transmuted Weibull distribution, Ebraheim [4] studied the exponentiated transmuted Weibull distribution, Pal and Tiensuwan [7] developed the beta transmuted Weibull distribution, Khan and King [6] investigated the transmuted modified

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Weibull distribution, and Ashour and Eltehiwy [3] proposed the transmuted exponentiated modified Weibull distribution.

In this paper, we introduce and study several mathematical properties of a new reliability model referred to as the exponentiated transmuted modified Weibull distribution. The modified Weibull distribution, introduced by Sarhan and Zaindin [8], has the cumulative distribution function (c.d.f.)

$$F_{MW}(t) = 1 - exp(-\alpha t - \gamma t^{\beta}), t \geq 0, \alpha, \beta, \gamma > 0. \tag{2}$$

The c.d.f. of the transmuted modified Weibull distribution [6] is given by

$$F_{TMW}(t) = [1 - exp(-\alpha t - \gamma t^{\beta})][1 + \lambda exp(-\alpha t - \gamma t^{\beta})], t \geq 0, \alpha, \beta, \gamma > 0. \tag{3}$$

The above distribution has three shape parameters α, β and γ, and λ denotes the transmuting parameter. The exponentiated transmuted modified Weibull distribution generalizes this distribution by introducing another shape parameter.

The paper is organized as follows. In Section 2 we introduce the distribution. In Sections 3, we obtain the quantile function of the distribution. The moment generating function and the moments are derived in Sections 4. Mean deviation is discussed in Section 5. Order statistics and their moments are studied in Sections 6. In Section 7, the stress-strength reliability is obtained. Estimation of parameters by the least square method and the maximum likelihood method are discussed in Sections 8 and 9. In Section 10, a simulation study is carried out to compare the two methods of estimation. The usefulness of the distribution for modeling real life data is illustrated in Section 11. Finally, in Section 12, we make some concluding remarks on our study.

2. Exponentiated Transmuted Modified Weibull Distribution

The five parameter exponentiated transmuted modified Weibull (ETMW) distribution is given by the c.d.f.

$$F(t) = [1 - exp(-\alpha t - \gamma t^{\beta})1 + \lambda exp(-\alpha t - \gamma t^{\beta})]^{\delta}, t \geq 0, \alpha, \beta, \gamma > 0, \lambda \in [-1, 1], \tag{4}$$

where α, β, γ, δ are all shape parameters, and λ is the transmuting parameter.

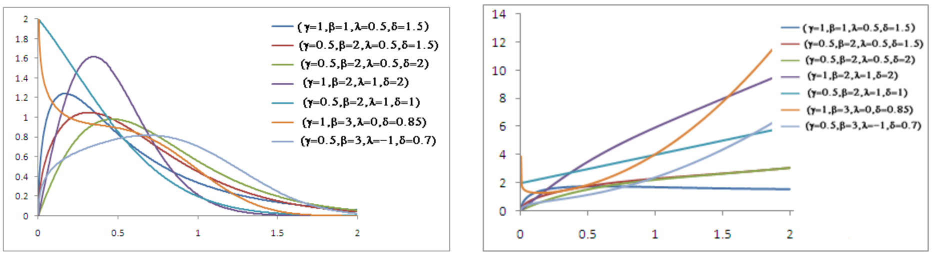
The density function of the distribution is obtained as

$$\begin{aligned} f(t) =& \delta[\{1 - exp(-\alpha t - \gamma t^{\beta})\}\{1 + \lambda exp(-\alpha t - \gamma t^{\beta})\}]^{\delta - 1}(\alpha + \beta\gamma t^{\beta - 1}) exp(-\alpha t - \gamma t^{\beta}) \\ & \times [1 - \lambda + 2\lambda exp(-\alpha t - \gamma t^{\beta})], t \geq 0. \end{aligned} \tag{5}$$

The hazard rate and the hazard function of the distribution are as follows:

$$\begin{aligned} r(t) =& \delta[\{1 - exp(-\alpha t - \gamma t^{\beta})\}1 + \lambda exp(-\alpha t - \gamma t^{\beta})]^{\delta - 1}(\alpha t + \beta\gamma t^{\beta - 1}) exp(-\alpha t - \gamma t^{\beta}) \\ & \times [1 - \lambda + 2\lambda exp(-\alpha t - \gamma t^{\beta})][1 - \{exp(-\alpha t - \gamma t^{\beta})\}^(\delta)\{1 + \lambda exp(-\alpha t - \gamma t^{\beta})\}^{\delta}]^(-1), t \geq 0, \\ H(t) =& -\ln[1 - \lambda + 2\lambda exp(-\alpha t - \gamma t^{\beta})][1 - \{-exp(-\alpha t - \gamma t^{\beta})\}^{\delta}\{1 + \lambda exp(-\alpha t - \gamma t^{\beta})\}^{\delta}], t \geq 0. \end{aligned}$$

Plots of the p.d.f. and the hazard rate are given in Figure 1. Figure 1a exhibits the diverse shapes of the exponentiated transmuted modified Weibull density for different choices of the parameters. Figure 1b shows that for almost all the parameter combinations considered the distribution exhibits monotonic hazard rate.



