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Observations on Some Special Matrices and Polynomials

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Abstract. In the current paper we focus on the study of three special matrices and two symmetric polynomials. As a consequence, a recurrence relation satisfied by the entries of the $n \times n$ inverse matrix, Q_n of the $n \times n$ symmetric Pascal matrix, P_n is obtained. Moreover, a new proof for El-Mikkawy conjecture [14] is investigated. Finally, some identities are discovered.

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1. Introduction and Basic Definitions

Matrices play an important role in all branches of science, engineering, social science and management. Matrices are used for data classification by which many problems are solved using computers. There are many special types of matrices such as the Pascal [1], Vandermonde [10], Stirling [12], and others. These matrices are of specific importance in many scientific and engineering applications. For example the Pascal matrix, which has been known since 1303, but has been studied carefully only recently [1], appears in combinatorics, image processing, signal processing, numerical analysis, probability and surface reconstruction. Much researches has been devoted to deal with such matrices and relations between them (see for instance, [3–5, 7, 9, 19, 21–25, 29–37]). The main objectives of the current paper is to introduce a recurrence relation for the inverse of the symmetric Pascal matrix, P_n of order n, some new identities and to give another proof for El-Mikkawy conjecture [14]. The paper is organized as follows: The main results are given in the next section. In Section 3, we present a new proof for El-Mikkawwy conjecture [14]. Moreover, new identities are obtained. Throughout this paper, δ_{ij} is the Kronecker delta which is equal to 1 or 0 according as $i = j$ or not. Also $\begin{pmatrix} n \\ r \end{pmatrix}$ denotes the binomial coefficient and $\mathbf{X}$ denotes the set $\{x_1, x_2, \ldots, x_n\}$.

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Definition 1. *[6]. For integer numbers n and k with $n \geq k \geq 0$, the Stirling numbers of the first kind, $s(n,k)$ and of the second kind, $S(n,k)$ are defined respectively by:*

$$(x)_n = [x-n+1]_n = \sum_{k=0}^{n} s(n,k)x^k, \tag{1}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \tag{2}$$

where the falling factorial of x, $(x)_n$ and the rising factorial of x, $[x]_n$ are given respectively by:

$$(x)_n = \begin{cases} 1 & \text{if } n = 0, \\ x(x-1)(x-2)\dots(x-n+1) & \text{if } n \geq 1, \end{cases} \tag{3}$$

and

$$[x]_n = \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)(x+2)\dots(x+n-1) & \text{if } n \geq 1 \end{cases}. \tag{4}$$

It is well known that for integers $n, k \geq 0$, the $s(n,k)$ and $S(n,k)$ satisfy the following Pascal-type recurrence relations:

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \tag{5}$$

and

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{6}$$

subject to:

$$s(k,k) = S(k,k) = 1, 1 \leq k \leq n \tag{7}$$

and

$$s(n,k) = S(n,k) = \delta_{nk}, \text{ if } k = 0 \text{ or } n = 0. \tag{8}$$

Replace x by $-x$ in (1) gives

$$[x]_n = \sum_{k=0}^{n} c(n,k)x^k, \tag{9}$$

where $c(n,k) = (-1)^{n-k}s(n,k)$ is called the unsigned or signless Stirling number of the first kind. It can be shown that the rising factorial, $[x]_n$ and the falling factorial, $(x)_n$ are also related by:

$$[x]_n = \sum_{k=0}^{n} H(n,k)(x)_k, \tag{10}$$

where

$$H(n,k) = \frac{n!}{k!} \left(\begin{array}{c} n-1 \\ k-1 \end{array} \right). \tag{11}$$

The coefficients, $H(n,k)$ in (10) are called the Lah numbers. These numbers satisfy the following Pascal-type recurrence relation:

$$H(n,k) = H(n-1,k-1) + (n+k-1)H(n-1,k). \tag{12}$$

The Lah numbers also satisfy:

$$H(i,j) = \sum_{k=1}^{n} (-1)^{i+k} s(i,k) S(k,j), \quad 1 \le i,j \le n. \tag{13}$$

