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FREE VIBRATIONS of VISCOELASTIC MATERIALS FOR ARBITRARY KERNEL

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Abstract

For an isotropic viscoelastic constitutive relation in Boltzmann-Volterra form the problems of vibrations of linear viscoelastic materials reduce to the solution of a certain integro-differential equation which coincides with the equation of vibrations of a viscoelastic system with single degree of freedom. Full solution of this equation for arbitrary kernel of relaxation is constructed in the present article. Iteration processes for calculating frequency and damping coefficient are given. There are two special cases of the relaxation kernel that a solution for the problem involved is given. One is the case of the sum of exponents, in the other the kernel is the sum of Dirac delta and an exponent. Analysis of obtained solutions and their comparisons with results available in literature are performed.

Key words: viscoelastic, damped vibration, any kernel, Laplace transform.

AMS subject classification: 45J05; 74H45.

1. Introduction

The theory of linear viscoelasticity finds numerous technical applications, connected with studies of the creeps of metals, plastics, concrete rock, polymers, composites and other solids. This theory found extensive development in the last half-century. The original methods of solutions are worked out to solve quasistatic and dynamic problems. Special place is devoted to transient dynamical problems often used in rheology for the determination of dynamic mechanical properties of viscoelastic materials. First research in this field belongs to 40's, when Ishlinskiy [1] explored vibrations of a viscoelastic bar of standard linear body. The method of separation of variables, operational method, and integral transforms, numerical methods are mainly used in solving dynamic problems. In [2] Rozovskiy investigate the problem of vibrations of viscoelastic body by using the operational method. Laplace transformation is used by many investigators [4]. This method is connected with inverse problem according to Mellin formula, evaluation of which is realized by the method of contour integral and by the theory of residues. The knowledge of all singularities of the integrand is necessary in the process of contour integral. This restriction necessitates a complete description of the kernel of stress-strain relation. The requirement of a more complex kernel to reflect the mechanical properties better makes it practically impossible to use the method of contour integrals. This situation is a cause of the appearance of approximate, asymptotic and numerical methods. In [5] method of averaging that belongs to Bogoliubov [6] is applied to vibration problems of viscoelasticity by Ilyushin and his colleagues. According to method of averaging, the viscous strength of material is small enough in comparison with elastic strength. The result obtained in [5] is found in [7] by Laplace transform method. The problem of free vibrations of viscoelastic system, with single degree of freedom has been analysed in [4] by method of complex modules. Here the ratio of imaginary part of complex module to its real part is considered to be small enough and beginning from the second, all powers of this ratio are neglected. Free vibrations of plates are investigated in [8] .This solution is constructed on the basis of generalization of results obtained in [5] and [7].

There are many works devoted to study of vibrations of viscoelastic bodies with specific kernels and models. The Voigt, Kelvin, Maxwell models and standard model of linear viscoelastic material are used in [9-12] for investigation of vibrations of viscoelastic Timoshenko beams. Kernels in the form of the sum of exponential functions with negative indices are often used [4].The problems are solved by the method of Laplace integral transform, but the inverses are found by using residue theory. The knowledge of poles of integrands is assumed. Using this method Struik [13] studied a problem of free damped vibrations of linear viscoelastic materials and used the result for the determination of mechanical properties of materials. Kernels in the form of the sum of exponential functions were used in [22] to solve the problems of free damped vibrations of viscoelastic rods, beams, plates and shells. Iteration processes for calculating frequency and damping coefficient are given.

This analysis shows the state of theoretical investigation of problems of transient vibrations of viscoelastic systems, which is in attention of scientists for more than 50 years. It offers no general theory about vibrations of viscoelastic systems with arbitrary mechanical properties. Full solution of this problem for any kernel of relaxation has been obtained in the given work. The problem is solved by means of Laplace transform and inverse transforms are found by the method of contour integration and convolution of functions. Existence of two complex-conjugate poles with negative real parts of integrand in Mellin formula for any monotonously decreasing kernel of relaxation has been proved. Real and imaginary parts of poles, which correspond to damping coefficient and frequency of viscoelastic vibrations respectively, are found by the method of iteration. Convergence of the iteration procedure is proved. Detaching the effect of complex poles from the denominator of the integrand in Mellin formula and factorising the rest part in geometric progression, using the convolution of functions, the solution for any kernel of relaxation is found. The absolute and uniform convergence of series in obtained formula and its derivatives of the first and second orders are proved. As an example the kernel of relaxation in the form of the sum of derivatives of Dirac delta with single exponent is considered.

2. Statement of Problem

Equations of transient vibrations of flexible string or longitudinal vibrations of homogeneous rod, transient vibrations of beam and plate of viscoelastic material are

$$\frac{\partial^2 w}{\partial x^2} - \varepsilon \int_o^t \Gamma(t-\tau) \frac{\partial^2 w}{\partial x^2} d\tau + \frac{Q}{E} = \frac{m}{E} \frac{\partial^2 w}{\partial t^2}, \tag{2.1}$$

$$\frac{\partial^4 w}{\partial x^4} - \varepsilon \int_o^t \Gamma(t-\tau) \frac{\partial^4 w}{\partial x^4} d\tau + \frac{m}{b} \frac{\partial^2 w}{\partial t^2} = \frac{Q}{b}, \tag{2.2}$$

$$\Delta^2 w - \varepsilon \int_o^t \Gamma(t-\tau) \Delta^2 w d\tau + \frac{m}{D} \frac{\partial^2 w}{\partial t^2} = \frac{Q}{D} \tag{2.3}$$

where w is displacement, E is the modulus of instantaneous elasticity, Q is the transverse load, m is the mass, b and D are flexural rigidities, $\varepsilon\Gamma(t)$ is the kernel of relaxation, ε is a positive parameter which we may put equal to one at the end of the operation. These equations can be found in many text books, such as [3, 15].

