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Transparency of Skew Polynomial Ring Over a Commutative Noetherian Ring

V. K. Bhat* and Kiran Chib

School of Mathematics, SMVD University, P/o SMVD University, Katra, J and K, India- 182320

Abstract. In this paper, we discuss a stronger type of primary decomposition (known as transparency) in noncommutative set up. One of the class of noncommutative rings are the skew polynomial rings. We show that certain skew polynomial rings satisfy this type of primary decomposition.
Recall that a right Noetherian ring R is said to be *transparent ring* if there exist irreducible ideals I_j, $1 \leq j \leq n$ such that $\cap_{j=1}^{n} I_j = 0$ and each R/I_j has a right artinian quotient ring.
Let R be a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers). Let σ be an automorphism of R and δ a σ-derivation of R. Then we show that the skew polynomial ring $R[x;\sigma,\delta]$ is a transparent ring.

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1. Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The set of minimal prime ideals of R is denoted by $Min.Spec(R)$. Prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$. The set of positive integers, the set of integers, the field of rational numbers, the field of real numbers and the field of complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively unless otherwise stated.

Let R be a ring, σ an automorphism of R and δ a σ-derivation of R; i.e. $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example let $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a map defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then ϕ is a ring homomorphism if and only if δ is a σ-derivation of R.

*Corresponding author.

Email address: `vijaykumarbhat2000@yahoo.com` (V. Bhat)

This article concerns the study of transparency of skew polynomial ring $R[x;\sigma,\delta]$ (also known as Ore extension).

We recall that the skew polynomial ring $R[x;\sigma,\delta]$ is the usual set of polynomials over R with coefficients in R with respect to usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$ as in McConnell and Robson [15]. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i.e. $\sigma(I) = I$) and is also δ-invariant (i.e. $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it by $O(I)$.

In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$. If J is an ideal of R such that J is δ-invariant (i.e. $\delta(J) \subseteq J$), then clearly $J[x;\delta]$ is an ideal of $D(R)$, and we denote it by $D(J)$. In case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. If K is an ideal of R such that K is σ-stable (i.e. $\sigma(K) = K$), then clearly $K[x;\sigma]$ is an ideal of $S(R)$, and we denote it by $S(K)$. The study of Skew polynomial rings have been of interest to many authors. For example [2–4, 6, 8, 11, 12, 14, 15]. The notion of the quotient ring of a ring appears in Chapter 9 of Goodearl and Warfield [12].

It is shown in Blair and Small [11] that if R is embeddable in a right artinian ring and has characteristic zero, then the differential operator ring $R[x;\delta]$ embeds in a right artinian ring. It is also shown in Blair and Small [11] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew polynomial ring $R[x;\sigma]$ embeds in an artinian ring. For more results on the existence of the artinian quotient rings, the reader is referred to Robson [15].

In Theorem (2.11) of Bhat [4], it is proved that if R is a ring which is an order in a right artinian ring. Then $O(R)$ is an order in a right artinian ring.

In this paper the above mentioned properties have been studied with emphasis on primary decomposition of $O(R)$, where R is a commutative Noetherian $\mathbb{Q}$-algebra. We actually discuss a stronger type of primary decomposition of a right Noetherian ring. We recall the following:

Definition 1. *A ring R is said to be an irreducible ring if the intersection of any two non-zero ideals of R is non-zero. An ideal I of R is called irreducible if $I = J \cap K$ implies that either $J = I$ or $K = I$. Note that if I is an irreducible ideal of R, then R/I is an irreducible ring.*

Proposition 1. *Let R be a Noetherian ring. Then there exist irreducible ideals I_j, $1 \leq j \leq n$ of R such that $\cap_{j=1}^{n} I_j = 0$.*

Proof. The proof is obvious and we leave the details to the reader. □

Definition 2. *[Definition 1.2 of [8]] A Noetherian ring R is said to be transparent ring if there exist irreducible ideals $I_j, 1 \leq j \leq n$ such that $\cap_{j=1}^{n} I_j = 0$ and each R/I_j has a right artinian quotient ring.*

It can be easily seen that a Noetherian integral domain is a *transparent ring*, a commutative Noetherian ring is a *transparent ring* and so is a Noetherian ring having an artinian quotient ring. A fully bounded Noetherian ring is also a *transparent ring*.

The following result has been proved in Bhat [2] towards the transparency of skew polynomial rings.

Result 1. *Let R be a commutative Noetherian ring and σ an automorphism of R and δ a σ-derivation of R. Then it is known that $S(R)$ and $D(R)$ are transparent (Bhat [2]).*

The following result has been proved in Bhat [8].