Definition 2 ([6]). *The Stirling matrix of the first kind, s_n and of the second kind, S_n are defined respectively by:*

$$s_n = \begin{cases} s(i,j) & \text{for } i \ge j, \\ 0 & \text{otherwise} \end{cases}, \tag{14}$$

and

$$S_n = \begin{cases} S(i,j) & \text{for } i \ge j, \\ 0 & \text{otherwise} \end{cases}. \tag{15}$$

For example

$$s_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix} \text{ and } S_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}.$$

It is known that $s_n S_n = S_n s_n = I_n$, where I_n is the $n \times n$ identity matrix. In other words, the matrices, s_n and S_n are inverse to each other. If $f_i(x) = x^i$, $1 \le i \le n$, then $S(i,k) = f_i[0,1,2,\ldots,k] = \frac{\Delta^k f_i(0)}{k!}$, where $f_i[0,1,2,\ldots,k]$ is the k-th divided difference of the function $f_i(x) = x^i$ which lies on the top of each column and Δ is the forward difference operator. For example, for $f_5(x) = x^5$, we have:

Table 1: Divided Differences and Stirling Numbers of the Second Kind

x	$f_5(x)$	First DD	Second DD	Third DD	Fourth DD	Fifth DD
0	0	1=S(5,1)	15=S(5,2)	25=S(5,3)	10=S(5,4)	1=S(5,5)
1	1	31	90	65	15	
2	32	211	285	125		
3	243	781	660			
4	1024	2101				
5	3125					

Thus,

$$f_5(x) = x^5 = (x)_1 + 15(x)_2 + 25(x)_3 + 10(x)_4 + (x)_5$$

and

$$\Delta f_5(0)=1,\Delta^2 f_5(0)=30,\Delta^3 f_5(0)=150,\Delta^4 f_5(0)=240,$$

and

$$\Delta^5 f_5(0)=120.$$

Definition 3 ([16]). *The elementary symmetric polynomial $\sigma_r^{(n)}$ and the complete symmetric polynomial $\tau_r^{(n)}$ in $x_1,x_2,\ldots,x_n$ are defined respectively by:*

$$\sigma_r^{(n)}(\mathbf{x}):=\begin{cases}0 & \text{if } r>n \text{ or } n<0 \text{ or } r<0,\\ 1 & \text{if } r=0,\\ \sum\limits_{1\le i_1<i_2<\ldots<i_r\le n} x_{i_1}x_{i_2}\ldots x_{i_r} & \text{if } 1\le r\le n.\end{cases} \tag{16}$$

and

$$\tau_r^{(n)}(\mathbf{X}):=\begin{cases}0 & \text{if } r<0 \text{ or } n<0 \text{ or } (n=0 \text{ and } r>0),\\ 1 & \text{if } r=0,\\ \sum\limits_{1\le i_1\le i_2\le\ldots\le i_r\le n} x_{i_1}x_{i_2}\ldots x_{i_r} & \text{if } r\ge 1.\end{cases} \tag{17}$$

It should be noticed that each $\sigma_r^{(n)}$ has $\binom{n}{r}$ terms and each $\tau_r^{(n)}$ has $\binom{n+r-1}{r}$ terms. Moreover these polynomials can be expressed as:

$$\sigma_r^{(n)}(\mathbf{x}):=\sum_{\substack{k_1+k_2+\ldots+k_n=r\\ k_1,k_2,\ldots,k_n\in\{0,1\}}} x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n},0\le r\le n. \tag{18}$$

and

$$\tau_r^{(n)}(\mathbf{X}):=\sum_{\substack{d_1+d_2+\ldots+d_n=r\\ d_1,d_2,\ldots,d_n\in\{0,1,\ldots,r\}}} x_1^{d_1}x_2^{d_2}\ldots x_n^{d_n},r\ge 0. \tag{19}$$

The falling factorial $(x)_n$ in (3) can be written in the form:

$$(x)_n=\sum_{k=0}^{n}(-1)^{n-k}\sigma_{n-k}^{(n)}(0,1,\ldots,n-1)x^k. \tag{20}$$

Comparing the coefficients of x^k in (1) and (20), yields

$$c(n,k)=\sigma_{n-k}^{(n)}(0,1,\ldots,n-1)=\sigma_{n-k}^{(n-1)}(1,2,\ldots,n-1). \tag{21}$$