We will consider a viscoelastic solid for which the kernel of relaxation $\varepsilon\Gamma(t)$ is a positive function which satisfies the condition [15]

$$\int_o^t \varepsilon\Gamma(\tau) d\tau << 1 \tag{2.4}$$

for any t. For this reason we will assume ε to be a small positive parameter.

To the equations (2.1)-(2.3) it is necessary to connect appropriate boundary and initial conditions. The initial conditions, appropriate to an initial stress-free state of rest, may be given by

$$w = W_0(x), \quad \frac{\partial w}{\partial t} = W_1(x) \quad \text{for } t = 0, \tag{2.5}$$

where $W_0(x)$ and $W_1(x)$ are given functions. As the boundary conditions, for example, we may put $w = 0$, $\frac{\partial w}{\partial x} = 0$, for the clamped edge, and $w = 0$, $\frac{\partial^2 w}{\partial x^2} = 0$, for the simply supported edge.

Let the load $Q(x,t)$ for simply supported beam be represented by the Fourier series on x

$$Q(x,t) = m \sum_{k=1}^{\infty} \varphi_k(t) \sin\frac{k\pi x}{l},$$

where l is the length of the beam. A solution of equation (2.2) is assumed to be in the form

$$w(x,t) = \sum_{k=1}^{\infty} T_k(t) \sin\frac{k\pi x}{l}. \tag{2.6}$$

For the unknown functions $T_k(t)$ we obtain an integro-differential equation

$$T_n'' + \lambda_n^2 T_n = \varepsilon \lambda_n^2 \int_0^t \Gamma(t-\tau) T_n(\tau) d\tau + \varphi_n(t), \tag{2.7}$$

where $\lambda_n = (\frac{n\pi}{l})^2 \sqrt{\frac{b}{m}} (n = 1,2,...)$

are the frequencies of elastic vibrations.

Let the functions $W_0(x)$ and $W_1(x)$ be represented by the Fourier series

$$W_0(x) = \sum_{k=1}^{\infty} T_{ko} \sin\frac{k\pi x}{l}, \quad W_1(x) = \sum_{k=1}^{\infty} T_{k1} \sin\frac{k\pi x}{l}.$$

Using (2.5) and (2.6) we find the initial conditions for the equation (2.7)

$$T_k(0) = T_{ko}, \quad T_\kappa'(0) = T_{\kappa 1}.$$

Exactly the same method without any alterations applies to vibrations of plates, shells, and arbitrary three-dimensional bodies if the eigenfunctions and eigenvalues of the elasticity problem are known. Using the method of separation of variables or Bubnov-Galerkin method, replacing differential operators with respect to space coordinates with finite differences and many other methods, the dynamical system of viscoelasticity can be reduced to the equations of form (2.7).

The problems for equation (2.7) are investigated by many authors [4, 5, 7, 8, 13, 15-19, 22], however the exact solutions have been constructed only for the exponential kernel.

3. Solution by Laplace Transform

The equation

$$T'' + \lambda^2 T = \varepsilon\lambda^2 \int_o^t \Gamma(t-\tau)T(\tau)d\tau , \quad t > 0 \tag{3.1}$$

will be solved for the following initial conditions

$$T(0) = T_o, \quad T'(0) = T_1. \tag{3.2}$$

Using Laplace transform we obtain the following image of the solution of the problem (3.1), (3.2)

$$\overline{T}(p) = \frac{pT_o + T_1}{p^2 + \lambda^2 - \varepsilon\lambda^2\overline{\Gamma}(p)} \tag{3.3}$$

where $\overline{T}(p)$ denotes the Laplace integral $\overline{T}(p) = \int_o^\infty T(t)e^{-pt}dt$,

p is the complex parameter of transformation. The function $\overline{T}(p)$ represented by (3.3) and the image of kernel of relaxation $\overline{\Gamma}(p)$ are analytic in the right half- plane $\operatorname{Re} p > 0$.

Assume that the Laplace transform $\overline{T}(p)$ is an analytic function in the whole of the complex p-plane except at isolated singular points. The inverse transformation of function (3.3) can be found by using the well-known Mellin formula

$$T(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(pT_o+T_1)e^{pt}}{p^2+\lambda^2-\varepsilon\lambda^2\overline{\Gamma}(p)}dp \tag{3.4}$$

here the integration is carried out in the plane of complex variable p along an infinite straight line parallel to the imaginary axis and situated so that all singular points of the function $\overline{T}(p)$ are located to the left of this straight line. The calculation of this integral is usually accomplished through the use of the residue theory. For this reason it is necessary to know the poles and the branch points of integrand considered before being analytically continued to the left-half p-plane. Poles are roots of the equation

$$p^2+\lambda^2-\varepsilon\lambda^2\overline{\Gamma}(p)=0. \tag{3.5}$$

For $\varepsilon=0$ the equation (3.5) has two solutions $p_1=i\lambda$ and $p_2=-i\lambda$.