(a) Probability density function. (b) Hazard rate function.

Figure 1: Exponentiated Transmuted Modified Weibull Distribution when $\alpha = 1$.

By proper selection of the model parameters we can get a number of distributions as special cases as shown below:

Parameters	Distribution
$\delta = 1$	Transmuted modified Weibull
$\delta = 1, \alpha = 0$	Transmuted Weibull
$\delta = 1, \beta = 1$	Transmuted exponential distribution
$\alpha = 0$	Exponentiated transmuted Weibull
$\beta = 1$	Exponentiated transmuted exponential
$\lambda = 0$	Exponentiated modified Weibull
$\lambda = 0, \alpha = 0$	Exponentiated Weibull
$\lambda = 0, \beta = 1$	Exponentiated exponential
$\delta = 1, \lambda = 0$	Modified Weibull
$\delta = 1, \lambda = 0, \beta = 2$	Linear failure rate distribution
$\delta = 1, \lambda = 0, \alpha = 0$	Weibull
$\delta = 1, \lambda = 0, \alpha = 0, \beta = 2$	Rayleigh
$\delta = 1, \lambda = 0, \beta = 1$	Exponential

3. Quantile Function and Simulation

For any random variable X with cumulative distribution function $F(\cdot)$, for a given q, $0 \leq q \leq 1$, the quantile function returns the threshold value x such that $F(x) = q$. This function is useful in statistical applications and Monte Carlo simulation. For statistical applications, one needs to know the key percentage points of a given distribution, like the median and the 25% and 75% quartiles. The quantile function is also helpful in examining the fit of a given data set to a theoretical distribution. This is done by using the quantile-quantile (Q-Q) plot.

The quantile function corresponding to the ETMW distribution (4) is given by

$$t_q = F^{-1}(q) = G^{-1}(q^{1/\delta}), \tag{6}$$

where $G(x) = [1 - exp(-\alpha t - \gamma t^{\beta})][1 + \lambda exp(-\alpha t - \gamma t^{\beta})]$ is the cumulative distribution function of the transmuted modified Weibull distribution investigated by Khan and King [6].

Thus, t_q is the $(q^{1/\delta})$-th quantile of a transmuted modified Weibull distribution, and, from Khan and King [6], t_q is the real solution to the equation

$$\gamma t_q^{\beta} + \alpha t_q + \ln(z^*) = 0, \tag{7}$$

where

$$z^* = 1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q^{1/\delta}}}{2\lambda}. \tag{8}$$

For $\beta = 2, t_q$ is given by

$$t_q = \frac{-\alpha + \sqrt{\alpha^2 - 4\gamma \ln(z^*)}}{2\gamma}, \tag{9}$$

where z^* is given by (8).

Thus, for $\beta = 2$, the median of the distribution has the form

$$t_{0.5} = \frac{-\alpha + \sqrt{\alpha^2 - 4\gamma \ln[\{\sqrt{(1+\lambda)^2 - 2^{2-1/\delta}\lambda} - (1-\lambda)\}/2\lambda]}}{2\gamma}.$$

In order to simulate from the ETMW distribution, we have to solve for t_q from (7) for a random proportion q. However, for $\beta = 2$, simulation is straight forward from (9).

4. Moment Generating Function

We can express the moment generating function $M(t^*)$ of the ETMW distribution in terms of the moment generating function of the modified Weibull distribution as follows:

$$\begin{aligned} M(t^*) =& \int_0^{\infty} exp(t^* t) f(t) dt \\ =& \delta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A(i,j;\delta,\lambda) [\frac{1-\lambda}{(i+j+1)} M_{MW}(t^*; (i+j+1)\alpha, (i+j+1)\gamma, \beta) \\ &+ \frac{2\lambda}{(i+j+2)} M_{MW}(t^*; (i+j+2)\alpha, (i+j+2)\gamma, \beta)], \end{aligned}$$

where $M_{MW}(t^*; \alpha, \gamma, \beta)$ denotes the moment generating function of a modified Weibull distribution with c.d.f. (2), and

$$A(i,j;\delta,\lambda) = (-1)^i \frac{\{\Gamma(\delta-1)\}^2}{\Gamma(i)\Gamma(j)\Gamma(\delta-i-1)\Gamma(\delta-j-1)} \lambda^j.$$

From Sarhan and Zaindin [8], we have

$$M_{MW}(t^*;\alpha,\gamma,\beta) = \sum_{k=0}^{\infty}\frac{(-\gamma)^k}{k!}\left[\frac{\alpha\Gamma(k\beta+1)}{(\alpha-t^*)^{k\beta+1}}+\frac{\gamma\beta\Gamma(k+1)\beta}{(\alpha-t^*)^{(k+1)\beta}}\right],\ \text{for } \alpha,\gamma>0,\alpha>t^*$$
$$= \sum_{k=0}^{\infty}\frac{t^{*k}\Gamma(\frac{k}{\beta+1})}{\gamma^{\frac{k}{\beta}}}\ \text{for } \alpha=0,\gamma>0$$
$$= \frac{\alpha}{\alpha-t^*},\ \text{for } \gamma=0,\alpha>t^*$$