It can also be shown that

$$S(n,k)=\tau_{n-k}^{(k)}(1,2,\ldots,k). \tag{22}$$

Also,

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n. \tag{23}$$

For any $x_j \in \mathbf{X}$, we have

$$\begin{aligned}\sigma_i^{(n)}(x_1, x_2, \ldots, x_n) =&\sigma_i^{(n-1)}(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \\ &+ x_j \sigma_{i-1}^{(n-1)}(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).\end{aligned} \tag{24}$$

Partial differentiation for both sides of (24) with respect to x_j, $j \in \{1, 2, \ldots, n\}$ gives

$$\frac{\partial}{\partial x_j}\sigma_i^{(n)} = \sigma_{i,j}^{(n)} = \sigma_{i-1}^{(n-1)}(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n). \tag{25}$$

Therefore by using (16), we obtain

$$\sigma_{ij}^{(n)}(\mathbf{X}) = \begin{cases} 0 & \text{if } i > n \text{ or } i < 0 \text{ or } n < 0, \\ 1 & \text{if } i = 1, \\ \displaystyle\sum_{\substack{1 \le r_1 < r_2 < \ldots < r_{i-1} \le n \\ r_1, r_2 \ldots, r_{i-1} \ne j}} x_{r_1} x_{r_2} \ldots x_{r_{i-1}} & \text{if } 2 \le i \le n. \end{cases} \tag{26}$$

Definition 4 ([17]). *The Vandermonde matrix V of order n is a matrix of the form:*

$$V = \left(x_j^{i-1}\right)_{i,j=1}^{n}. \tag{27}$$

The Vandermonde determinant formula is well known, see for instance [26], in many text books and articles. It is given by:

$$\det(V) = \left|x_j^{i-1}\right|_{i,j=1}^{n} = \prod_{1 \le j < i \le n} (x_i - x_j). \tag{28}$$

From (28), we see that if the x_i are distinct, then this determinant, $\det(V)$ is nonzero and hence V is invertible. The explicit form of the inverse matrix, $V^{-1} = (\alpha_{ij})_{i,j=1}^{n}$ of the Vandermonde matrix V is given by [11]:

$$\alpha_{ij} =(-1)^{n-j} \frac{\sigma_{n-j+1,i}^{(n)}(x_1, x_2, \ldots, x_n)}{F_i} \tag{29}$$

$$= (-1)^{n-j} \frac{\sigma_{n-j}^{(n-1)}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{F_i}, \quad 1 \le i, j \le n, \tag{30}$$

where

$$F_i = \prod_{\substack{r=1 \\ r \ne i}}^{n} (x_i - x_r), \quad i = 1, 2, \ldots, n. \tag{31}$$

Consequently, the cost of the solution of the Vandermonde linear system

$$V[u_1 u_2 \ldots u_n]^T = [d_1 d_2 \ldots d_n]^T \tag{32}$$

is $O(n^2)$. Setting $x_k = k$, $1 \le k \le n$, in (30), yields:

$$\alpha_{ij} = \frac{(-1)^{i+j}}{(n-1)!}\binom{n-1}{i-1}\sigma_{n-j}^{(n-1)}(1,2,\ldots,i-1,i+1,\ldots,n), \quad 1 \le i,j \le n. \tag{33}$$

The Vandermonde matrix, V in (27) satisfies:

$$V = LU, \tag{34}$$

where $L = (L_{ij})_{i,j=1}^{n}$ is an $n \times n$ lower triangular matrix given by:

$$L_{ij} = \begin{cases} \tau_{i-j}^{(j)}(x_1, x_2, \ldots, x_j) & \text{for } i \ge j \\ 0 & \text{for } i < j \end{cases}, \quad 1 \le i,j \le n \tag{35}$$

and $U = (U_{ij})_{i,j=1}^{n}$ is an $n \times n$ upper triangular matrix given by:

$$U_{ij} = \begin{cases} 0 & \text{for } i > j, \\ \prod_{r=1}^{i-1}(x_j - x_r) & \text{for } i \le j \end{cases}, \quad 1 \le i,j \le n \tag{36}$$

