Spectrum and Spectral Singularities of Quadratic Pencil of Klein-Gordon Operators with General Boundary Condition

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ABSTRACT

In this atricle we investigated the spectrum and the spectral singularities of the Quadratic pencil of Klein-Gordon Operator L generated in $L^2(R_+)$ by the differential expression

$$l(y) = y'' + (\lambda - Q(x))^2 y, \qquad x \in R_+ = (0,\infty)$$

and the boundary condition

$$\int_0^\infty K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0$$

where Q and K are coplex valued functions $K \in L_2(R_+)$ and $\alpha, \beta \in C$, with $|\alpha| + |\beta| \neq 0$. Discussing the spectrum, we proved that L has a finite number of eigenvalues and spectral singulatities with finite multuplicities, if the conditions

$$\lim_{x\to\infty} Q(x) = 0, \qquad \sup_{x\in R_+}\left\{e^{\varepsilon\sqrt{x}}\left[|Q'(x)| + |K(x)|\right]\right\} < \infty, \quad \varepsilon > 0$$

hold. Later we have investigated the properties of the principal functions corresponding to the spectral singularities. Moreover, some results about the spectrum of L have also been applied to non-selfadjoint Sturm-Liouville.

Genel Sınır Koşulu ile Verilen Quadratic Pencil Klein-Gordon Operatörünün Spectrumu ve Spectral Tekillikleri

ÖZET

Bu çalışmada $L^2(R_+)$ uzayında

$$l(y) = y'' + (\lambda - Q(x))^2 y, \qquad x \in R_+ = (0,\infty) \text{ denklemi ve}$$

$$\int_0^\infty K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0$$

genel sınır koşulu ile verilen Quadratic Pencil Klein-Gordon L operatörünün spektral tekillikleri ve spektrumu incelenmiştir. Burada K ve Q kompleks değerli fonksiyonlar, $K \in L_2(R_+)$ ve $\alpha, \beta \in C$ olup, $|\alpha| + |\beta| \neq 0$.

Spectrum incelendiğinde

$$\lim_{x\to\infty} Q(x) = 0, \qquad \sup_{x\in R_+}\left\{e^{\varepsilon\sqrt{x}}\left[|Q'(x)| + |K(x)|\right]\right\} < \infty, \quad \varepsilon > 0$$

koşulunun sağlanması durumunda L operatörünün sonlu sayıda sonlu katlı özdeğerlere ve spektral tekilliklere sahip olduğu araştırılmıştır. Sonra spektral tekilliklere göre baş fonksiyonlar incelenmiştir. Ayrıca L operatörünün bazı sonuçları non-selfadjoint Sturm-Liouville denklemine de her zaman uygulanabilir.

INTRODUCTION

Let L_0 denote the operator generated $K \in L_2(R_+)$ by the differential expression

$$l_0(y) = -y'' + q(x)y, \; x \in R_+ = (0,\infty) \tag{1.1}$$

and the boundary condition $y'(0) - hy(0) = 0$, where q is a complex valued function and $h \in C$. The study of the spectral analysis of L_0 has been started by Naimark in 1954(Naimark, 1960). In this article, the spectrum of L_0 has been investigated and shown that it is composed of the eigenvalues, the continuous spectrum and the spectral singularities. It has been proved that the spectral singularities are on the continuous spectrum and are the poles of the resolvent's kernel; but they are not the eigenvalues. Also, if

$$\int_0^\infty e^{\varepsilon x} |q(x)| dx < \infty$$

holds, for some $\varepsilon > 0$, then it has been obtained that L_0 has a finite number of eigenvalues and spectral singularities with finite multiplicities. Moreover, the spectral expansion has been derived in some particular cases.

The above results of Naimark has been generalized to the differential operators on the whole real axis by Kemp(Kemp, 1958).

One of the very important steps in the spectral analysis of L_0 has been taken by Pavlov(Pavlov, 1967). In this article the dependence of the structure of spectral singularities of L_0 to the behaviour of the potential function at infinity has been studied.

