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ON A SYMMETRIC TENSOR FIELD IN AREAL SPACE OF SUBMETRIC CLASS

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ABSTRACT

The theory of isotropic areal space of submetric class was studied by Kawaguchi and Tandai (1952) , Kikuchi (1968) and others . A symmetric curvature tensor field was defined and studied by the present author (1993) in generalized Finsler space. S.M.Uppal and the author (1996) have obtained Veblen identities in special Kawaguchi space . The purpose of the present paper is to define a symmetric tensor field and obtain its Bianchi and Veblen identities . The recurrence property of this tensor is dealt with in the last section of this paper .

1 . INTRODUCTION

Let $A_n^{(m)}$ be an areal space of the submetric class whose normalized metric tensor g_{ij} satisfies the relations :

$$(1.1)\qquad g_{ij}p^i_\alpha p^j_\beta = g_{\alpha\beta} \quad , \quad |g_{\alpha\beta}| = F^2 \ ,$$

$$(1.2)\qquad g_{ij}\gamma^i_h p^j_\alpha = 0 \ , \quad \gamma^i_h = \delta^i_h - \beta^i_h \quad , \quad \beta^i_h = p^i_\alpha p^\alpha_h$$

and

$$(1.3)\qquad g_{ij};^\alpha_k \gamma^i_h p^j_\alpha = 0 \ ,$$

where the Latin indices run from 1 to n and the Greek indices from 1 to m , and $;^\alpha_k$ denotes $\partial/\partial p^k_\alpha$.

The covariant derivative of X^i , with the symmetric connection parameters Γ^{*i}_{jk} is defined as

(1.4) $$X^i_{|k} = X^i_{,k} - X^i ;^{\alpha}_{l} \Gamma^{*l}_{\alpha k} + \Gamma^{*i}_{jk} X^j ,$$

where $\Gamma^{*l}_{\alpha k} = \Gamma^{*l}_{jk} p^j_\alpha$ and $,j$ denotes $\partial/\partial x^j$.

The corresponding curvature tensor field is defined by

(1.5)
$$K^i_{jkh} = \Gamma^{*i}_{jk,h} - \Gamma^{*i}_{jh,k} - \Gamma^{*i}_{jk};^{\alpha}_{m} \Gamma^{*m}_{\alpha h} + \Gamma^{*i}_{jh};^{\alpha}_{m} \Gamma^{*m}_{\alpha k} + \Gamma^{*i}_{mh}\Gamma^{*m}_{jk} - \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} ,$$

which satisfies the identities

(1.6) $$K^i_{jkh} = -K^i_{jhk}$$

and

(1.7) $$K^i_{jkh} + K^i_{khj} + K^i_{hjk} = 0$$

The commutation formulae satisfied by the curvature tensor field are

(1.8) $$X_{i|k|h} - X_{i|h|k} = -K^l_{ikh} X_l$$

and

(1.9)
$$T_{ik|h|r} - T_{ik|r|h} = -K^l_{ihr} T_{lk} - K^l_{khr} T_{il} - T_{ik};^{\alpha}_{m} K^m_{thr} p^t_\alpha ,$$

where $X_i(x)$ and $T_{ij}(x,p)$ are arbitrary vector and tensor fields respectively
.
If we substitute $T_{ij} = g_{ij}$ in (1.9), we have

(1.10) $$K_{ijkh} + K_{jikh} = -g_{ij};^{\alpha}_{m} K^m_{\alpha kh} ,$$

where $K_{ijkh} = g_{il} K^l_{jkh}$ and $K^m_{\alpha kh} = K^m_{jkh} p^j_\alpha$.