The inverse matrices L^{-1} and U^{-1} of the matrices L and U in (35) and (36) are given respectively by:

$$L^{-1} = \left[(-1)^{i-j}\sigma_{i-j}^{(i-1)}(x_1, x_2, \ldots, x_{i-1})\right]_{i,j=1}^{n} \tag{37}$$

and

$$U^{-1} = \left[\frac{1}{\prod_{\substack{r=1 \\ r \ne i}}^{j}(x_i - x_r)}\right]_{i,j=1}^{n}. \tag{38}$$

In particular, if $x_k = k$, $1 \le k \le n$, then we have [6]:

$$V = S_n \tilde{L}^T,$$

where S_n is the Stirling matrix of the second kind and $\tilde{L} = (\tilde{L}_{ij})_{i,j=1}^{n}$ is an $n \times n$ lower triangular matrix given by:

$$\tilde{L}_{ij} = \begin{cases} [i-1]_{j-1} & \text{for } i \ge j \\ 0 & \text{for } i < j \end{cases}, \quad 1 \le i,j \le n \tag{39}$$

The inverse of the matrix $\tilde{L}$ is given by:

$$\tilde{L}^{-1} = \left[(-1)^{i-j}\frac{1}{(i-1)!}\binom{i-1}{j-1}\right]_{i,j=1}^{n}.$$

Definition 5 ([18]). *An $n \times n$ matrix A is called totally positive if all its minors of all sizes are positive.*

Definition 6 ([8])**.** *The symmetric matrix $A=(a_{ij})_{i,j=1}^{n}$ is called positive definite if and only if*

$$\mathbf{x}^T A\mathbf{x} > 0, \textit{ for all } \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x} \neq 0.$$

Definition 7. *The $n \times n$ permutation matrix, J is defined by:*

$$J = \left[e_n, e_{n-1}, \ldots, e_1\right], \tag{40}$$

where $e_i = \left[\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}\right]^T$, and δ_{ij} is the kronecker delta.

The permutation matrix J of order n enjoys the following properties:

- $J = J^T = J^{-1}$.
- $J^k = \begin{cases} I_n & k \text{ even} \\ J & k \text{ odd} \end{cases}$, where I_n is the identity matrix of order n.
- $\det(J) = (-1)^{\frac{n(n-1)}{2}} = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \ mod(4) \\ -1 & \text{if } n \equiv 2 \text{ or } 3 \ mod(4). \end{cases}$.

Lemma 1. *For nonnegative integer numbers n and k, we have*

$$\sum_{r=1}^{n} \binom{r}{k} = \binom{n+1}{k+1} - \delta_{k0}. \tag{41}$$

Proof. By using the following Pascal's rule:

$$\binom{r}{k} = \binom{r+1}{k+1} - \binom{r}{k+1},$$

we get:

$$\begin{aligned} \sum_{r=1}^{n} \binom{r}{k} &= \sum_{r=1}^{n} \left[\binom{r+1}{k+1} - \binom{r}{k+1}\right] \\ &= \binom{n+1}{k+1} - \binom{1}{k+1} \\ &= \binom{n+1}{k+1} - \delta_{k0}, \end{aligned}$$

having used the telescoping sum. □

Definition 8 ([20])**.** *A real $n \times n$ matrix $A = (a_{ij})_{i,j=1}^{n}$ is called row stochastic matrix if*

(i) $a_{ij} \geq 0$, *for* $1 \leq i, j \leq n$

(ii) $\sum_{j=1}^{n} a_{ij} = 1$, *for* $1 \leq i \leq n$

Note that *(ii)* is equivalent to $AE = E$, where $E = [1, 1, \dots, 1]^T$.

Definition 9 ([15]). *The symmetric Pascal matrix $P_n = (P_{ij})_{i,j=1}^n$ of order n is a matrix of integers defined by:*

$$P_{ij} = \binom{i+j-2}{i-1}, \quad 1 \leq i, j \leq n. \tag{42}$$

The Pascal matrix P_n enjoys the following properties:

(i) P_n is totally positive;

(ii) P_n is positive definite;

(iii) The eigenvalues of the matrix P_n are real and positive;

(iv) If $\lambda \neq 0$ is an eigenvalue of P_n, then $\frac{1}{\lambda}$ is also an eigenvalue of P_n;

(v) $\det(P_n) = 1$;