The spectral analysis of the non-selfadjoint operator L_1 generated in $L^2(R_+)$ by (1.1) and the boundary condition

$$\int_0^\infty K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0,$$

in which $K \in L_2(R_+)$ is a coplex valued function and $\alpha, \beta \in C$ has been investigated in detail by Kral (Krall, 1965).

Let us consider a differential expression of the form

$$l(y) = y'' + (\lambda - Q(x))^2 y, \qquad x \in R_+,$$

where Q is a complex valued function and is continuously differentiable on R_+ and bounded.

We denote by D_0 those functions f defined on R_+ and satisfying

i. $f \in L^2(R_+)$,

ii. f' exists and absolutely continuous on every finite subinterval of R_+ **,**

iii. $l(f) \in L^2(R_+)$.

Let K be an arbitrary coplex-valued function in $L^2(R_+)$, $\alpha, \beta \in C$ and $|\alpha| + |\beta| \neq 0$. We denote by D those functions f satisfying

i. $f \in D_0$,

ii. $\int_0^\infty K(x)f(x)dx + \alpha f'(0) - \beta f(0) = 0$.

It is clear that, D is dense in $L^2(R_+)$ We define the operator L by $Lf = l(f)$ for all f in D.

In this paper, we discuss the discrete spectrum of L and prove that this operator has a finite number of eigenvalues and spectral singularities and each of them is a finite multiplicity, under the conditions

$$\lim_{x\to\infty} Q(x) = 0, \qquad \sup_{x\in R_+}\left\{e^{\varepsilon\sqrt{x}}\left[\left|Q'(x)\right| + \left|K(x)\right|\right]\right\} < \infty, \quad \varepsilon > 0$$

Afterwards the properties of the principal functions corresponding to the spectral singularities of L are obtained.

2. Solutions of *l(y)=0*

Let us suppose the functions Q satisfy

$$\int_0^\infty x\left\{\left|Q(x)\right| + \left|Q'(x)\right|\right\}dx < \infty . \tag{2.1}$$

Under the condition (2.1) the equation *l(y)=0* has the solution

$$e^+(x,\lambda) = e^{iw(x)+i\lambda x} + \int_x^\infty A^+(x,t)e^{i\lambda t}dt \tag{2.2}$$

and

$$e^-(x,\lambda) = e^{-iw(x)-i\lambda x} + \int_x^\infty A^-(x,t)e^{-i\lambda t}dt \tag{2.3}$$

for $\lambda \in \overline{C}_+ = \{\lambda : \lambda \in C,\ \operatorname{Im}\lambda \geq 0\}$ and $\lambda \in \overline{C}_- = \{\lambda : \lambda \in C,\ \operatorname{Im}\lambda \leq 0\}$, respectively, where $w(x) = \int_x^\infty Q(t)dt$ and the kernels $A^\pm(x,t)$ may be expressed in terms of Q. $A^\pm(x,t)$ are continuously differentiable with respect to their arguments and

$$\left|A^\pm(x,t)\right| \leq C\xi\left(\frac{x+t}{2}\right)\exp\{\zeta(x)\} \tag{2.4}$$

$$\left|A_x^\pm(x,t)\right|, \left|A_t^\pm(x,t)\right| \leq C\left[\xi^2\left(\frac{x+t}{2}\right) + \theta\left(\frac{x+t}{2}\right)\right] \tag{2.5}$$

hold, where

$$\xi(x) = \int_x^\infty \left\{\left|Q(t)\right|^2 + \left|Q'(t)\right|\right\}dt,\ \zeta(x) = \int_x^\infty \left\{\left|Q(t)\right|^2 + 2\left|Q(t)\right|\right\}dt, \tag{2.6}$$

$$\theta(x) = \frac{1}{4}\left[\left|Q(x)\right|^2 + \left|Q'(x)\right|\right], \tag{2.7}$$

and C>0 is a constant. Therefore $e^+(x, \lambda)$ and $e^-(x, \lambda)$ are analytic with respect to λ in $\lambda \in \overline{C}_+ = \{\lambda : \lambda \in C,\ \operatorname{Im}\lambda \geq 0\}$ and $\lambda \in \overline{C}_- = \{\lambda : \lambda \in C,\ \operatorname{Im}\lambda \leq 0\}$, respevtively and continuous up to the real axis. $e^\pm(x,\lambda)$ also satisfy