Thus,

$$M(t^*) = \delta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}A(i,j;\delta,\lambda)[\frac{1-\lambda}{(i+j+1)}\sum_{k=0}^{\infty}\frac{(-(i+j+1)\gamma)^k}{k!}\{\frac{(i+j+1)\alpha\Gamma(k\beta+1)}{((i+j+1)\alpha-t^*)^{k\beta+1}}$$
$$= +\frac{(i+j+1)\gamma\beta\Gamma(k+1)\beta}{((i+j+1)\alpha-t^*)^{(k+1)\beta}}+\frac{2\lambda}{(i+j+2)}\sum_{k=0}^{\infty}\frac{(-(i+j+2)\gamma)^k}{k!}\times$$
$$\{\frac{(i+j+2)\alpha\Gamma(k\beta+1)}{((i+j+2)\alpha-t^*)^{k\beta+1}}+\frac{(i+j+2)\gamma\beta\Gamma(k+1)\beta}{((i+j+2)\alpha-t^*)^{(k+1)\beta}}\}],\ \text{for } \alpha,\gamma>0,\alpha>t^*,$$
$$= \delta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}A(i,j;\delta,\lambda)[\frac{1-\lambda}{(i+j+1)}\sum_{k=0}^{\infty}\frac{t^{*k}\Gamma(\frac{k}{\beta+1})}{\{(i+j+1)\gamma\}^{\frac{k}{\beta}}}$$
$$+\frac{2\lambda}{(i+j+2)}\sum_{k=0}^{\infty}\frac{t^{*k}\Gamma(\frac{k}{\beta+1})}{\{(i+j+2)\gamma\}^{\frac{k}{\beta}}},\ \text{for } alpha=0,\gamma>0,$$
$$= \delta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}A(i,j;\delta,\lambda)[\frac{1-\lambda}{(i+j+1)}\frac{(i+j+1)\alpha}{\{(i+j+1)\alpha-t^*\}}$$
$$+\frac{2\lambda}{(i+j+2)}\frac{(i+j+2)\alpha}{\{(i+j+2)\alpha-t^*\}}],\ \text{for } \gamma=0,\alpha>t^* \tag{10}$$

The moments can be independently derived as follows:

$$\mu_r = E\{X^r\}$$
$$= \delta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}A(i,j;\delta,\lambda)[\frac{1-\lambda}{i+j+1}\mu_r^{MW}((i+j+1)\alpha,(i+j+1)\gamma,\beta)$$
$$+\frac{2\lambda}{i+j+2}\mu_k^{MW}((i+j+2)\alpha,(i+j+2)\gamma,\beta),$$

where $\mu_r^{MW}(\alpha,\gamma,\beta)$ denotes the r-th moment of the modified Weibull distribution (2).

From Sarhan and Zaindin [8] we have

$$\mu_r^{MW}(\alpha,\gamma,\beta) = \begin{cases} \sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\left[\frac{\Gamma(k\gamma+r+1)}{\alpha^{k\gamma+r}}+\beta\gamma\frac{\Gamma(r+k\gamma+\gamma)}{\alpha^{k\gamma+\gamma+r}}\right], & \text{for } \alpha,\beta>0 \\ \frac{\Gamma(\frac{r}{\gamma}+1)}{\beta^{\frac{r}{\gamma}}}, & \text{for } \alpha=0,\beta>0 \\ \frac{\Gamma(r+1)}{\alpha^r}, & \text{for } \alpha>0,\beta=0. \end{cases} \tag{11}$$

Hence, we get

$$
\begin{aligned}
\mu_r =&\delta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A(i,j;\delta,\lambda)[\frac{1-\lambda}{i+j+1}\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\{\frac{\Gamma(k(i+j+1)\gamma+r+1)}{\{(i+j+1)\alpha\}^{k(i+j+1)\gamma+r}}\\
&(i+j+1)\beta\gamma\frac{\Gamma(r+\{k(i+j+1)+1\}\gamma)}{\{(i+j+1)\alpha\}^{\{k(i+j+1)+1\}\gamma+r}}\}+\frac{2\lambda}{i+j+2}\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\\
&\times\frac{\Gamma(k(i+j+2)\gamma+r+2)}{\{(i+j+2)\alpha\}^{k(i+j+2)\gamma+r}}+(i+j+2)\beta\gamma\frac{\Gamma(r+\{k(i+j+2)+1\}\gamma)}{\{(i+j+2)\alpha\}^{\{k(i+j+2)+1\}\gamma+r}}\}, \text{for } \alpha,\beta>0\\
=&\delta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A(i,j;\delta,\lambda)[\frac{1-\lambda}{i+j+1}(\frac{r}{(i+j+1)\gamma}+1)/\beta^{\frac{r}{(i+j+1)\gamma}}\\
&+\frac{2\lambda}{i+j+2}(\frac{r}{(i+j+2)\gamma}+1)/\beta^{\frac{r}{(i+j+2)\gamma}}, \text{for } \alpha=0,\beta>0\\
=&\delta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A(i,j;\delta,\lambda)[\frac{1-\lambda}{i+j+1}\frac{\Gamma(r+1)}{\{(i+j+1)\alpha\}^r}+\frac{2\lambda}{i+j+2}\frac{\Gamma(r+1)}{\{(i+j+2)\alpha\}^r}], \text{for } \alpha>0,\beta=0
\end{aligned}
\tag{12}
$$