(vi) The Cholesky's factorization [2] of the matrix, P_n is given by:

$$P_n = AA^T,$$

where A is the Pascal matrix defined by:

$$A = \left[\binom{i-1}{j-1} \right]_{i,j=1}^n; \tag{43}$$

(vii) The explicit form of the inverse matrix, $P_n^{-1} = Q_n = (\beta_{ij})_{i,j=1}^n$ is given by [15]:

$$\beta_{ij} = (-1)^{i+j} \sum_{k=\max(i,j)}^{n} \binom{k-1}{i-1} \binom{k-1}{j-1}; \tag{44}$$

(viii) The matrix, P_n satisfies:

$$P_n = AB^{-1} s_n V = TV, \tag{45}$$

where $T = (T_{ij})_{i,j=1}^n$ is an $n \times n$ lower triangular stochastic matrix given by:

$$T_{ij} = \frac{1}{(i-1)!} c(i-1, j-1), \quad 1 \leq i, j \leq n, \quad i \geq j, \tag{46}$$

$V = \left(j^{i-1}\right)_{i,j=1}^n$, A is given in (43) and $B = \operatorname{diag}(0!, 1!, 2!, \dots, (n-1)!)$ (for more details, see [13, 14, 27, 28]);

(ix) The entries of the matrix, P_n satisfy the recurrence relation:

$$P_{i1} = P_{1j} = 1, \tag{47}$$

and

$$P_{ij} = P_{i,j-1} + P_{i-1,j}. \tag{48}$$

(x) Let R_n be the matrix obtained from the Pascal matrix P_n by subtracting one from the element in position (n, n) of P_n, then $\det(R_n) = 0$.

2. Main Results

This section is mainly devoted to study the $n \times n$ inverse matrix Q_n, in (44), of the $n \times n$ symmetric Pascal matrix P_n in (42). The main object is to find a recurrence relation satisfied by the entries of Q_n. Setting $j = 1$ in (44), we obtain:

$$\beta_{i1} = (-1)^{i+1} \binom{n}{i} = \beta_{1i}, \quad 1 \leq i \leq n, \tag{49}$$

having used Lemma 1. Putting $j = n$ in (44), yields:

$$\beta_{in} = (-1)^{i+n} \binom{n-1}{i-1} = \beta_{ni}, \quad 1 \leq i \leq n. \tag{50}$$

At this point we may formulate the following result.

Theorem 1. *The entries of the inverse matrix* $Q_n = P_n^{-1} = (\beta_{ij})_{i,j=1}^{n}$, *in* (44)*, satisfy:*

$$\beta_{i,j} = \beta_{i,j+1} + \beta_{i+1,j} + (-1)^{i+j} \binom{n}{i}\binom{n}{j}, \quad 1 \leq i, j \leq n. \tag{51}$$

Proof. It is well-known that

$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}. \tag{52}$$

From (44), we have

$$\beta_{i,j} = (-1)^{i+j} \sum_{k=1}^{n} \binom{k-1}{i-1}\binom{k-1}{j-1}. \tag{53}$$