$$e^{\pm}(x,\lambda) = e^{\pm iw(x) \pm i\lambda x} + O\left(\frac{e^{\mp x \operatorname{Im}\lambda}}{|\lambda|}\right), \ \lambda \in \overline{C}_{\pm}, \ |\lambda \to \infty|, \tag{2.8}$$

$$e_x^{\pm}(x,\lambda) = \pm i[\lambda - Q(x)]e^{\pm iw(x) \pm i\lambda x} + O(1), \ \lambda \in \overline{C}_{\pm}, \ |\lambda \to \infty|, \tag{2.9}$$

Let $\varphi^{\pm}(x,\lambda)$ denote the solutions of *l(y)=0* subject to the conditions

$$\lim_{x\to\infty} e^{\pm i\lambda x}\varphi^{\pm}(x,\lambda) = 1, \ \lim_{x\to\infty} e^{\pm i\lambda x}\varphi_x^{\pm}(x,\lambda) = \mp i\lambda, \quad \lambda \in \overline{C}_{\pm} \tag{2.10}$$

moreover

$$W\left[e^{\pm}(x,\lambda), \varphi^{\pm}(x,\lambda)\right] = \mp 2i\lambda, \quad \lambda \in \overline{C}_{\pm} \tag{2.11}$$

$$W\left[e^{+}(x,\lambda), e^{-}(x,\lambda)\right] = -2i\lambda, \quad \lambda \in R = (0,\infty) \tag{2.12}$$

hold, where $W[f_1, f_2]$ is the Wronskian of f_1 and f_2 . The results started above have been obtained by Jaulent-Jean.

1. The Resolvent and The Continuous Spectrum of L

Let us consider the following functions

$$N^{\pm}(\lambda) = \int_0^\infty K(x)e^{\pm}(x,\lambda)dx + \alpha e_x^{\pm}(0,\lambda) - \beta e^{\pm}(0,\lambda), \tag{3.1}$$

$$N_1^{\pm}(\lambda) = \int_0^\infty K(x)\varphi^{\pm}(x,\lambda)dx + \alpha \varphi_x^{\pm}(0,\lambda) - \beta \varphi^{\pm}(0,\lambda)$$

$$g^{\pm}(t,\lambda) = \pm\frac{1}{2i\lambda}\left\{N^{\pm}(\lambda)\varphi^{\pm}(t,\lambda) - N_1^{\pm}(\lambda)e^{\pm}(t,\lambda) - \varphi^{\pm}(t,\lambda)\int_t^\infty e^{\pm}(x,\lambda)K(x)dx\right\}$$
$$\pm\frac{1}{2i\lambda}\left\{e^{\pm}(t,\lambda)\int_t^\infty \varphi^{\pm}(x,\lambda)K(x)dx\right\}$$

Let

$$G(x,t,\lambda) = \begin{cases} G^{+}(x,t,\lambda), & \lambda \in C_{+} \\ G^{-}(x,t,\lambda), & \lambda \in C_{-} \end{cases} \tag{3.2}$$

be the Green's function of L obtained by the standart techniques, where

$$G^{\pm}(x,t,\lambda) = G_1^{\pm}(x,t,\lambda) + G_2^{\pm}(x,t,\lambda) \tag{3.3}$$

$$G_1^{\pm}(x,t,\lambda) = \frac{e^{\pm}(x,\lambda)g^{\pm}(t,\lambda)}{N^{\pm}(\lambda)}, \tag{3.4}$$

$$G_2^{\pm}(x,t,\lambda) = \begin{cases} 0, & 0 \le t < x \\ -\frac{e^{\pm}(x,\lambda)\varphi^{\pm}(t,\lambda)}{2i\lambda} + \frac{e^{\pm}(t,\lambda)\varphi^{\pm}(x,\lambda)}{2i\lambda}, & x \le t < \infty \end{cases}. \tag{3.5}$$

We will denote the continuous spectrum of L by $\sigma_c(L)$. Using (3.2)-(3.5) in a similar way to Theorem4.4 in Krall(Krall, 1965) we have the following