The covariant differentiation of (1.8) with respect to x^r gives

$$(1.11) \qquad X_{i|k|h|r} - X_{i|h|k|r} = -K^l_{ihk|r} X_l - K^l_{ikh} X_{l|r} \ .$$

We put $T_{ik} = X_{i|k}$ in (1.9) , to get

$$(1.12) \qquad X_{i|k|h|r} - X_{i|k|r|h} = -K^l_{ihr} X_{l|k} - K^i_{khr} X_{i|l} - \left(X_{i|k}\right);^{\alpha}_{m} K^m_{\alpha hr} \ .$$

The cyclic permutation of indices k ,h, r in (1.11) yields Bianchi identities

$$(1.13)$$
$$K^i_{jkh|h} + K^i_{jhr|k} + K^i_{jrk|h} + \Gamma^{*i}_{jk};^{\alpha}_{m} K^m_{\alpha hr} + \Gamma^{*i}_{jh};^{\alpha}_{m} K^m_{\alpha rk} + \Gamma^{*i}_{jr};^{\alpha}_{m} K^m_{\alpha kh} = 0 \ .$$

The Veblen identities satisfied by the curvature tensor K^i_{jkh} are

$$(1.14) \qquad \begin{aligned} &K^i_{jkh|r} + K^i_{hjr|k} + K^i_{rhk|j} + K^i_{krj|h} + \Gamma^{*i}_{jk};^{\alpha}_{m} K^m_{\alpha hr} \\ &+ \Gamma^{*i}_{rh};^{\alpha}_{m} K^m_{\alpha kj} + \Gamma^{*i}_{hj};^{\alpha}_{m} K^m_{\alpha rk} + \Gamma^{*i}_{kr};^{\alpha}_{m} K^m_{\alpha jh} = 0. \end{aligned}$$

2. SYMMETRIC CURVATURE TENSOR FIELD J^i_{jkh}

In an areal space of submetric class $A_n^{(m)}$ we notice that the curvature tensor K^i_{jkh} is skew-symmetric in the last pairs of covariant indices . Following the method of Gh. Vranceanu (1957) , we define symmetric curvature tensor J^i_{jkh} in $A_n^{(m)}$ as under :

$$(2.1) \qquad J^i_{jkh} = K^i_{jkh} + K^i_{kjh} \quad ,$$

where K^i_{jkh} are the components of the skew-symmetric tensor . From (2.1), it is obvious that

$$(2.2) \qquad J^i_{jkh} = J^i_{kjh} \quad ,$$

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which shows that J^i_{jkh} is symmetric in the first pair of covariant indices since $K^i_{jkh} + K^i_{kjh} = K^i_{kjh} + K^i_{jkh}$.

The cyclic permutation of indices j, k ,h in (2.1) yields the identity

$$(2.3) \qquad J^i_{jkh} + J^i_{khj} + J^i_{hjk} = 0$$

in view of (1.7) .

Theorem 2.1 In an areal space of submetric class $A_n^{(m)}$, the symmetric curvature tensor J^i_{jkh} satisfies the Bianchi identities

$$(2.4) \qquad J^i_{jkh|r} + J^i_{jhr|k} + J^i_{jrk|h} = 0 \ .$$

Proof . Using definition (2.1) in the equation (2.4) , it becomes

$$(2.5) \qquad \left(K^i_{jkh} + K^i_{kjh}\right)_{|h} + \left(K^i_{jhr} + K^i_{hjr}\right)_{|k} + \left(K^i_{jrk} + K^i_{rjk}\right)_{|h} = 0 \ .$$

In equation (2.5) , if we develop the calculus by considering the properties (1.6) and (1.7), this relation is identically verified .

Theorem 2.2 In an areal space of submetric class $A_n^{(m)}$ the symmetric curvature tensor J^i_{jkh} satisfies the Veblen identities

$$(2.6) \qquad J^i_{jkh|r} + J^i_{jkh|r} + J^i_{rhk|j} + J^i_{krj|h} = 0 \ .$$

Proof . On account of (2.1), the equation (2.6) assumes the form

(2.7)

$$K^i_{jkh|r} + K^i_{kjh|r} + K^i_{hjr|k} + K^i_{jhr|k} + K^i_{rhk|j} + K^i_{hrk|j} + K^i_{krj|h} + K^i_{rkj|h} = 0 \ .$$

We observe that the relation (2.7) is an identity in view of the properties $(1.6),(1.7)$ and (1.13) of the curvature tensor K^i_{jkh} .