Lemma. *Let $\Gamma(t)$ be a positive monotonously decreasing convex and continuous function for $t\geq 0$, let it vanish for $t<0$ and let the inequality (2.4) hold. The equation (3.5) has just two complex roots having negative real parts.*

Proof. Let $-\alpha\pm i\beta$ be the roots of the equation (3.5). Substituting $p=-\alpha+i\beta$ to (3.5) and splitting in real and imaginary parts, gives

$$\hat{\Gamma}_c\equiv\int_o^\infty e^{\alpha\tau}\Gamma(\tau)\cos\beta\tau d\tau=\frac{\alpha^2+\lambda^2-\beta^2}{\varepsilon\lambda^2},\hat{\Gamma}_s\equiv\int_o^\infty e^{\alpha\tau}\Gamma(\tau)\sin\beta\tau d\tau=\frac{2\alpha\beta}{\varepsilon\lambda^2}. \tag{3.6}$$

Thus the equation (3.5) is equivalent to the system of two equations (3.6).For convenience integrals in (3.6) are denoted by $\hat{\Gamma}_c$ and $\hat{\Gamma}_s$ respectively. The integrals $\hat{\Gamma}_c$ and $\hat{\Gamma}_s$ are convergent if the function $e^{\alpha t}\Gamma(t)$ satisfies the Dirichlet conditions for $0<t<\infty$, or if this function is monotonously decreasing. It is evident that each of the functions $\hat{\Gamma}_c(\alpha,\beta)$ and $\hat{\Gamma}_s(\alpha,\beta)$ tends to zero as β tends to infinity through any set of values

[20]. Using these limits from (3.6) and (2.4) we deduce the well-known results of the literature [3, 21]

$$\lim_{\lambda\to\infty}\frac{\beta}{\lambda}=1, \quad \operatorname{hm}_{\lambda\to\infty}\alpha=\frac{\varepsilon\Gamma(0)}{2}; \tag{3.7}$$

$$\beta^2\approx\lambda^2[1-\varepsilon\int_0^\infty\Gamma(\tau)d\tau],\ \alpha\approx 0\ \text{ for }\ \lambda<<1. \tag{3.8}$$

From (3.6) we find

$$\varepsilon\sqrt{\hat{\Gamma}_c^2+\hat{\Gamma}_s^2}=\lambda^{-2}\left[\left(\alpha^2+\lambda^2-\beta^2\right)^2+4\alpha^2\beta^2\right]^{1/2}=\frac{2\varepsilon}{\lambda}\sqrt{\alpha^2+\gamma^2}\left(1-\frac{\gamma}{\lambda}+\frac{\alpha^2+\gamma^2}{4\lambda^2}\right)^{1/2}$$

By the help of (3.7) and (3.8) we may put

$$\varepsilon\sqrt{\hat{\Gamma}_c^2+\hat{\Gamma}_s^2}<1. \tag{3.9}$$

The inequalities $0<\varepsilon\hat{\Gamma}_c<1$ give $\beta^2>\alpha^2>0$.Thus α and β are real and we may put $\beta>0$. As $\hat{\Gamma}_s>0$, from the second of the equality (3.6) we deduce $\alpha>0$.

The roots α and β may be calculated by iteration procedure. From (3.6) we define the following system of iterations

$$\alpha_{n+1}=\frac{\varepsilon\lambda^2}{2\beta_n}\hat{\Gamma}_s(\alpha_n,\beta_0),\beta_{n+1}=[\lambda^2+\alpha_{n+1}^2-\varepsilon\lambda^2\hat{\Gamma}_c(\alpha_{n+1},\beta_n)]^{1/2}\ n=0,1,2,\ldots,$$

$$\alpha_0=0,\beta_0=\lambda\ . \tag{3.10}$$

From (3.10) we obtain

$$\alpha=\frac{\varepsilon\Gamma_s\lambda}{2}+\varepsilon^2\lambda\omega_1+\varepsilon^3\theta_1+\ldots,\quad \beta=\lambda-\gamma\quad,\gamma=\frac{\varepsilon\Gamma_c\lambda}{2}+\varepsilon^2\lambda\omega_2+\varepsilon^3\theta_2+\ldots$$

Where

$$\omega_1=\frac{1}{4}(\Gamma_s\Gamma_c-\lambda\Gamma_c\Gamma_{1c}+\lambda\Gamma_s\Gamma_{1s}),$$

$$\omega_2=\frac{1}{8}(\Gamma_c^2-\Gamma_s^2+2\lambda\Gamma_s\Gamma_{1c}+2\lambda\Gamma_c\Gamma_{1s}),$$

$$\theta_1=\frac{\lambda}{2}[\Gamma_s\omega_2+\Gamma_s\omega_1-\lambda\omega_2\Gamma_{1c}+\lambda\omega_1\Gamma_{1s}-\frac{\lambda^2}{4}\Gamma_s\Gamma_c\Gamma_{2c}+\frac{\lambda^2}{8}\Gamma_{2s}(\Gamma_s^2-\Gamma_c^2)],$$

$$\theta_2 = \frac{\lambda}{2}[\Gamma_c\omega_2 - \Gamma_s\omega_1 + \lambda\omega_1\Gamma_{1c} + \lambda\omega_2\Gamma_{1s} + \frac{\lambda^2}{4}\Gamma_s\Gamma_c\Gamma_{2s} + \frac{\lambda^2}{8}\Gamma_{2c}(\Gamma_s^2 - \Gamma_c^2)],$$

$$\Gamma_{kc} = \int_0^\infty t^k\Gamma(t)\cos\lambda t dt, \Gamma_{ks} = \int_0^\infty t^k\Gamma(t)\sin\lambda t dt, k = 0,1,2,$$

$\Gamma_{0c} \equiv \Gamma_c$ and $\Gamma_{0s} \equiv \Gamma_s$ are sine and cosine Fourier transforms of $\Gamma(t)$.