By using the identity (52), we get

$$\begin{aligned}
\beta_{i,j} =& (-1)^{i+j}\Bigg[\sum_{k=1}^{n}\left\{\binom{k}{i}-\binom{k-1}{i}\right\}\left\{\binom{k}{j}-\binom{k-1}{j}\right\}\Bigg] \\
=& (-1)^{i+j}\Bigg[\sum_{k=1}^{n}\binom{k}{i}\binom{k}{j}-\sum_{k=1}^{n}\binom{k}{i}\binom{k-1}{j} \\
& -\sum_{k=1}^{n}\binom{k-1}{i}\binom{k}{j}+\sum_{k=1}^{n}\binom{k-1}{i}\binom{k-1}{j}\Bigg] \\
=& (-1)^{i+j}\Bigg[\binom{n}{i}\binom{n}{j}+\sum_{k=1}^{n-1}\binom{k}{i}\binom{k}{j}-\sum_{k=1}^{n}\binom{k}{i}\binom{k-1}{j} \\
& -\sum_{k=1}^{n}\binom{k-1}{i}\binom{k}{j}+\sum_{k=1}^{n}\binom{k-1}{i}\binom{k-1}{j}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&=(-1)^{i+j}\Bigg[\binom{n}{i}\binom{n}{j}+\sum_{k=1}^{n}\binom{k-1}{i}\binom{k-1}{j}-\sum_{k=1}^{n}\binom{k}{i}\binom{k-1}{j}\\
&\quad-\sum_{k=1}^{n}\binom{k}{j}\binom{k-1}{i}+\sum_{k=1}^{n}\binom{k-1}{i}\binom{k-1}{j}\Bigg]\\
&=(-1)^{i+j}\Bigg[\binom{n}{i}\binom{n}{j}-\sum_{k=1}^{n}\binom{k-1}{j}\Bigg\{\binom{k}{i}-\binom{k-1}{i}\Bigg\}\\
&\quad-\sum_{k=1}^{n}\binom{k-1}{i}\Bigg\{\binom{k}{j}-\binom{k-1}{j}\Bigg\}\Bigg]\\
&=(-1)^{i+j}\Bigg[\binom{n}{i}\binom{n}{j}-\sum_{k=1}^{n}\binom{k-1}{j}\binom{k-1}{i-1}-\sum_{k=1}^{n}\binom{k-1}{i}\binom{k-1}{j-1}\Bigg]\\
&=(-1)^{i+j}\Bigg[\binom{n}{i}\binom{n}{j}+(-1)^{i+j}\beta_{i,j+1}+(-1)^{i+j}\beta_{i+1,j}\Bigg]. \qquad (54)
\end{aligned}
$$

Therefore

$$
\beta_{i,j}=\beta_{i,j+1}+\beta_{i+1,j}+(-1)^{i+j}\binom{n}{i}\binom{n}{j}. \tag{55}
$$

□

Using (49), (50) and (55), we see that in order to obtain Q_n, we only need to compute the $\frac{1}{2}(n-1)(n-2)$ elements $\beta_{22},\beta_{32},\ldots,\beta_{n-1,n-1}$ and taking into account the fact that Q_n is symmetric. Also, the solution of the linear system of the Pascal type

$$
P_n[u_1u_2\ldots u_n]^T=[f_1f_2\ldots f_n]^T \tag{56}
$$

may be obtained in $O(n^2)$ operations by using (49), (50) and (55). The following is a MAPLE code to compute the inverse matrix of the Pascal matrix, P_n for $n=6$, as an example.

MAPLE Code for Inverting Pascal Matrix.

```
> restart:
> n:=6: Q:=array(1..n,1..n,symmetric):
> for i to n do
    Q[i,n]:=(-1)^(n+i)*binomial(n-1,i-1):
    Q[i,1]:=(-1)^(i+1)*binomial(n,i):
  od:
> for i from n-1 by -1 to 2 do
    for j from i by -1 to 2 do
      Q[i,j]:=Q[i,j+1]+Q[i+1,j]+(-1)^(i+j)*binomial(n,i)*binomial(n,j):
    od:
  od:
> Q:=evalm(Q):
># Check using the linalg package.
> P:=linalg[inverse](Q):
```

The results are printed as shwon.

$$Q = \begin{bmatrix} 6 & -15 & 20 & -15 & 6 & -1 \\ -15 & 55 & -85 & 69 & -29 & 5 \\ 20 & -85 & 146 & -127 & 56 & -10 \\ -15 & 69 & -127 & 117 & -54 & 10 \\ 6 & -29 & 56 & -54 & 26 & -5 \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix}$$

3. Applications

In this section we are going to give a new proof for El-Mikkawy conjecture [14]. Moreover, some new identities will be obtained.

3.1. A New Proof for El-Mikkawy Conjecture [14]

Theorem 2. *Let G be the $n \times n$ matrix whose (i, j) entry is given by $\sigma_{i,j}^{(n)}(x_1, x_2, \ldots, x_n)$ and let V be the $n \times n$ Vandermonde matrix defined by* (27). *Then we have*

$$\det(G) = (-1)^{\frac{n(n-1)}{2}} \det(V) = \begin{cases} \det(V) & \text{if } n \equiv 0 \text{ or } 1 \ mod(4), \\ -\det(V) & \text{if } n \equiv 2 \text{ or } 3 \ mod(4). \end{cases} \tag{57}$$

Proof. For $i = 1, 2, \ldots, n$, let F_i as given in (31), D and $\tilde{D}$ are $n \times n$ diagonal matrices given by: $D = \text{diag}(\frac{1}{F_1}, \frac{1}{F_2}, \ldots, \frac{1}{F_n})$ and $\tilde{D} = \text{diag}((-1)^{n-1}, (-1)^{n-2}, \ldots, (-1)^0)$.