Theorem 2.3 In an areal space of submetric class $A_n^{(m)}$, if we denote

$$(2.8) \qquad B^i_{jkhr} = J^i_{jkh|r} + J^i_{jhr|k} + J^i_{jrk|h}$$

and

$$(2.9) \qquad V^i_{jkhr} = J^i_{hjr|r} + J^i_{hjr|k} + J^i_{rhk|j} + J^i_{krj|h}$$

then the following relations

$$(2.10) \qquad V^i_{jkhr} = B^i_{jkhr} + B^i_{hrjk}$$

and

$$(2.11) \qquad 2B^i_{jkhr} = V^i_{jkhr} + V^i_{jrkh} + V^i_{jhrk}$$

hold good ,which show the equivalence of Bianchi and Veblen identities .

Proof. Applying (2.8) and (2.9) in the equations (2.10) and(2.11), we get

$$(2.12) \qquad \begin{aligned} J^i_{jkh|r} + J^i_{hjr|k} + J^i_{rhk|j} + J^i_{krj|h} = J^i_{jkh|r} \\ + J^i_{jrk|h} + J^i_{jhr|k} + J^i_{hrj|k} + J^i_{hkr|j} + J^i_{hjk|r} \end{aligned}$$

and

(2.13)

$$2\left(J^i_{jkh|r} + J^i_{jrk|h} + J^i_{jhr|k}\right) = J^i_{jkh|r} + J^i_{hjr|k} + J^i_{rhk|j} + J^i_{krj|h}$$
$$+ J^i_{jrk|h} + J^i_{kjh|r} + J^i_{hkr|j} + J^i_{rhj|k} + J^i_{jhr|k} + J^i_{rjk|h} + J^i_{krh|j} + J^i_{hkj|r}$$

respectively .
The relations (2.12) and (2.13) are identically verified by applying $(1.6),(1.7),(1.13)$,
(1.14) and (2.1) .

3. RECURRENT SYMMETRIC CURVATURE TENSOR FIELD

A recurrent and symmetrically recurrent areal spaces of submetric class are characterized by

$$K^i_{jkh|m} = V_m K^i_{jkh} \quad , \quad K^i_{jkh} \neq 0 \tag{3.1}$$

and

$$J^i_{jkh|m} = V_m J^i_{jkh} \quad , \quad J^i_{jkh} \neq 0 \tag{3.2}$$

respectively . The non-zero vector field V_m is called recurrence vector field .

LEMMA 3.1 The necessary and sufficient condition for an areal space of submetric class to be symmetrically recurrent is that it is a recurrent areal space of submetric class.

Proof . Let us assume that the condition (3.1) is true . In view of (2.1) , we have

$$J^i_{jkh|m} = \left(K^i_{jkh} + K^i_{kjh}\right)_{|m} \tag{3.3}$$

which yields (3.2) , that is , J^i_{jkh} is recurrent and the space is a symmetric recurrent space.

Conversely , if (3.2) is true , we find

$$\left(K^i_{jkh} + K^i_{kjh}\right)_{|m} = V_m\left(K^i_{jkh} + K^i_{kjh}\right) \tag{3.4}$$

or equivalently

(3.5) $$\left(K^{i}_{jkh|m} - V_m K^{i}_{jkh}\right) + \left(K^{i}_{kjh|m} - V_m K^{i}_{kjh}\right) = 0 \ ,$$

which implies (3.1).

COROLLARY 3.1 In symmetrically recurrent areal space of submetric class , the Bianchi and Veblen identities assume the forms

(3.6) $$V_r J^{i}_{jkh} + V_k J^{i}_{jhr} + V_h J^{i}_{jrk} = 0$$

and

(3.7) $$V_r J^{i}_{jkh} + V_k J^{i}_{hjr} + V_j J^{i}_{rhk} + V_h J^{i}_{krj} = 0$$

respectively . .

Proof . It is evident from (2.4),(2.6) and (3.2) .

Now we shall prove certain theorems on the recurrence vector field V_m in the symmetrically recurrent areal space of submetric class . We shall denote hereafter by (J) the property : the value of the symmetric tensor field J^{i}_{jkh} is not the zero tensor at each element (x,p) of the space .

Theorem 3.1 In a symmetrically recurrent areal space of submetric class having the property (J) , if J^{i}_{jkh} is independent of p^{i}_{α} , the relation

(3.8) $$\left(V_{m|n} - V_{n|m}\right)_{|l} = V_l\left(V_{m|n} - V_{n|m}\right)$$

holds good .