5. Mean Deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. If X has a ETMW distribution, then we can derive the mean deviations about the mean $\mu = E(X)$ and about the median M as

$$\eta_1 = \int_0^{\infty} |\, x-\mu \,| f(x)dx, \eta_2 = \int_0^{\infty} |\, x-M \,| f(x)dx.$$

The mean of the distribution is obtained from (12) by putting $r = 1$, and the median is obtained by solving the equation

$$\gamma M^{\beta} + \alpha M = -\ln(\nu_0),$$

where ν_0 is given by

$$
\begin{aligned}
\nu_0 &= \frac{-(1-\lambda)+\sqrt{(1-\lambda)^2+4\lambda(1-(0.5)^{1/\delta})}}{2\lambda}, \text{ if } \lambda \neq 0\\
&= 1-(0.5)^{1/\delta}, \text{ if } \lambda = 0.
\end{aligned}
$$

6. Order Statistics

Let $T_{(1)} < T_{(2)} < \ldots < T_{(n)}$ be the ordered observations in a random sample of size n drawn from the exponentiated transmuted modified Weibull distribution with cdf $F(t)$, given by (4) and density $f(t)$, given by (5).

The pdf of $T(r)$, $1 \le r \le n$, is given by

$$\begin{aligned} f_{(r)}(t) =& \frac{n!}{(r-1)!(n-r)!}[F(t)]^{r-1}[1-F(t)]^{n-r}f(t) \\ =& \frac{n!}{(r-1)!(n-r)!}\delta[\{1-exp(-\alpha t-\gamma t^{\beta})\}\{1+\lambda exp(-\alpha t-\gamma t^{\beta}\}]^{\delta r-1} \\ & \times[1-\{1-exp(-\alpha t-\gamma t^{\beta})\}^{\delta}\{1+\lambda exp(-\alpha t-\gamma t^{\beta})\}^{\delta}]^{n-r}(\alpha t+\beta\gamma t^{\beta-1})exp(-\alpha t-\gamma t^{\beta}) \\ & \times[1-\lambda+2\lambda exp(-\alpha t-\gamma t^{\beta})], t\ge 0, \alpha,\beta,\delta>0, \lambda\in[-1,1]. \end{aligned}$$

Hence the pdf of the smallest and the largest order statistics are as follows:

$$\begin{aligned} f_{(1)}(t) =& n\delta[\{1-exp(-\alpha t-\gamma t^{\beta})\}\{1+\lambda exp(-\alpha t-\gamma t^{\beta})\}]^{\delta-1} \\ & \times[1-\{1-exp(-\alpha t-\gamma t^{\beta})\}^{\delta}\{1+\lambda exp(-\alpha t-\gamma t^{\beta})\}^{\delta}]^{n-1}(\alpha t+\beta\gamma t^{\beta-1})exp(-\alpha t-\gamma t^{\beta}) \\ & \times[1-\lambda+2\lambda exp(-\alpha t-\gamma t^{\beta})], \\ f_{(n)}(t) =& n\delta[\{1-exp(-\alpha t-\gamma t^{\beta})\}\{1+\lambda exp(-\alpha t-\gamma t^{\beta})\}]^{\delta n-1}(\alpha t+\beta\gamma t^{\beta-1}) \\ & \times[1-\lambda+2\lambda exp(-\alpha t-\gamma t^{\beta}], \\ & t\ge 0, \alpha,\beta,\delta>0, \lambda\in[-1,1]. \end{aligned}$$

The density of the $(r+1)$-th order statistic can be expressed as a function of the density of the r-th order statistic from the following relation:

$$f_{(r+1)}(t) = \frac{n-r}{r}\left[\{1-(1-exp(-\alpha t-\gamma t^{\beta}))^{\delta}(1+\lambda exp(-\alpha t-\gamma t^{\beta}))^{\delta}\}]^{-1}-1\right]f_{(r)}(t).$$