It can be shown that for the matrices D, J and $\tilde{D}$, we have

$$\det(D) = \frac{1}{\prod_{i=1}^{n} F_i} = \frac{1}{\left((-1)^{\frac{n(n-1)}{2}} \left[\det(V)\right]^2\right)}, \tag{58}$$

having used (28) and (31),

$$\det(J) = (-1)^{\frac{n(n-1)}{2}} \quad \text{and} \quad \det(\tilde{D}) = (-1)^{\frac{n(n-1)}{2}}. \tag{59}$$

By noticing that the matrix V^{-1} in (30) can be written in the form:

$$V^{-1} = DGJ\tilde{D}, \tag{60}$$

the result follows. □

3.2. New Identities

In [13] it is shown that the Pascal matrix, P_n and the Vandermonde matrix, $V = (j^{i-1})_{i,j=1}^n$ are related by:

$$P_n = TV. \tag{61}$$

From (61), we get:

$$Q_n = (\beta_{ij})_{i,j=1}^n = P_n^{-1} = V^{-1}T^{-1}, \tag{62}$$

where

$$T^{-1} = (\gamma_{ij})_{i,j=1}^n, \tag{63}$$

with

$$\gamma_{ij} = (-1)^{i+j}(j-1)!S(i-1,j-1), \quad 1 \le i,j \le n, i \ge j. \tag{64}$$

Thus

$$\begin{aligned}\beta_{ij} =& \sum_{k=1}^{n} \alpha_{ik}\gamma_{kj} = \sum_{k=1}^{n} \left[\frac{(-1)^{i+k}}{(n-1)!} \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} \sigma_{n-k+1,i}^{(n)} \right] \left[(-1)^{j+k}(j-1)!S(k-1,j-1) \right] \\ =&(-1)^{i+j} \frac{(j-1)!}{(i-1)!(n-i)!} \sum_{k=1}^{n} \sigma_{n-k+1,i}^{(n)} S(k-1,j-1).\end{aligned} \tag{65}$$

Rewriting (44) in the form:

$$\beta_{ij} = (-1)^{i+j} \sum_{k=1}^{n} \begin{pmatrix} k-1 \\ i-1 \end{pmatrix} \begin{pmatrix} k-1 \\ j-1 \end{pmatrix}, \tag{66}$$

then using (65) and (66), yields:

$$\sum_{k=\max(i,j)}^{n} \begin{pmatrix} k-1 \\ i-1 \end{pmatrix} \begin{pmatrix} k-1 \\ j-1 \end{pmatrix} = \frac{(j-1)!}{(i-1)!(n-i)!} \sum_{k=j}^{n} \sigma_{n-k+1,i}^{(n)} S(k-1,j-1). \tag{67}$$

Setting $i = n$ on both sides of (67) gives

$$\begin{aligned}\begin{pmatrix} n-1 \\ j-1 \end{pmatrix} =& \frac{(j-1)!}{(n-1)!0!} \sum_{k=j}^{n} \sigma_{n-k+1,n}^{(n)} S(k-1,j-1) \\ =& \frac{(j-1)!}{(n-1)!} \sum_{k=j}^{n} \sigma_{n-k}^{(n-1)}(1,2,\ldots,n-1) S(k-1,j-1) \\ =& \frac{(j-1)!}{(n-1)!} \sum_{k=j}^{n} c(n,k) S(k-1,j-1) \\ =& \frac{1}{(n-j)! \begin{pmatrix} n-1 \\ j-1 \end{pmatrix}} \sum_{k=j}^{n} c(n,k) S(k-1,j-1).\end{aligned} \tag{68}$$

Hence

$$\begin{aligned}\sum_{k=j}^{n} c(n,k)S(k-1,j-1) =&(n-j)!\begin{pmatrix} n-1 \\ j-1 \end{pmatrix}^2 \\ =&(n-1)_{n-j}\begin{pmatrix} n-1 \\ n-j \end{pmatrix}. \quad \blacksquare\end{aligned} \tag{69}$$