Theorem 3.1 $\sigma_c(L) = (-\infty, \infty)$

4. The Eigenvalues and the Spectral Singularities of L

It is trivial from (2.11) and (2.12) that the functions $\psi^{\pm}(x,\lambda)$ and $\psi(x,\lambda)$ defined by

$$\psi^{\pm}(x,\lambda) = N_1^{\pm} e^{\pm}(x,\lambda) - N^{\pm}\psi^{\pm}(x,\lambda), \quad \lambda \in C_{\pm} \tag{4.1}$$

$$\psi(x,\lambda) = N^{+} e^{-}(x,\lambda) - N^{-} e^{+}(x,\lambda), \quad \lambda \in R^{*} = R/\{0\} \tag{4.2}$$

are the solutions of the boundary value problem

$$y'' + (\lambda - Q(x))^2 y = 0, \qquad x \in R_+,$$

$$\int_0^{\infty} K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0,$$

where

$$N_1^{\pm}(\lambda) = \int_0^{\infty} K(x)\varphi^{\pm}(x,\lambda)dx + \alpha\varphi_x^{\pm}(0,\lambda) - \beta\varphi^{\pm}(0,\lambda).$$

Let us denote the eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From (2.8), (2.10), (3.4) and (4.2) it is trivial that

$$\sigma_d(L) = \{\lambda : \lambda \in C_+,\ N^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in C_-,\ N^{-}(\lambda) = 0\} \tag{4.3}$$

$$\sigma_{ss}(L) = \{\lambda : \lambda \in R^{*},\ N^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in R^{*},\ N^{-}(\lambda) = 0\} \tag{4.4}$$

$$\{\lambda : \lambda \in R^{*},\ N^{+}(\lambda) = 0\} \cap \{\lambda : \lambda \in R^{*},\ N^{-}(\lambda) = 0\} = \varnothing$$

Definition 4.1 The multiplicity of a zero $N^{+}(\lambda)$(or $N^{-}(\lambda)$) in $\overline{C}_+$ (or $\overline{C}_-$) is called as the multiplicity of corresponding eigenvalue or spectral singularity of L

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of $N^{+}(\lambda)$ and $N^{-}(\lambda)$ in $\overline{C}_+$ and $\overline{C}_-$, respectively. For the sake of siplicity we will consider only the zeros of $N^{+}(\lambda)$ in $\overline{C}_+$. A similar procedure may also be employed for zeros of $N^{-}(\lambda)$ in $\overline{C}_-$. Let us define

$$M_1^{+} = \{\lambda : \lambda \in C_+,\ N^{+}(\lambda) = 0\},\ M_2^{+} = \{\lambda : \lambda \in R^{*},\ N^{+}(\lambda) = 0\}.$$

Lemma 4.2. Let $K \in L^1(R_+) \cap L^2(R_+)$. Under the condition (2.1) we have

i. The set M_1^+ is bounded and has at most a countable number of elements and its limit points can lie only in a bounded subinterval of real axis

ii. The set M_2^+ is compact.

Proof: From (2.2) we obtain that $N^+(\lambda)$ is analytic in $\overline{C}_+$ and has the form

$$N^+(\lambda) = i\lambda\alpha e^{iw(0)} - \alpha\left[iQ(0)e^{iw(0)} + A^+(0,0)\right] + \int_0^\infty f^+(t)e^{i\lambda t}dt, \tag{4.5}$$

where

$$f^+(t) = K(t)e^{iw(t)} + \int_0^\infty K(x)A^+(x,t)dx + \alpha A_x^+(0,t) - \beta A^+(0,t)\cdot \tag{4.6}$$

since $f^+(t) \in L^1(R_+)$ then

$$N^+(\lambda) = i\lambda\alpha e^{iw(0)} - \alpha\left[iQ(0)e^{iw(0)} + A^+(0,0)\right] - \beta e^{iw(0)} + o(1), |\lambda| \to \infty \tag{4.7}$$

holds by (4.5). The proof may be completed by (4.7).