The moments of the order statistics can be easily written in terms of the moments of the modefied Weibull distribution by proceeding as follows:
We can write $f_{(r)}(t)$ as

$$\begin{aligned} f_{(r)}(t) =& \frac{n!}{(r-1)!(n-r)!}\delta\sum_{i=0}^{\infty}(-1)^i\binom{n-r}{i}[\{1-exp(-\alpha t-\gamma t^{\beta})\}\{1+\lambda exp(-\alpha t-\gamma t^{\beta})\}]^{\delta(i+r)-1} \\ & \times(\alpha+\gamma\beta t^{\beta-1})exp(-\alpha t-\gamma t^{\beta})[1-\lambda+2\lambda exp(-\alpha t-\gamma t^{\beta})] \\ =& \frac{n!}{(r-1)!(n-r)!}\sum_{i=0}^{\infty}(-1)^i\binom{n-r}{i}g(t;\alpha,\gamma,\beta,\lambda,\delta(i+r)), t\ge 0, \alpha,\beta,\gamma,\delta>0, \lambda\in[-1,1], \end{aligned}$$

where $g(t;\alpha,\gamma,\beta,\lambda,\delta(i+r))$ denote the density function of an exponentiated transmuted modified Weibull distribution with shape paameters (α,β,γ), transmuting parameter λ and exponentiatinf parameter $\delta(i+r)$.

Hence, using (12) we have

$$E\left(T_{(s)}^{r}\right)=\frac{n!}{(s-1)!(n-s)!}\sum_{u=0}^{\infty}(u+s)\delta$$
$$\times[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}A(i,j;(u+s)\delta,\lambda)[\frac{1-\lambda}{i+j+1}\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\{\frac{\Gamma(k(i+j+1)\gamma+r+1)}{\{(i+j+1)\alpha\}^{k(i+j+1)\gamma+r}}$$
$$+(i+j+1)\beta\gamma\frac{\Gamma(r+\{k(i+j+1)+1\}\gamma)}{\{(i+j+1)\alpha\}^{\{k(i+j+1)+1\}\gamma+r}}\}+\frac{2\lambda}{i+j+2}\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}$$
$$\times\{\frac{\Gamma(k(i+j+2)\gamma+r+2)}{\{(i+j+2)\alpha\}^{k(i+j+2)\gamma+r}}+(i+j+2)\beta\gamma\frac{\Gamma(r+\{k(i+j+2)+1\}\gamma)}{\{(i+j+2)\alpha\}^{\{k(i+j+2)+1\}\gamma+r}}\}],$$
$$\text{for } \alpha,\beta>0$$
$$=\frac{n!}{(s-1)!(n-s)!}\sum_{u=0}^{\infty}(u+s)\delta[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}A(i,j;(u+s)\delta,\lambda)$$
$$\times\{\frac{1-\lambda}{i+j+1}(\frac{r}{(i+j+1)\gamma}+1)/\beta^{\frac{r}{(i+j+1)\gamma}}+\frac{2\lambda}{i+j+2}(\frac{r}{(i+j+2)\gamma}+1)/\beta^{\frac{r}{(i+j+2)\gamma}}\}],$$
$$\text{for } \alpha=0,\beta>0$$
$$=\frac{n!}{(s-1)!(n-s)!}\sum_{u=0}^{\infty}(u+s)\delta[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}A(i,j;(u+s)\delta,\lambda)\{\frac{1-\lambda}{i+j+1}\frac{\Gamma(r+1)}{\{(i+j+1)\alpha\}^r}$$
$$+\frac{2\lambda}{i+j+2}\frac{\Gamma(r+1)}{\{(i+j+2)\alpha\}^r}\}]text,for\alpha>0,\beta=0.$$

In addition, we can calculate the L-moments [5], which are summary statistics for probability distributions and data samples but have several advantages over ordinary moments. For example, they apply for any distribution having a finite mean and no higher-order moments need be finite. The rth L-moment is computed from the linear combinations of the ordered data values as given below:

$$\rho_r=\sum_{u=0}^{\infty}(-1)^{r-u-1}\binom{r-1}{u}\binom{r+u-1}{u}\theta_u,$$

where $\theta_u=E[TF(T)^u]$.

Thus, $\rho_1=\theta_0$, $\rho_2=2\theta_1-\theta_0$, $\rho_3=6\theta_2-6\theta_1+\theta_0$, and $\rho_4=20\theta_3-30\theta_2+12\theta_1-\theta_0$. In general, $\theta_k=(k+1)^{-1}E(T_{k+1:k+1})$, which can be computed from (6) by substituting $n=s=k+1$ and $r=1$.

7. Reliability

A stress-strength model describes the life of a component having a random strength X_1 and subjected to a random stress X_2. The component functions satisfactorily for $X_1 > X_2$ and fails when $X_1 < X_2$. The probability $R = Pr(X_1 > X_2)$ defines the component reliability. Stress-strength models have many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