Theorem 1. *[Theorem (3.4) of [8]] Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R. Then there exists an integer $m \geq 1$ such that the skew-polynomial ring $R[x;\alpha,\delta]$ is a transparent ring, where $\sigma^m = \alpha$ and δ is an α-derivation of R such that $\alpha(\delta(a)) = \delta(\alpha(a))$, for all $a \in R$.*

In this paper, we generalize Theorem (3.4) of [8]. But before we state the main result, we need the following:

1.1. 2-Primal Rings

A ring R is called a 2-primal if $P(R) = N(R)$, i.e. if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ for $a \in R$. Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [14]. 2-primal near rings have been discussed by Argac and Groenewald in [1].

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [14], Greg Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ an automorphism of R and δ a σ-derivation of R. In Greg Marks [14], it has been shown that for a local ring R with a nilpotent maximal ideal, the ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $R[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

We note that a reduced ring (i.e. a ring with no non-zero nilpotent elements) is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [1, 5, 7, 13, 14].

We note that a Noetherian ring need not be 2-primal. For this we have the following:

Example 1. *[Example (4) of Bhat [10]] Let $R = \mathbb{Q} \oplus \mathbb{Q}$ with $\sigma(a,b) = (b,a)$. Then the only σ-invariant ideals of R are 0 and R, and so R is σ-prime. Let $\delta : R \to R$ be defined by $\delta(r) = ra - a\sigma(r)$, where $a = (0,\alpha) \in R$. Then δ is a σ-derivation of R and $R[x;\sigma,\delta]$ is prime and $P(R[x;\sigma,\delta]) = 0$. But $(x(1,0))^2 = 0$ as $\delta(1,0) = -(0,\alpha)$. Therefore $R[x;\sigma,\delta]$ is not 2-primal. If δ is taken to be the zero map, then even $R[x;\sigma]$ is not 2-primal.*

Example 2. *[Example (5) of Bhat [10]]Let $R = M_2(\mathbb{Q})$, the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and so can not be 2-primal.*

From these examples we conclude that if R is a Noetherian ring, then $R[x;\sigma,\delta]$ and $R[x]$ need not be 2-primal. But it is known that if R is a 2-primal Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $R[x;\delta]$ is 2-primal Noetherian (Theorem (1.2) of Bhat [7]).

1.2. Weak σ-Rigid Ring

Let R be a ring and σ be an endomorphism of R. Recall that in [8], σ is called a rigid endomorphism if $a\sigma(a)=0$ implies $a=0$ for $a\in R$, and R is called a σ-rigid ring.

Example 3. *Let $R=\mathbb{C}$, and $\sigma : R\to R$ be the map defined by $\sigma(a+ib)=a-ib$, $a,b\in\mathbb{R}$. Then it can be seen that R is a σ-rigid ring.*

Definition 3. *[Ouyang [16]] Let R be a ring and σ an endomorphism of R such that $a\sigma(a)\in N(R)$ if and only if $a\in N(R)$ for $a\in R$. Then R is called a weak σ-rigid ring.*

Example 4 (Example (2.1) of Ouyang [16]). *Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let*

$$A=\left\{\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix} \mid a,b,c,d\in R\right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij}))=(\sigma(a_{ij}))$. Then it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

We now state the main result of this paper in the form of the following statement which we will prove in Section 3: "Let R be a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. Then $O(R)=R[x;\sigma,\delta]$ is a transparent ring."

2. Preliminaries

Recall that an ideal I of a ring R is said to be completely semiprime if $a^2\in I$ implies that $a\in I$.

We now have the following Theorem.

Proposition 2. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring if and only if $N(R)$ is completely semiprime ideal of R.*

Proof. R is commutative Noetherian implies that $N(R)$ is an ideal of R. It is easy to see that $\sigma(N(R))=N(R)$. Now let R be a weak σ-rigid ring. We will show that $N(R)$ is completely semiprime. Let $a\in R$ be such that $a^2\in N(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a))=a\sigma(a)\sigma(a)\sigma^2(a)\in\sigma(N(R))=N(R).$$

Therefore, $a\sigma(a)\in N(R)$ and hence $a\in N(R)$. So $N(R)$ is completely semiprime. Conversely let $N(R)$ be completely semiprime. We will show that R is a weak σ-rigid ring. Let $a\in R$ be such that $a\sigma(a)\in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a))\in N(R)$ implies that $a^2\in N(R)$, and so $a\in N(R)$. Hence R is a weak σ-rigid ring. □

In this paper we investigate the transparency for $O(R)=R[x;\sigma,\delta]$. Before we prove the main result, we have the following:

Proposition 3. *Let R be a commutative Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R. Then $\sigma(U)=U$ for all $U \in Min.Spec(R)$.*

Proof. The proof is on the same lines as in Bhat and Kiran [9] R is a commutative Noetherian ring implies R is 2-primal. This implies that $P(R)=N(R)$ i.e., $P(R)$ is completely semiprime. Therefore, if $a \in R$ is such that $a^2 \in P(R)$, then $a \in P(R)$. Now, $P(R)=N(R)$, therefore $N(R)$ is also completely semiprime. Now Proposition 2 implies that R is a weak σ-rigid ring.