Now, let us assume that

$$\int_0^\infty \left\{e^{\varepsilon x}\left[|Q'(x)| + |K(x)|\right]\right\}dx < \infty \qquad \varepsilon > 0 \tag{4.8}$$

holds.

Theorem 4.3. Under the condition (4.8) the operator L has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity.

We easily prove Theorem 4.3 using the analytic continuation technique.

Now let us discuss whether the hyphothesis of Theorem 4.3 can be weakened to attain the same result. For this we will assume that

$$\lim_{x\to\infty} Q(x) = 0, \quad \sup_{x\in R_+}\left\{e^{\varepsilon\sqrt{x}}\left[|Q'(x)| + |K(x)|\right]\right\} < \infty, \quad \varepsilon > 0 \tag{4.9}$$

hold. From (2.2), (2.4) and (2.5), it is clear that under the condition (4.9) the function $N^+(\lambda)$ is analytic in C_+ and all of its derivatives are continuous in $\overline{C}_+$. So

$$\left|N^+(\lambda) - i\lambda\alpha e^{iw(0)}\right| < \infty, \quad \lambda \in \overline{C}_+ \tag{4.10}$$

$$\left|\frac{d^r}{d\lambda^r}N^+(\lambda)\right| \le A_r, \quad \lambda \in \overline{C}_+, \quad r = 1,2,... \tag{4.11}$$

hold, where

$$A_r = 2^r c\int_0^\infty t^r \exp\left\{-\frac{\varepsilon}{2}\sqrt{t}\right\}dt, \quad r = 1,2,... \tag{4.12}$$

and c>0 is a constant.

Let us denote the set of all limit points of M_1^+ and M_2^+ by M_3^+ and M_4^+, respectively and set of all zeros $N^+(\lambda)$ with infinity multiplicity in $\overline{C}_+$ by M_5^+. It is clear that

$$M_3^+ \subset M_2^+, \quad M_4^+ \subset M_2^+, \quad M_5^+ \subset M_2^+ .$$

Using of the continuity of all derivatives of $N^+(\lambda)$ on the real axis we obtain

$$M_3^+ \subset M_5^+, \quad M_4^+ \subset M_5^+ . \tag{4.13}$$

We will use the following uniqueness theorem of Pavlov for the analytic functions on the upper half-plane, to prove the next result.

Pavlov's Theorem: Let us assume that the function *f* is analytic in C_+, all of its derivatives are continuous up to the real axis and there exist T>0 such that

$$\left|f^{(r)}(z)\right| \le A_{r,\ r=0,1,...} \quad z \in \overline{C}_+,\ |z| < 2 \tag{4.14}$$

and

$$\left|\int_{-\infty}^{\infty} \frac{\ln|f(x)|}{1+x^2}dx\right| < \infty, \ \left|\int_{T}^{\infty} \frac{\ln|f(x)|}{1+x^2}dx\right| < \infty \tag{4.15}$$

hold. If the set Q with linear Lebesque measure zero is the set of all zeros of the function *f* with infinity multiplicity and if

$$\int_0^h \ln E(s) d\mu(Q_s) = -\infty$$

holds, then $f(z) \equiv 0$, where $E(s) = \inf_r \frac{A_r s^r}{r!}$, $r = 0,1,..., \mu(Q_s)$ is the linear Lebesque measure of s-neighbourhood of Q and h is an arbitrary positive constant(Pavlov, 1967).

Lemma 4.4. $M_5^+ = \varnothing$.

Proof: It is trivial from Lemma 4.2 and (4.10), (4.11) that $N^+(\lambda)$ satisfies (4.14) and (4.15). Since the function $N^+(\lambda)$ is not equal to zero identically, then by the Pavlov's Theorem M_5^+ satisfies

$$\int_0^h \ln E(s) d\mu(M_{5,s}^+) > -\infty , \tag{4.16}$$

where $E(s) = \inf_r \frac{A_r s^r}{r!}$, $\mu(M_{5,s}^+)$ is the linear Lebesque measure of s-neighbourhood of M_5^+ and the constant A_r is defined by (4.12).