Consider the integrals

$$\hat{\Gamma}_{1c} = \int_o^\infty \tau e^{\alpha\tau}\Gamma(\tau)\cos\beta\tau d\tau, \quad \hat{\Gamma}_{1s} = \int_o^\infty \tau e^{\alpha\tau}\Gamma(\tau)\sin\beta\tau d\tau .$$

If the integrals (3.6) are convergent then using Dirichlet theorem it is not difficult to prove the uniform convergences with respect to α and β for $0 \le \alpha \le \varepsilon\Gamma(0)/2,\ \lambda[1-\varepsilon\int_0^\infty\Gamma(\tau)d\tau]^{1/2} \le \beta \le \lambda$ of integrals $\hat{\Gamma}_{1s}$ and $\hat{\Gamma}_{1c}$.

Therefore, by the theorem on differentiation of an improper integral with respect to the parameter, the differentiation under the integral sign in (3.10) with respect to α and β is valid. The derivatives of iteration vector-function with respect to α and β are described by the integrals $\hat{\Gamma}_{1s}$ and $\hat{\Gamma}_{1c}$.To estimate these integrals, consider the product of the integrals $\hat{\Gamma}_s$ and $\hat{\Gamma}_c$

$$2\hat{\Gamma}_s\hat{\Gamma}_c = 2\int_0^\infty\int_0^\infty e^{\alpha(\varsigma+\tau)}\Gamma(\varsigma)\Gamma(\tau)\sin\beta\varsigma\cos\beta\tau d\varsigma d\tau = \int_0^\infty\int_0^\infty \Omega(\varsigma,\tau)e^{\alpha(\varsigma+\tau)}\Gamma(\varsigma+\tau)\sin\beta(\varsigma+\tau)d\varsigma d\tau,$$

where $\Omega(\varsigma,\tau) = \Gamma(\varsigma)\Gamma(\tau)/\Gamma(\varsigma+\tau)$ is a continuous monotonously decreasing convex function of two variables in the domain $X = \{(\varsigma,\tau): \varsigma \ge 0, \tau \ge 0\}$. Besides, $\Omega(0,\tau) = \Omega(\varsigma,0) = \Omega(0,0) = \Gamma(0),\ \lim_{\varsigma\to\infty,\tau\to\infty}\Omega(\varsigma,\tau) = \Omega_0 > 0$ (For the exponential kernel the function $\Omega = \Gamma(0)$ is constant). By the second mean-value theorem of integral calculus there is a point $(\varsigma^*,\tau^*) \in X$ such that

$$2\hat{\Gamma}_s\hat{\Gamma}_c = \Omega(\varsigma^*,\tau^*)\int_0^\infty\int_0^\infty e^{\alpha(\varsigma+\tau)}\Gamma(\varsigma+\tau)\sin\beta(\varsigma+\tau)d\varsigma d\tau .$$

Using functions

$$\varsigma = \frac{s\cos\varphi}{\sqrt{2}\cos(\pi/4-\varphi)}, \quad \tau = \frac{s\sin\varphi}{\sqrt{2}\cos(\pi/4-\varphi)} \quad 0 \le s < +\infty, 0 \le \varphi \le \pi/2,$$

and taking into account that $\varsigma + \tau = s$, after changing variables in the double integral, we find $2\hat{\Gamma}_s\hat{\Gamma}_c = \Omega(\varsigma^*, \tau^*)\hat{\Gamma}_{1s}$. By the same way $\hat{\Gamma}_c^2 - \hat{\Gamma}_s^2 = \Omega(\varsigma^{**}, \tau^{**})\hat{\Gamma}_{1c}$ is obtained, here $(\varsigma^{**}, \tau^{**}) \in X$. The last two relations together with the inequality (3.9) secure the convergence of iteration (3.10) to a unique limit.

The proof of the lemma is complete.

Let us represent the image of the solution in the form

$$\overline{T}(p) = \frac{pT_o + T_1}{p^2 + \lambda^2 - \varepsilon\lambda^2\overline{\Gamma}} = \frac{pT_o + T_1}{(p+\alpha)^2 + \beta^2 - (\varepsilon\lambda^2\overline{\Gamma} + 2\alpha p + \alpha^2 - 2\lambda\gamma + \gamma^2)} =$$

$$= \frac{pT_o + T_1}{(p+\alpha)^2 + \beta^2}\frac{1}{1 - \overline{B}(p)}, \tag{3.11}$$

where

$$\overline{B}(p) = \frac{\varepsilon\lambda^2\overline{\Gamma} + 2\alpha p + \alpha^2 + \beta^2 - \lambda^2}{(p+\alpha)^2 + \beta^2}.$$

In the half-plane $\operatorname{Re} p > 0$ we have $|\overline{B}(p)| < 1$, thus (3.11) may be expanded into geometrical series

$$\overline{T}(p) = \overline{A}(p)(1 + \overline{B} + \overline{B}^2 + \ldots), \tag{3.12}$$

where $$\overline{A}(p) = \frac{pT_o + T_1}{(p+\alpha)^2 + \beta^2}.$$

The inverse transforms of $\overline{A}(p)$ and $\overline{B}(p)$ are

$$A(t) = e^{-\alpha t}[T_o\cos\beta t + \frac{T_1 - \alpha T_o}{\beta}\sin\beta t] = A_0 e^{-\alpha t}\cos(\beta t - \psi), \tag{3.13}$$

$$B(t) = \frac{\varepsilon\lambda^2}{\beta}\int_o^t \Gamma(\tau)e^{-\alpha(t-\tau)}\sin\beta(t-\tau)d\tau + e^{-\alpha t}(2\alpha\cos\beta t + \frac{\beta^2 - \lambda^2 - \alpha^2}{\beta}\sin\beta t),$$

where $A_0 = \sqrt{T_o^2 + (T_1 - \alpha T_o)^2 \beta^{-2}}$ and $\psi = \arctan[(T_1 - \alpha T_o)/\beta T_o]$. As we see, α is the damping coefficient and β is the frequency of viscoelastic vibrations. $2\pi\alpha/\beta$ is the natural logarithm of the decay per period of the damped vibration in (3.13).