Consider X_1 and X_2 to be independently distributed, with $X_1 \sim ETMW(\alpha_1, \gamma_1, \beta, \lambda_1, \delta_1)$ and $X_2 \sim ETMW(\alpha_2, \gamma_2, \beta, \lambda_2, \delta_2)$. The c.d.f. F_1 of X_1 and the pdf f_2 of X_2 are obtained from (4) and (5), respectively. Then,

$$
\begin{aligned}
R =& Pr(X_1 > X_2) = \int_0^\infty f_2(y)[1 - F_1(y)]dy \\
=& 1 - \sum_{k,l=0}^{\infty} w_{k,l}^{(1)} A(k,l),
\end{aligned}
$$

where

$$
w_{k,l}^{(1)} = (-1)^k \binom{\delta_1}{k}\binom{\delta_1}{l}\lambda_1^l, w_{k,l}^{(2)} = (-1)^k \binom{\delta_2 - 1}{k}\binom{\delta_2 - 1}{l}\lambda_2^l,
$$

and

$$
\begin{aligned}
A(k,l) =& \int_0^\infty f_2(y) exp(-(k+l)\alpha_1 y - (k+l)\gamma_2 y^\beta) dy \\
=& \sum_{i,j=0}^{\infty} w_{i,j}^{(2)} \int_0^\infty (\alpha_2 + \gamma_2 \beta y^{\beta-1}) exp(-\{(k+l)\alpha_1 + (i+j+1)\alpha_2\}y \\
& - \{(k+l)\gamma_1 + (i+j+1)\gamma_2\} y^\beta)(1 - \lambda_2 + 2\lambda_2 exp(-\alpha_2 y - \gamma_2 y^\beta)) dy \\
=& \sum_{i,j=0}^{\infty} w_{i,j}^{(2)} [\alpha_2 \{(1-\lambda_2)\mu_1^{MW}((k+l)\alpha_1 + (i+j+1)\alpha_2, (k+l)\gamma_1 + (i+j+1)\gamma_2, \beta) \\
& + 2\lambda_2 \mu_1^{MW}((k+l)\alpha_1 + (i+j+2)\alpha_2, (k+l)\gamma_1 + (i+j+2)\gamma_2, \beta)\} \\
& + \gamma_2 \{(1-\lambda_2)\mu_\beta^{MW}((k+l)\alpha_1 + (i+j+1)\alpha_2, (k+l)\gamma_1 + (i+j+1)\gamma_2, \beta) \\
& + 2\lambda_2 \mu_\beta^{MW}((k+l)\alpha_1 + (i+j+2)\alpha_2, (k+l)\gamma_1 + (i+j+2)\gamma_2, \beta)\}],
\end{aligned}
$$

with $\mu_r^{MW}(\alpha, \gamma, \beta)$ given by (11).

In particular, if $\alpha_1 = \alpha_2$ and $\gamma_1 = \gamma_2$, we have

$$
A(k,l) = \sum_{i,j=0}^{\infty} w_{i,j}^{(2)} [\frac{(1-\lambda_2)}{(i+j+k+l+1)} + \frac{2\lambda_2}{(i+j+k+l+2)}].
$$

8. Least Squares Estimation

Let $T_{(1)} < T_{(2)} < \ldots < T_{(n)}$ denote the ordered observations in a random sample of size n drawn from the ETMW$(\alpha, \gamma, \beta, \lambda, \delta)$ distribution with distribution function $F(\cdot)$, given by (4). Then,

$$E[F(T_{(i)})] = \frac{i}{n+1}, i = 1, 2, \ldots, n.$$

The least square estimators are obtained by minimizing

$$\begin{aligned} D(\boldsymbol{\theta}) &= \sum_{i=1}^{n}\left(F(T_{(i)}) - \frac{i}{n+1}\right)^2 \\ &= \sum_{i=1}^{n}\left((1 - exp(-\alpha T_{(i)} - \gamma T_{(i)}^{\beta}))^{\delta}(1 + \lambda exp(-\alpha T_{(i)} - \gamma T_{(i)}^{\beta}))^{\delta} - \frac{i}{n+1}\right)^2. \end{aligned}$$

Writing $V_i = exp(-\alpha T_{(i)} - \gamma T_{(i)}^{\beta})$, the normal equations to be satisfied by the estimators are as follows:

$$\sum_{i=1}^{n}\left((1-V_i)^{\delta}(1+\lambda V_i)^{\delta} - \frac{i}{n+1}\right)(1-V_i)^{\delta-1}(1+\lambda V_i)^{\delta-1}T_{(i)}V_i(1-\lambda+2\lambda V_i) = 0$$

$$\sum_{i=1}^{n}\left((1-V_i)^{\delta}(1+\lambda V_i)^{\delta} - \frac{i}{n+1}\right)(1-V_i)^{\delta-1}(1+\lambda V_i)^{\delta-1}T_{(i)}^{\beta}V_i(1-\lambda+2\lambda V_i) = 0$$

$$\sum_{i=1}^{n}\left((1-V_i)^{\delta}(1+\lambda V_i)^{\delta} - \frac{i}{n+1}\right)(1-V_i)^{\delta-1}(1+\lambda V_i)^{\delta-1}T_{(i)}^{\beta}\ln(T_{(i)})V_i(1-\lambda+2\lambda V_i) = 0$$

$$\sum_{i=1}^{n}\left((1-V_i)^{\delta}(1+\lambda V_i)^{\delta} - \frac{i}{n+1}\right)(1-V_i)^{\delta}(1+\lambda V_i)^{\delta-1}V_i = 0$$

$$\sum_{i=1}^{n}\left((1-V_i)^{\delta}(1+\lambda V_i)^{\delta} - \frac{i}{n+1}\right)(1-V_i)^{\delta}(1+\lambda V_i)^{\delta}[\ln(1-V_i) + \ln(1+\lambda V_i)] = 0.$$

9. Maximum Likelihood Method of Estimation

Consider a random sample $(T_1, T_2, \ldots, T_n)$ of size n taken from the distribution ETMW$(\alpha, \gamma, \beta, \lambda, \delta)$ with density function (5).