Now we will obtain the following estimates for A_r :

$$A_r = 2^r c \int_0^\infty t^r \exp\left\{-\frac{\varepsilon}{2}\sqrt{t}\right\} dt \leq B b^r r^r r!, \tag{4.17}$$

where B and b are constants depending on c and ε. From (4.17) we arrive to

$$E(s) = \inf_r \frac{A_r s^r}{r!} \leq B \inf_r \left\{ b^r s^r r^r \right\} \leq B \exp\left\{ -b^{-1} e^{-1} s^{-1} \right\}$$

or by (4.16)

$$\int_0^h \frac{1}{s} d\mu(M_{5,s}^+) < \infty \cdot \tag{4.18}$$

holds for an arbitrary s, if and only if $\mu(M_{5,s}^+) = 0$ or $M_5^+ = \varnothing$.

Theorem 4.5. Under the condition (4.9) the operator L has a finite number of eigenvalues and spectral singularities and eah of them is of a finite multiplicity.

Proof: To be able to prove theorem we have to show that the function N^+ and N^- have finite number of zeros with the finite multiplicities in $\overline{C}_+$ and $\overline{C}_-$, respectively. We are going to prove it only for N^+. The similar proof can be given for N^-. From Lemma 4.4 and (4.13) we obtain that $M_3^+ = M_4^+ = \varnothing$. So the bounded sets M_1^+ and M_2^+ have no limit points(see lemma 4.2), ie., the function N^+ has only a finite number of zeros in $\overline{C}_+$. Since $M_5^+ = \varnothing$ these zeros are of finite multiplicity.

Now we will discuss the structure of M_5^+ under the weaker codition (4.9).

Theorem 4.6. If

$$\lim_{x\to\infty} Q(x) = 0, \quad \sup_{x\in R_+} \left\{ e^{\varepsilon x^\delta} \left[\left| Q'(x) \right| + \left| K(x) \right| \right] \right\} < \infty, \quad \varepsilon > 0, 0 < \delta < \frac{1}{2} \tag{4.19}$$

hold, then

$$\sum_n \left(l_n^+ \right)^{\frac{1-2\delta}{1-\delta}} < \infty$$

holds, where $\left\{ l_n^+ \right\}$ is the sequence of lengths of all finite complementary intervals of M_5^+.

Proof: From (2.4)-(2.7) and (4.19) we obtain

$$\left| \frac{d^r}{d\lambda^r} N^+(\lambda) \right| \leq B b^r r! r^{r^{\frac{1-\delta}{\delta}}}, \quad \lambda \in \overline{C}_+ \tag{4.20}$$

where B and b constants depending on ε and δ. Let $G_\theta = G_\theta(C_+),\ 0 < \theta < 1$ be the Gevrey classof analytic functions in C_+ and Φ_θ be the system of all sets of uniqueness for G_θ (Hruscev, 1977), Hence it is trivial from (4.20) that

$N^{+} \in G_{\frac{1-\delta}{\delta}},\ M_5^{+} \notin \Phi_{\frac{1-\delta}{\delta}}$.

Then by the Carlesson Theorem we ontain

$\sum_n \left(l_n^{+}\right)^{\frac{1-2\delta}{1-\delta}} < \infty$.

The above theorem shows that the eigenvalues of L may not be of finite numbers if the Condition (4.19) is satisfied

5. Principal Functions.

In this section we assume (4.9) holds. Let $\lambda_1^{+}, \lambda_2^{+} \ldots \lambda_j^{+}$ and $\lambda_1^{-}, \lambda_2^{-} \ldots \lambda_k^{-}$ denote the zeros of the functions N^{+} in C_{+} (which are the eigenvalues of L) with multiplicities $\mathrm{m}_1^{+}, \mathrm{m}_2^{+} \ldots \mathrm{m}_j^{+}$ and $m_1^{-}, m_2^{-} \ldots m_k^{-}$, respevtively. Similary, let $\lambda_1, \lambda_2 \ldots \lambda_v$ and $\lambda_{v+1}, \lambda_{v+2} \ldots \lambda_l$ be zeros of $N^{+}(\lambda)$ and $N^{-}(\lambda)$ in R* with multiplicities $n_1, n_2 \ldots n_v$ and $n_{v+1}, n_{v+2} \ldots n_l$, respectively.