The function $B(t)$ may be written by means of improper integrals

$$B(t) = -\frac{\varepsilon\lambda^2}{\beta}\int_t^\infty \Gamma(\tau)e^{-\alpha(t-\tau)}\sin\beta(t-\tau)d\tau + \frac{\varepsilon\lambda^2}{\beta}\int_o^\infty \Gamma(\tau)e^{-\alpha(t-\tau)}\sin\beta(t-\tau)d\tau +$$

$$+e^{-\alpha t}(2\alpha\cos\beta t + \frac{\gamma^2 - \alpha^2 - 2\lambda\gamma}{\beta}\sin\beta t).$$

According to (3.6) the sum of the last two terms is zero, then by substitution $\tau - t = s$ we have

$$B(t) = \frac{\varepsilon\lambda^2}{\beta}\int_o^\infty \Gamma(t+s)e^{\alpha s}\sin\beta s ds \tag{3.14}$$

It is easy to see that $\frac{dB(t)}{dt} < 0$ for $t > 0$, i.e. $B(t)$ is a monotonously decreasing positive function. Using (3.6) we find $B(0) = 2\alpha$ and the equalities

$$B(t) = -\frac{\varepsilon\lambda^2}{\beta}\int_t^\infty \Gamma(s)e^{-\alpha(t-s)}\sin\beta(t-s)ds = \frac{\varepsilon\lambda^2}{\beta}e^{-\alpha t}\int_t^\infty \Gamma(s)e^{\alpha s}\sin\beta(s-t)ds,$$

give

$$B(t) \le 2\alpha e^{-\alpha t}, \tag{3.15}$$

for all $t \ge 0$. So that $B(t)$ approaches to zero more quickly than $e^{-\alpha t}$ and $B(\infty) = 0$.

Remark1. *For given $\Gamma(t)$ the calculation of $B(t)$ from (3.14) and then finding its transform $\overline{B}(p)$ is not difficult. Thus, if we know the original*

$$\frac{\overline{B}(p)}{1-\overline{B}(p)} \doteq \Phi(t), \tag{3.16}$$

from(3.11) we find required solution of the problem (3.1),(3.2) as

$$T(t) = A(t) + \int_0^t A(t-\tau)\Phi(\tau)d\tau. \qquad (3.17)$$

According to (3.11) equation (3.5) may be written as

$$p^2 + \lambda^2 - \varepsilon\lambda^2\overline{\Gamma}(p) = [(p+\alpha)^2 + \beta^2][1 - \overline{B}(p)] = 0 \qquad (3.18)$$

As a result of Lemma1, equation $1 - \overline{B}(p) = 0$ *has only real roots and may be solved easily by comparison with (3.5).*

Let us introduce kernels in such a way that

$$\Gamma_1(t+\tau_1) = \Gamma(t+\tau_1),\ \Gamma_2(t+\tau_1+\tau_2) = \int_0^t \Gamma_1(t+\tau_1-s)\Gamma_1(s+\tau_2)ds, \ldots,$$

$$\Gamma_n(t+\tau_1+\ldots+\tau_n) = \int_0^t \Gamma_{n-1}(t+\tau_1+\ldots+\tau_{n-1}-s)\Gamma_1(s+\tau_n)ds,\ n = 2,3,\ldots \quad (3.19)$$

Original $B_n(t)$ of the function $\overline{B}^n(p)$

$$B_n(t) = \frac{\varepsilon^n\lambda^{2n}}{\beta^n}\int_0^\infty \ldots \int_0^\infty \Gamma_n(t+\tau_1+\ldots+\tau_n)e^{\alpha(\tau_1+\ldots+\tau_n)}\sin\beta\tau_1\ldots\sin\beta\tau_n d\tau_1\ldots d\tau_n,$$

$$(n = 2,3,\ldots) \qquad (3.20)$$

is found by using (3.14),(3.19) and the convolution of functions. Here $B_1(t) \equiv B(t)$.

Using (3.15) and convolution of functions we find

$$B_n(t) \le 2\alpha\frac{(2\alpha t)^{n-1}}{(n-1)!}e^{-\alpha t} \qquad (3.21)$$

for all $t \ge 0$. That $B_n(t)$ is a positive function such that $B_n(0) = 0\ (n \ge 2), B_n(\infty) = 0.$

Now we may write the inverse of (3.12) as

$$T(t) = A(t) + \sum_{n=1}^{\infty}\frac{\varepsilon^n\lambda^{2n}}{\beta^n}\int_0^\infty \ldots \int_0^\infty\int_0^t \Gamma_n(\tau+\tau_1+\ldots+\tau_n)A(t-\tau)e^{\alpha(\tau_1+\ldots+\tau_n)}\sin\beta\tau_1\ldots\sin\beta\tau_n d\tau d\tau_1\ldots d\tau_n$$

$$(3.22)$$

Theorem. *Let the conditions of Lemma 1 hold. Function* $T(t)$ *defined by the formula (3.22) is the solution of the problem (3.1), (3.2).*