For given $T_i = t_i, i = 1, 2, \ldots, n$, the log-likelihood function for $\boldsymbol{\theta} = (\alpha, \gamma, \beta, \lambda, \delta)$ is

$$\begin{aligned} l(\boldsymbol{\theta}) =& n\ln\delta + (\delta - 1)\{\sum_{i=1}^{n}\ln(1 - exp(-\alpha t_i - \gamma t_i^{\beta})) + \sum_{i=1}^{n}\ln(1 + \lambda exp(-\alpha t_i - \gamma t_i^{\beta}))\} \\ &+ \sum_{i=1}^{n}\ln(\alpha + \gamma\beta t_i^{\beta-1}) + \sum_{i=1}^{n}\ln(1 - \lambda + 2\lambda exp(-\alpha t_i - \gamma t_i^{\beta})) - \sum_{i=1}^{n}(\alpha t_i + \gamma t_i^{\beta}). \end{aligned}$$

Writing $\nu_i = exp(-\alpha t_i - \gamma t_i^{\beta})), i = 1, 2, \ldots, n$, the log-likelihood equations are obtained as follows:

$$\sum_{i=1}^{n} t_i = -2\lambda \sum_{i=1}^{n} \frac{t_i \nu_i}{(1-\lambda+2\lambda\nu_i)} + (1-\delta)\sum_{i=1}^{n} \frac{t_i \nu_i (1-\lambda+2\lambda\nu_i)}{(1-\nu_i)(1+\lambda\nu_i)} + \sum_{i=1}^{n} \frac{1}{(\alpha+\gamma\beta t_i^{\beta-1})} \tag{13}$$

$$\sum_{i=1}^{n} t_i^{\beta} = -2\lambda \sum_{i=1}^{n} \frac{t_i^{\beta} \nu_i}{(1-\lambda+2\lambda\nu_i)} + (1-\delta)\sum_{i=1}^{n} \frac{t_i^{\beta} \nu_i (1-\lambda+2\lambda\nu_i)}{(1-\nu_i)(1+\lambda\nu_i)} + \beta \sum_{i=1}^{n} \frac{t_i^{\beta-1}}{(\alpha+\gamma\beta t_i^{\beta-1})} \tag{14}$$

$$\sum_{i=1}^{n} t_i^{\beta} \ln t_i = -2\lambda \sum_{i=1}^{n} \frac{(t_i^{\beta} \ln t_i)\nu_i}{(1-\lambda+2\lambda\nu_i)} + (1-\delta)\sum_{i=1}^{n} \frac{(t_i^{\beta} \ln t_i)\nu_i (1-\lambda+2\lambda\nu_i)}{(1-\nu_i)(1+\lambda\nu_i)} \tag{15}$$

$$+ \sum_{i=1}^{n} \frac{t_i^{\beta-1}(1+\beta \ln t_i)}{(\alpha+\gamma\beta t_i^{\beta-1})} \tag{16}$$

$$(\delta-1)\sum_{i=1}^{n} \frac{\nu_i}{1+\lambda\nu_i} = \sum_{i=1}^{n} \frac{1-2\nu_i}{1-\lambda+2\lambda\nu_i} \tag{17}$$

$$\sum_{i=1}^{n} \ln(1-\nu_i) + \sum_{i=1}^{n} \ln(1+\lambda exp\, \nu_i) = -\frac{n}{\delta}. \tag{18}$$

Solving the non-linear system of equations (13)-(18) we obtain the maximum likelihood estimate $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\gamma}, \hat{\beta}, \hat{\lambda}, \hat{\delta})$ of $\boldsymbol{\theta}$.

Under certain regularity conditions, $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} \text{Normal}(0, I^{-1}(\boldsymbol{\theta}))$ (here $\xrightarrow{d}$ stands for convergence in distribution), where $I(\boldsymbol{\theta})$ denotes the information matrix given by

$$I(\boldsymbol{\theta}) = E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta\partial\theta'}\right). \tag{19}$$

This information matrix $I(\boldsymbol{\theta})$ may be approximated by the observed information matrix

$$I(\hat{\boldsymbol{\theta}}) = E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta\partial\theta'}\right)_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}. \tag{20}$$

Hence, using the approximation $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \sim Normal(0, I^{-1}(\hat{\boldsymbol{\theta}}))$, one can carry out tests and find confidence regions for functions of some or all of the parameters in $\boldsymbol{\theta}$.