We now show that $\sigma(U)=U$ for all $U \in Min.Spec(R)$. Let $U=U_1$ be a minimal prime ideal of R. Now Theorem (2.4) of Goodearl and Warfield [12] implies that $Min.Spec(R)$ is finite. Let $U_2, U_3, \ldots, U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U)=U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Now $P(R)$ is completely semiprime implies that $a \in P(R)$ and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$ which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence, $\sigma(U)=U$ for all $U \in Min.Spec(R)$. □

Theorem 2. *[Hilbert Basis Theorem] Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then $O(R)=R[x;\sigma,\delta]$ is a right/left Noetherian.*

Proof. See Theorem (2.6) of Goodearl and Warfield [12]. □

Proposition 4. *Let R be a Noetherian ring having an artinian quotient ring. Then R is a transparent ring.*

Proof. See Lemma (2.8) of Bhat [8]. □

Definition 4. *Let P be a prime ideal of a commutative ring R. Then the symbolic power of P for any $n \in \mathbb{N}$ is denoted by P^n and is defined as*

$$P^n = \{a \in R \text{ such that there exists some } d \in R, d \notin P \text{ such that } da \in P^n\}.$$

Also if I is an ideal of R, define as usual

$$\sqrt{I} = \{a \in R \text{ such that } a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Lemma 1. *Let R be a commutative Noetherian ring, and σ an automorphism of R. If P is a prime ideal of R such that $\sigma(P)=P$, then $\sigma(P^n)=P^n$ for all integers $n \geq 1$.*

Proof. See Lemma (2.10) of Bhat [8]. □

Lemma 2. *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R. If $P \in Min.Spec(R)$ is such that $\sigma(P)=P$, then $\delta(P) \subseteq P$.*

Proof. See Lemma (2.6) of Bhat [8]. □

Lemma 3. *Let R be a commutative Noetherian ring; σ and δ as usual. Let P be a prime ideal of R such that $\sigma(P)=P$ and $\delta(P) \subseteq P$. Then $\delta(P^{(k)}) \subseteq P^{(k)}$ for all integers $k \geq 1$.*

Proof. See Lemma (2.11) of Bhat [8]. □

3. Proof of the Main Theorem

We are now in a position to prove the main result in the form of the following theorem:

Theorem 3. *[repeated from Introduction] Let R be a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. Then $O(R) = R[x;\sigma,\delta]$ is a transparent ring.*

Proof. $R[x;\sigma,\delta]$ is Noetherian ring by Hilbert Basis Theorem, namely Theorem (1.12) of Goodearl and Warfield [12]. Now R is a commutative Noetherian $\mathbb{Q}$-algebra, therefore, the ideal $\{0\}$ has a reduced primary decomposition. Let $I_j, 1 \leq j \leq n$ be irreducible ideals of R such that $(0) = \cap_{j=1}^{n} I_j$. For this see Theorem (4) of Zariski and Samuel [17]. Let $\sqrt{I_j} = P_j$, where P_j is a prime ideal belonging to I_j. Now $P_j \in Ass(R_R), 1 \leq j \leq n$ by first uniqueness Theorem. Now by Theorem (23) of Zariski and Samuel [17], there exists a positive integer k such that $P_j^{(k)} \subseteq I_j, 1 \leq j \leq n$. Therefore we have $\cap_{j=1}^{n} P_j^k = 0$. Now each P_j contains a minimal prime ideal U_j by Proposition (2.3) of Goodearl and Warfield [12], therefore $\cap_{j=1}^{n} U_j^k = 0$. Now R is commutative, therefore, Proposition 3 implies that $\sigma(U_j) = U_j$, for all $j, 1 \leq j \leq n$. Also, by Lemma 2, we have $\delta(U_j) \subseteq U_j$, for all $j, 1 \leq j \leq n$. Now Lemma 1 implies that $\sigma(U_j)^{(k)} = U_j^{(k)}$ and Lemma 3 implies that $\delta(U_j^{(k)}) \subseteq U_j^{(k)}$, for all $j, 1 \leq j \leq n$ and for all $k \geq 1$. Therefore, $O(U_j^{(k)})$ is an ideal of $O(R)$ and $\cap_{j=1}^{n} O(U_j^{(k)}) = 0$.

Now $R/U_j^{(k)}$ has an artinian quotient ring, as it has no embedded primes, therefore $O(R)/O(U_j^{(k)})$ has also an artinian quotient ring by Theorem (2.11) of Bhat [4]. Hence $O(R) = R[x;\sigma,\delta]$ is *transparent ring*. □

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