Proof. First let us show the convergence of the series in (3.22).Since $\Gamma(t)$ is a monotonously decreasing function, then

$$\Gamma_1(t+\tau_1) \le \Gamma(\tau_1), \Gamma_2(t+\tau_1+\tau_2) \le \Gamma(\tau_1)\Gamma(\tau_2)\int_o^t ds = \Gamma(\tau_1)\Gamma(\tau_2)t, .., \quad (3.23)$$

$$\Gamma_n(t+\tau_1+...+\tau_n) \le \Gamma(\tau_1)...\Gamma(\tau_n)\frac{1}{(n-2)!}\int_o^t (t-s)^{n-2}ds = \Gamma(\tau_1)...\Gamma(\tau_n)\frac{t^{n-1}}{(n-1)!}$$

,...

Using (3.23) and (3.6) we find

$$|T(t)| \le |A(t)| + \sum_{n=1}^{\infty}\frac{\varepsilon^n \lambda^{2n}}{\beta^n}\left|\int_o^{\infty}...\int_o^{\infty}\Gamma(\tau_1)...\Gamma(\tau_n)e^{\alpha(\tau_1+...+\tau_n)}\sin\beta\tau_1...\sin\beta\tau_n\left(\int_o^t \frac{\tau^{n-1}}{(n-1)!}A(t-\tau)d\tau\right)d\tau_1...d\tau_n\right|$$

$\le$

$$\le \left(|T_o| + \frac{|T_1-\alpha T_o|}{\beta}\right)\sum_{n=o}^{\infty}\frac{\varepsilon^n\lambda^{2n}}{\beta^n}\frac{t^n}{n!}\left(\int_o^{\infty}\Gamma(s)e^{\alpha s}\sin\beta s ds\right)^n = \left(|T_o| + \frac{|T_1-\alpha T_o|}{\beta}\right).$$

$$.\sum_{n=o}^{\infty}\frac{(2\alpha t)^n}{n!} = \left(|T_o| + \frac{|T_1-\alpha T_o|}{\beta}\right)e^{2\alpha t},$$

i.e. (3.22) is absolutely and uniformly convergent for any finite interval on $t>0$.

From (3.22) we get

$$T'(t) = A'(t) + \sum_{n=1}^{\infty}\frac{\varepsilon^n\lambda^{2n}}{\beta^n}\int_o^{\infty}...\int_o^{\infty}[T_o\Gamma_n(t+\tau_1+...+\tau_n) + \int_o^t \Gamma_n(\tau+\tau_1+...+\tau_n)A'(t-\tau)d\tau].$$

$$.e^{\alpha(\tau_1+...+\tau_n)}\sin\beta\tau_1...\sin\beta\tau_b d\tau_1...d\tau_n, \quad (3.24)$$

where

$$A'(t) = -\alpha A(t) + \beta e^{-\alpha t}\left(-T_o\sin\beta t + \frac{T_1-\alpha T_o}{\beta}\cos\beta t\right).$$

As above mentioned, we derive

$$|T'^{(t)}| \le [(3\alpha+\beta)|T_o| + \frac{\alpha+\beta}{\beta}|T_1 - \alpha T_o|]e^{2\alpha t},$$

i.e. series (3.24) is also absolutely and uniformly convergent for any finite $t > o$.

From (3.22) we get $T(0) = A(0) = T_o$. Using (3.19), we find from (3.24)

$$T'(0) = T_1 - 2\alpha T_o + \frac{\varepsilon\lambda^2 T_o}{\beta}\int_o^\infty \Gamma(\tau_1)e^{\alpha\tau_1}\sin\beta\tau_1 d\tau_1.$$

The use of (3.6) gives $T'(0) = T_1$. Thus the function $T(t)$ satisfies initial conditions. From (3.24) we calculate

$$T''(t) = -(\alpha^2+\beta^2)T(t) - 2\alpha T'(t) + \sum_{n=1}^{\infty}\frac{\varepsilon^n\lambda^{2n}}{\beta^n}\int_o^\infty\ldots\int_o^\infty [T_1\Gamma_n(t+\tau_1+\ldots+\tau_n)+$$

$$+T_o\Gamma_n'(t+\tau_1+\ldots+\tau_n)]e^{\alpha(\tau_1+\ldots+\tau_n)}\sin\beta\tau_1\ldots\sin\beta\tau_n d\tau_1\ldots d\tau_n. \qquad (3.25)$$

It is easy to show that the absolute value of series in (3.25) is less than

$$(2\alpha|T_1-\alpha T_o| + |T_o(\lambda^2+\alpha^2-\beta^2)|)e^{2\alpha t}.$$

Writing the integral on the interval $[0,t]$ as the difference of integrals on $[0,\infty)$ and (t,∞) and using (3.6) we find after some calculations

$$\varepsilon\lambda^2\int_o^t\Gamma(\tau)T(t-\tau)d\tau = -2\alpha T'(t) + (2\lambda\gamma-\alpha^2-\gamma^2)T(t) + \sum_{n=1}^{\infty}\frac{\varepsilon^n\lambda^{2n}}{\beta^n}\int_o^\infty\ldots\int_o^\infty[T_1\Gamma_n(t+$$

$$+\tau_1+\ldots+\tau_n)+T_o\Gamma_n'(t+\tau_1+\ldots+\tau_n)]e^{\alpha(\tau_1+\ldots+\tau_n)}\sin\beta\tau_1\ldots\sin\beta\tau_n d\tau_1\ldots d\tau_n. \; (3.26)$$

Formulas (3.22), (3.25) and (3.26) invert equation (3.1) to identity.