Proof . Differentiating (3.2) covariantly with respect to x^{n} , we get

(3.9) $$J^{i}_{jkh|m|n} = V_{m|n} J^{i}_{jkh} + V_m V_n J^{i}_{jkh}$$

which yields

$$(3.10) \qquad \left(J^i_{jkh|m|n} - J^i_{jkh|n|m}\right) = \left(V_{m|n} - V_{n|m}\right)J^i_{jkh} \ .$$

Using the commutation formula (1.9) in the equation (3.10) , it gives

$$(3.11)$$
$$\left(V_{m|n} - V_{n|m}\right)J^i_{jkh} = -K^p_{jmn}J^i_{pkh} - K^p_{kmn}J^i_{jph} - K^p_{hmn}J^i_{jkp} + K^i_{pmn}J^p_{jkh}$$

by considering the fact that J^i_{jkh} is independent of p^i_α .
Now differentiating (3.11) covariantly with respect to x^l and applying (3.1) and (3.2), we obtain

$$(3.12) \qquad \left(V_{m|n} - V_{n|m}\right)_{|l} J^i_{jkh} = \left(V_{m|n} - V_{n|m}\right)V_l J^i_{jkh} \ .$$

In view of the property (J) , it immediately yields (3.8) .

COROLLARY 3.2 In a symmetrically recurrent areal space of submetric class having the property (J) , if J^i_{jkh} is independent of p^i_α , the recurrence vector field V_l satisfied the identity

$$(3.13) \qquad \left(V_{m|n} - V_{n|m}\right)V_l + \left(V_{n|l} - V_{l|n}\right)V_m + \left(V_{l|m} - V_{m|n}\right)V_n = 0 \ .$$

Proof . Adding the expressions obtained by a cyclic change in the indices l,m and n in (3.8) , we have

$$(3.14)$$
$$\left(V_{m|n} - V_{n|m}\right)_{|l} + \left(V_{n|l} - V_{l|n}\right)_{|m} + \left(V_{l|m} - V_{m|l}\right)_{|n}$$
$$= V_l\left(V_{m|n} - V_{n|m}\right) + V_m\left(V_{n|l} - V_{l|n}\right) + V_n\left(V_{l|m} - V_{m|n}\right)$$

On account of (1.7) and (1.8) , the above equation yields (3.13) .

Theorem 3.2 In a symmetrically recurrent areal space of submetric class having the property (J) , if J^{l}_{jkh} is independent of p^{i}_{α} , the recurrence vector field V_l satisfies the relation

$$(3.15)\qquad V_{[l|m|n]|r} = V_{[n|m|l]|r} \quad ,$$

where the symbol $[lmn]$ represents skew-symmetric part with respect to the indices l,m,n .

Proof . Taking covariant differentiation of (3.8) with respect to x^r , we get

$$(3.16)\qquad V_{m|n|l|r} - V_{n|m|l|r} = V_{l|r}\left(V_{m|n} - V_{n|m}\right) + V_l\left(V_{m|n|r} - V_{n|m|r}\right)$$

which yields

$$(3.17)\qquad \begin{aligned}&\left(V_{m|n|l|r} - V_{m|l|n|r}\right) + \left(V_{l|m|n|r} - V_{n|m|l|r}\right) = \left(V_{m|n} - V_{n|m}\right)V_{l|r}\\ &-\left(V_{m|l} - V_{l|m}\right)V_{n|r} - \left(V_{m|n|r} - V_{n|m|r}\right)V_l - \left(V_{m|l|r} - V_{l|m|r}\right)V_n\end{aligned}$$

In view of (1.11) , it assumes the form

$$(3.18)\qquad \begin{aligned}&-K^{i}_{mnl|r}V_p - K^{i}_{mnl}V_{p|r} + \left(V_{l|m|n|r} - V_{n|m|l|r}\right) = \left(V_{m|n} - V_{n|m}\right)V_{l|r}\\ &-\left(V_{m|l} - V_{l|m}\right)V_{n|r} + \left(V_{m|n|r} - V_{n|m|r}\right)V_l - \left(V_{m|l|r} - V_{l|m|r}\right)V_n\end{aligned}$$

.

Taking cyclic permutation of the indices l,m,n in (3.18) and using (1.7), (3.1) and (3.13) in the obtained result ,it reduces to (3.15) .

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