From (61), we see that

$$\begin{pmatrix} i+j-2 \\ i-1 \end{pmatrix} = \sum_{k=1}^{n}\left(\frac{1}{(i-1)!}c(i-1,k-1)\right)\left(j^{k-1}\right).$$

Then

$$\begin{pmatrix} i+j-2 \\ i-1 \end{pmatrix} = \sum_{k=1}^{n}\frac{j^{k-1}}{(i-1)!}c(i-1,k-1). \tag{70}$$

Setting $j=2$ in (70), we obtain

$$\sum_{k=1}^{n} 2^{k-1}c(i-1,k-1) = (i-1)!\begin{pmatrix} i \\ i-1 \end{pmatrix} = i!. \tag{71}$$

Therefore, for any positive integers n and $1 \le i \le n$, we have

$$\sum_{k=1}^{n} 2^{k-1}c(i-1,k-1) = i!. \quad \blacksquare \tag{72}$$

We list the following additional identities, without proof, for the sake of space requirements. In all cases n is a positive integer and i, j are integers such that $1 \le i, j \le n$.

- $\sum\limits_{r=1}^{n}(-1)^{r+1}\begin{pmatrix} i+r-2 \\ i-1 \end{pmatrix}\begin{pmatrix} n \\ r \end{pmatrix} = \delta_{i1}$.
- $\sum\limits_{k=1}^{n}(-1)^{k+1}k^{i-1}\begin{pmatrix} n \\ k \end{pmatrix} = \delta_{i1}$.
- $\sum\limits_{k=1}^{n}(-1)^{k+1}\sigma_{n-k}^{(n-1)}(2,3,\ldots,n) = (n-1)!$.
- $\sum\limits_{k=1}^{n} s(n,k)j^{k-1} = (n-1)!\delta_{nj}$.
- $\sum\limits_{r=j}^{n}\begin{pmatrix} n-1 \\ n-r \end{pmatrix}(n-1)_{n-r}s(r-1,j-1) = c(n,j)$.
- $\sum\limits_{r=j}^{n}\sum\limits_{k=1}^{n}(-1)^{j+k}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}\sigma_{n-r+1,k}^{(n)}S(r-1,j-1) = 0$.

- $\sum_{r=1}^{n}\sum_{k=1}^{n}(-1)^k\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)\left(\begin{array}{c} i+k-2 \\ k-1 \end{array}\right)\sigma_{n-r+1,k}^{(n)} = \begin{cases} -(n-1)! & \text{if } i=1 \\ (n-1)! & \text{if } i=2 \\ 0 & \text{otherwise} \end{cases}.$

- $\sum_{r=1}^{n}\sum_{k=1}^{n}(-1)^k k^2\left(\begin{array}{c} n \\ k \end{array}\right)\sigma_{n-r+1,k}^{(n)} = n!.$

- $\sum_{k=\max(i,j)}^{n}\frac{\left(\begin{array}{c} k \\ i \end{array}\right)}{k!}c(k,j) = \frac{\left(\begin{array}{c} n \\ i \end{array}\right)}{n!}\sigma_{n-j}^{(n-1)}(1,2,\ldots,i-1,i+1,\ldots,n);$

- $\sum_{k=\max(i,j)}^{n}(n-i)_{n-k}c(k,j) = \sigma_{n-j}^{(n-1)}(1,2,\ldots,i-1,i+1,\ldots,n);$

- $\sum_{k=i}^{n}\frac{\left(\begin{array}{c} n-i \\ n-k \end{array}\right)}{\left(\begin{array}{c} n-1 \\ n-k \end{array}\right)} = \sum_{k=i}^{n}\frac{(n-i)_{n-k}}{(n-1)_{n-k}} = \frac{n}{i};$

- $\sum_{k=i}^{n}\frac{\left(\begin{array}{c} k \\ i \end{array}\right)}{k} = \sum_{k=i}^{n}\frac{(k-1)_{i-1}}{i!} = \frac{\left(\begin{array}{c} n \\ i \end{array}\right)}{i};$

- $\sum_{k=i}^{n}\frac{(k)_i}{k} = \frac{(n)_i}{i}.$

- $\sum_{k=1}^{n}ks(n+1,k+1) = (-1)^{n-1}(n-1)! = s(n,1).$

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