Using (3.19), (3.20) and (3.22) we define the remainder term

$$R_{n+1}(t) = \frac{\varepsilon^{n+1}\lambda^{2n+2}}{\beta^{n+1}}\int_o^\infty\ldots\int_o^\infty\left(\int_o^t\Gamma_{n+1}(\tau+\tau_1+\ldots+\tau_{n+1})T(t-\tau)d\tau\right)e^{\alpha(\tau_1+\ldots+\tau_{n+1})}.$$

$$.\sin\beta\tau_1\ldots\sin\beta\tau_{n+1}d\tau_1\ldots d\tau_{n+1.} = \int_o^t B_{n+1}(\tau)T(t-\tau)d\tau.$$

By the help of (3.21) we see that the remainder term of (3.22) has the same power of ε as the first neglected term of series and tends to zero for $n\to\infty$. It is easy to see that $R_{n+1}(0)=0, R_{n+1}(\infty)=0$.

The proof of the theorem is complete.

The function (3.22) consists of two parts. The first one describes a damped vibrations process with frequency β and damping coefficient α. Second part is called transient part of solution.

If in the formula (3.22) we neglect all terms under the summation sign and take into account only the terms linear in ε in (3.10), i.e. if we put $\alpha = \varepsilon\lambda\Gamma_s/2, \beta = \lambda(1-\varepsilon\Gamma_c/2)$, we will get the result of Ilyushin and his colleagues [5], obtained by Bogolyubov's averaging method The approach [4] leads to the same result, obtained by the method of complex modules, where the ratio of the imaginary part of complex module to its real part is considered small enough and all of its powers over the first are neglected. The frequency and damping coefficient obtained in [8] correspond to $\beta = \beta_1$ and $\alpha = \alpha_1$ in (3.10).

Remark2.*Equation (3.1) can be reduced to a linear differential equation of* $(N+2)th$ *order with constant coefficients for the kernel* $\Gamma(t) = \sum_{k=1}^{N} q_k e^{-\eta_k t}$ *by successively eliminating the integral terms. The initial conditions also become known during the process [4, 16, 17, and 22]. Its solution is naturally sought in the form of an exponential function. Characteristic equation of this differential equation is equivalent to (3.5) and has the same roots* $-\alpha \pm \beta, -\rho_k \, (k = 1,2,\dots,N)$, *so the general solution of differential equation may be represented as*

$$T(t) = e^{-\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t) + \sum_{k=1}^{N} C_{2+k} e^{-\rho_k t}.$$

Arbitrary constants $C_k \, (k = 1,2,\dots,N+2)$ *are determined from initial conditions. The solution constructed in this way coincides with the solution obtained in [22].*

4. Model Kernel

Let $\Gamma(t) = -q_1\delta'(t) + q_2 e^{-\eta t}$, where q_1, q_2, η are positive numbers, $\delta'(t)$ is the derivative of Dirac delta. For $q_2 = o$ we get the Voigt model and when $q_1 = o, \eta = q_2\varepsilon$ Maxwell model is obtained. For other values of parameters q_1, q_2 and η we may get more complicated models [14]. This kernel is not monotonously decreasing and has a singularity at $t = 0$, so we will construct the solution of the problem as below.

Let us represent the image of the solution like

$$\bar{T}(p) = \frac{pT_o + T_1 + \varepsilon\lambda^2 q_1 T_o}{p^2 + \lambda^2 - \varepsilon\lambda^2\bar{\Gamma}} = \frac{pT_o + T_1 + \varepsilon\lambda^2 q_1 T_o}{\left(p + \frac{\varepsilon\lambda^2 q_1}{2}\right)^2 + \lambda^2\left(1 - \frac{\varepsilon^2\lambda^2 q_1^2}{4}\right) - \frac{\varepsilon\lambda^2 q_2}{p+\eta}} \doteq e^{-\frac{\varepsilon\lambda^2 q_1 t}{2}}\hat{T}(t) \quad (4.1)$$

where $\hat{T}(t)$ is the inverse transform of the function

$$\bar{\hat{T}}(p) = \frac{pT_o + \hat{T}_1}{p^2 + \Lambda^2 - \frac{\varepsilon\Lambda^2 q}{p+\xi}}.$$

Here the notations

$$\Lambda^2 = \lambda^2\left(1 - \frac{\varepsilon^2\lambda^2 q_1^2}{4}\right),\ \xi = \eta - \frac{\varepsilon\lambda^2 q_1}{2},\ \hat{T}_1 = T_1 + \frac{\varepsilon\lambda^2 q_1}{2}T_o,\ q = q_2\left(\frac{\lambda}{\Lambda}\right)^2$$

are introduced. As we see, the function $\hat{T}(t)$ corresponds to the solution of the problem of the vibrations of a viscoelastic system with the elastic frequency Λ (if $\Lambda^2 > 0$) and the kernel of relaxation $\varepsilon\hat{\Gamma}(t) = \varepsilon q e^{-\xi t}$ for the initial conditions $\hat{T}(0) = T_o, \hat{T}'(0) = \hat{T}_1$. As in the previous section it is easy to show that the equation (3.5) has the complex roots $-\alpha \pm i\beta (\alpha > 0)$ and a real negative root $-\rho (\rho > 0)$, which satisfies the equality $\rho + 2\alpha = \xi$. If $\Lambda^2 \le 0$ the function $\hat{T}(t)$ is expressed by exponential functions with negative powers and the vibrations are absent.