$$\psi^{+}(x,\lambda_i^{+}), \left\{\frac{d}{d\lambda}\psi^{+}(x,\lambda)\right\}_{\lambda=\lambda_i^{+}}, \ldots \left\{\frac{d^{m_i^{+}-1}}{d\lambda^{m_i^{+}-1}}\psi^{+}(x,\lambda)\right\}_{\lambda=\lambda_i^{+}}$$

and

$$\psi^{-}(x,\lambda_i^{+}), \left\{\frac{d}{d\lambda}\psi^{-}(x,\lambda)\right\}_{\lambda=\lambda_i^{+}}, \ldots \left\{\frac{d^{m_i^{+}-1}}{d\lambda^{m_i^{+}-1}}\psi^{-}(x,\lambda)\right\}_{\lambda=\lambda_i^{+}}$$

are called the principal functions corresponding to the eigenvalues $\lambda = \lambda_i^{+},\ i = 1,2,\ldots,j$ and $\lambda = \lambda_i^{-},\ i = 1,2,\ldots,k$ of L, respectively, where $\psi^{\pm}$ are defined by (4.1). Similarly

$$\psi(x,\lambda_i^{+}), \left\{\frac{d}{d\lambda}\psi(x,\lambda)\right\}_{\lambda=\lambda_i^{+}}, \ldots \left\{\frac{d^{m_i^{+}-1}}{d\lambda^{m_i^{+}-1}}\psi(x,\lambda)\right\}_{\lambda=\lambda_i^{+}}, \quad i = 1,\ldots v, v+1,\ldots l$$

are the prencipal functions corresponding to the spectral singularities of L, where ψ is defined by (4.2). From (2.8), (4.1) and (4.3) we obtain that the principal functions corresponging to the eigenvalues of L are in $L^2(R_+)$.

Let us introduce the Hilbert spaces

$$H_{+} = \left\{ f : \int_0^{\infty} (1+|x|)^{2n_0} |f(x)|^2 dx < \infty \right\},$$

$$H_{-} = \left\{ g : \int_0^{\infty} (1+|x|)^{-2n_0} |g(x)|^2 dx < \infty \right\}$$

with

$$\|f\|_{+}^2 = \int_0^{\infty} (1+|x|)^{2n_0} |f(x)|^2 dx, \quad \|g\|_{-}^2 = \int_0^{\infty} (1+|x|)^{-2n_0} |g(x)|^2 dx,$$

respectively, where $n_0 = \max\{n_1, n_2, ..., n_l\}$. It is clear that

$$H_+ \underset{\neq}{\subset} L_2 R_+) \underset{\neq}{\subset} H_- .$$

Obviously H_- is isomorphic to the dual of H_+(Berenzanski, 1968).

Therorem 5.1.

$$\left\{\frac{d^n}{d\lambda^n}\varphi(.,\lambda)\right\}_{\lambda=\lambda_i} \notin L^2(R_+),\ n = 0,...,n_i - 1,\ i = 1,...,v, v+1,...,l$$

$$\left\{\frac{d^n}{d\lambda^n}\varphi(.,\lambda)\right\}_{\lambda=\lambda_i} \in H_-,\ n = 0,...,n_i - 1,\ i = 1,...,v, v+1,...,l$$

Proof: Let $0 \le n \le n_i - 1$ and $1 \le i \le v$. Using (4.2) we have

$$\left\{\frac{d^n}{d\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i} = \sum_{j=0}^{n} A_j(\lambda_i)\left\{\frac{d^j}{d\lambda^j}e^+(x,\lambda)\right\}_{\lambda=\lambda_i}, \qquad (5.1)$$

where

$$A_j(\lambda_i) = \binom{n}{j}\left\{\frac{d^{n-j}}{d\lambda^{n-j}}N^-(\lambda)\right\}_{\lambda=\lambda_i}.$$

The proof of theorem is obtained from (2.2), (2.8) and (5.1). In a similar way we may also prove the result for $0 \le n \le n_i - 1$ and $v+1 \le i \le l$.

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