10. Simulation Study

A simulation study is carried out to investigate the performance of the least square (LS) estimators and the ML estimators. We take sample sizes to be $n = 15, 25, 50, 100, 250, 500$ and generate observations from a ETMW distribution with parameters $\alpha = 1$, $\gamma = 1$, $\beta = 2.5$, $\lambda = 0.5$, $\delta = 0.75$. For each sample we compute the estimates of the parameters using the two methods. The process is repeated $N = 1000$ times, and the average mean squared error is computed as follows:

$$AMSE(T) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\boldsymbol{\theta}}_i(T) - \boldsymbol{\theta})'(\hat{\boldsymbol{\theta}}_i(T) - \boldsymbol{\theta}), \tag{21}$$

where T denotes the method used to estimate $\boldsymbol{\theta} = (\alpha, \gamma, \beta, \lambda, \delta), i$ denotes the sample repetition number, $i = 1, 2, \ldots, N$, and $\hat{\boldsymbol{\theta}}_i(T)$ is the corresponding estimate. Table 1 gives the AMSEs for the two methods of estimation.

Table 1: Average Mean Squared Errors for the Method of Least Squares and Maximum Likelihood Method for Estimating $\boldsymbol{\theta}$

Sample size	***AMSE(LS)***	***AMSE(ML)***
10	3.2651	0.9985
15	1.4247	0.4386
25	0.8769	0.1027
50	0.4317	0.0285
100	0.0621	0.0075
250	0.0860	0.0010
100	0.0009	0.0001

The above table shows that as the sample size increases, the average mean squared errors decrease. This verifies the consistency properties of the estimates. Further, for each sample size $AMSE(LS) > AMSE(ML)$. Thus, we may conclude that the maximum likelihood method provides better estimators than the least square method. However, the difference between $AMSE(LS)$ and $AMSE(ML)$ decrease as the sample size increases.

11. Application of the Exponentiated Transmuted Modified Weibull Distribution

In this section we illustrate the usefulness of the exponentiated transmuted modified Weibull distribution for modelling real life data. The data set relates to the time-to-failure of 50 devices, and is taken from Aarset [1]:

Table 2: The time-to-failure of 50 devices

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

The Weibull (W), modified Weibull (M), transmuted modified Weibull(T) and exponentiated transmuted modified Weibull (E) distributions are fitted to the data and the MLEs of the parameters are given in Table 3. The values of the log-likelihood, Kolmogorov-Smirnov statistic (K-S), Akaike information criterion (AIC) and Bayesian information criterion (BIC) for the different fitted distributions are also given, and show that the ETMW distribution gives a better fit than the others. The same is also evident from Figure 2, which compares the cumulative distribution curves of the fitted distributions with that of the empirical distribution.

Table 3: The estimated parameters and the log-likelihood, K-S, AIC, BIC values for the different fitted distributions

	α	γ	β	λ	δ	$-l(\boldsymbol{\theta})$	**K-S**	***AIC***	***BIC***
W	0	0.0268	0.9499	0	1	241.002	0.0750	486.004	489.8280
M	0.012	2.159×10^{-8}	4.014	0	1	230.15	0.0739	466.30	472.0361
T	0.0122	1.0006×10^{-8}	4.1924	0.0747	1	229.06	0.0730	466.12	473.7681
E	0.0056	8.96×10^{-9}	4.2448	-0.4706	0.4553	222.61	0.0655	455.22	464.7801

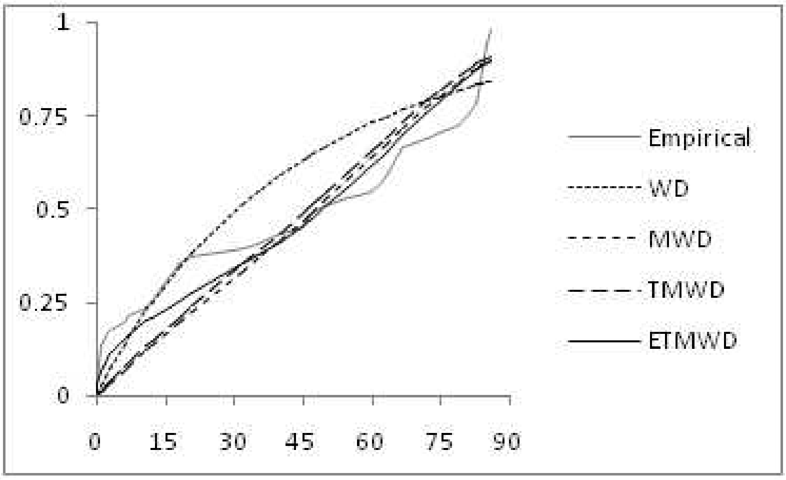


Figure 2: Comparison of the CDFs of the Fitted Distributions with the Empirical CDF

12. Discussion

In this paper, we introduce a new generalization of the Weibull distribution called exponentiated transmuted modified Weibull distribution and discuss its intrinsic properties. The distribution is very flexible in the sense that it exhibits both increasing and decreasing failure rates depending on its parameters.

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