The equations (3.6) for obtaining the values α and β corresponding to the kernel $\hat{\Gamma}$ become

$$\frac{q\beta}{\beta^2 + (\xi - \alpha)^2} = \frac{2\alpha\beta}{\varepsilon\Lambda^2}, \quad \frac{q(\xi - \alpha)}{\beta^2 + (\xi - \alpha)^2} = \frac{\alpha^2 + \Lambda^2 - \beta^2}{\varepsilon\Lambda^2}. \quad (4.2)$$

Here we find $\beta^2 = \Lambda^2 - 2\alpha\xi + 3\alpha^2$. The iteration process may be determined as

$$\alpha_{n+1} = \frac{\varepsilon\Lambda^2 q}{2}\frac{1}{(\xi-2\alpha_n)^2+\Lambda^2}, \alpha_o = 0; \beta_n^2 = \Lambda^2 - 2\alpha_n\xi + 3\alpha_n^2; n = 0,1,2,... \quad (4.3)$$

From (4.3) we obtain

$$\alpha = \frac{\varepsilon\Lambda^2 q}{2\psi} + \frac{\varepsilon^2\Lambda^4 q^2\xi}{\psi^3} + \frac{\varepsilon^3 q^3\Lambda^6}{2\psi^5}(7\xi^2 - \Lambda^2) + ... \gamma = \frac{\varepsilon q\ \Lambda^2\xi}{2\psi\Lambda} +$$

$$\frac{\varepsilon^2 q^2\Lambda}{8\psi^3}(\xi^4 - 3\Lambda^4 + 6\Lambda^2\xi^2) + \frac{\varepsilon^3 q^3\Lambda\xi}{16\psi^5}(-35\Lambda^6 + 35\Lambda^4\xi^2 + 7\Lambda^2\xi^4 + \xi^6) + ...$$

$$\beta = \Lambda - \gamma, \psi = \Lambda^2 + \xi^2.$$

The derivative of the iteration function with respect to α is estimated as

$$\frac{2\varepsilon\Lambda^2 q\ |\xi - 2\alpha|}{\left[\Lambda^2 + (\xi-2\alpha)^2\right]^2} = \frac{4\alpha|\xi-2\alpha|}{\Lambda^2+(\xi-2\alpha)^2} = \frac{2\alpha}{\Lambda}\frac{2\Lambda|\xi-2\alpha|}{\Lambda^2+(\xi-2\alpha)^2} < 1.$$

This shows the convergence of the iteration (4.3) to unique limit.

For $\hat{\Gamma}(t)$ the function $B(t)$ is:

$$B(t) = \frac{\varepsilon\Lambda^2 q}{\beta}\int_o^\infty e^{-\xi(t+s)+\alpha s}\sin\beta s ds = \frac{\varepsilon\Lambda^2 q}{(\xi-\alpha)^2+\beta^2}e^{-\xi t}.$$

The use of the first of the formulas (4.2) gives $B(t) = 2\alpha e^{-\xi t}$.

The function $B_n(t)$ is $\quad B_n(t) = \frac{(2\alpha)^n t^{n-1}}{(n-1)!}e^{-\xi t}, n = 1,2,...$

From (3.22) we obtain

$$\hat{T}(t) = A(t) + 2\alpha\int_o^t A(t-s)e^{-\xi s}[1 + 2s\alpha + ... + \frac{(2s\alpha)^n}{n!} + ...]ds =$$

$$= A(t) + 2\alpha\int_o^t A(t-s)e^{-(\xi-2\alpha)s}ds \quad (4.4)$$

where $A(t)$ is obtained from (3.13) by replacing T_1 to the $\hat{T}_1$.

Using (3.13), (5.1) and (5.4) we find

$$T(t) = e^{-(\alpha+\varepsilon\eta_1\lambda^2/2)t}\cos\beta t\left\{T_o\left[1 + \frac{2\alpha\xi - 4\alpha^2}{(\xi-3\alpha)^2+\beta^2}\right] - \frac{2\alpha\hat{T}_1}{(\xi-3\alpha)^2+\beta^2}\right\} +$$

$$+e^{-(\alpha+\varepsilon q_1\lambda^2/2)t}\frac{\sin\beta t}{\beta}\left\{\hat{T}_1\left[1+\frac{2\alpha\xi-6\alpha^2}{(\xi-3\alpha)^2+\beta^2}\right]-\alpha T_o\left[1+\frac{2\alpha\xi-6\alpha^2-2\beta^2}{(\xi-3\alpha)^2+\beta^2}\right]\right\}+$$

$$+\frac{2\alpha[(2\alpha-\xi)T_o+\hat{T}_1]}{(\xi-3\alpha)^2+\beta^2}e^{-(\eta-2\alpha)t}. \qquad (4.5)$$

It may be proved by direct substitution that the function (4.5) is the solution of the Problem (3.1), (3.2) for the considered kernel.

The solution (4.5) may be easily obtained if we use the formula (3.17).

The first two terms in (4.5) describe a damped vibration process with the frequency β and the last term shows the transient part of solution. The damping coefficient is equal to $\alpha+\varepsilon\lambda^2 q_1/2$ .Here α is a limited number but the second term becomes unbounded for increasing λ. In this case the motion corresponding to these λ (λ>>1) disappears almost completely.

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