

BOUNDEDNESS AND STABILITY RESULTS FOR A CERTAIN SYSTEM OF FIFTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract : The paper studies equation (1.1) in two cases : (i) $P \neq 0$, (ii) $P(\neq 0)$ satisfies, $\|P(t, X, Y, Z, W, U)\| \leq [\delta_1 + \delta_2 (\|Y\| + \|Z\| + \|W\| + \|U\|)]\theta(t)$, where $\theta(t)$ is a nonnegative function of t . In case (i) the asymptotic stability in the large of the trivial solution of (1.1) is investigated; in case (ii) a boundedness result is deduced for solutions of (1.1). These results generalize some of the results obtained earlier.

1. Introduction and statement of the results

We shall consider the real non-linear autonomous vector differential equation of fifth order

$$X^{(5)} + F(X, \dot{X}, \ddot{X}, \ddot{X})X^{(4)} + \Phi(\ddot{X}, \ddot{X}) + G(\dot{X}, \ddot{X}) + H(\dot{X}) + \Psi(X) = P(t, X, \dot{X}, \ddot{X}, X^{(4)}) \tag{1.1}$$

or its equivalent system

$$\begin{aligned} \dot{X} &= Y, \dot{Y} = Z, \dot{Z} = W, \dot{W} = U, \\ \dot{U} &= -F(X, Y, Z, W)U - \Phi(Z, W) - G(Y, Z) - H(Y) - \Psi(X) + P(t, X, Y, Z, W, U) \end{aligned} \tag{1.2}$$

where $X \in R^n$, R^n denotes the real n -dimensional Euclidean space $R \times R \times \dots \times R$ (n factors), $R = (-\infty, \infty)$, F is an $n \times n$ -matrix function, $\Phi: R^n \times R^n \rightarrow R^n$, $G: R^n \times R^n \rightarrow R^n$, $H: R^n \rightarrow R^n$, $\Psi: R^n \rightarrow R^n$ and $P: R \times R^n \times R^n \times R^n \times R^n \times R^n \rightarrow R^n$.

The dots as usual indicate differentiation with respect to t . The non-linear functions F , Φ , G , H and Ψ are continuous and so constructed such that the uniqueness theorem is valid. The equation (1.1) represents a system of real fifth-order differential equations of the form :

$$\begin{aligned} x_i^{(5)} + \sum_{k=1}^n f_{i,k}(x_1, \dots, x_n; \dot{x}_1, \dots, \dot{x}_n; \ddot{x}_1, \dots, \ddot{x}_n; \ddot{x}_1, \dots, \ddot{x}_n) x_i^{(4)} + \phi_i(\ddot{x}_1, \dots, \ddot{x}_n; \ddot{x}_1, \dots, \ddot{x}_n) \\ + g_i(\dot{x}_1, \dots, \dot{x}_n; \ddot{x}_1, \dots, \ddot{x}_n) + h_i(\dot{x}_1, \dots, \dot{x}_n) + \psi_i(x_1, \dots, x_n) \\ = p_i(x_1, \dots, x_n; \dot{x}_1, \dots, \dot{x}_n; \ddot{x}_1, \dots, \ddot{x}_n; \ddot{x}_1, \dots, \ddot{x}_n; x_1^{(4)}, \dots, x_n^{(4)}), \quad (i = 1, 2, \dots, n). \end{aligned} \tag{1.3}$$

Key words : System of non-linear differential equations of the fifth order, Lyapunov function, Global stability.

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Finding sufficient conditions for stability and boundedness of solutions is a problem of general interest in theory of differential equations. The problem in this paper, in the case $n=1$, has been the subject of intensive investigation in recent years (see, in particular, Abou-El-Ela & Sadek [3], Chukwu [5], Tunç ([10], [12]), Yuanhong [14]). Recently, Abou-El-Ela & Sadek [4] and Sadek [8] established sufficient conditions for the asymptotic stability in the large of the zero solution of the vector differentials equations

$$X^{(5)} + AX^{(4)} + \Phi(\ddot{X}) + G(\dot{X}) + H(X) + BX = 0$$

and

$$X^{(5)} + F(\ddot{X})X^{(4)} + \Phi(\ddot{X}) + G(\dot{X}) + H(X) + \Psi(X) = 0,$$

respectively.

Tunç [13] derived similar results for the problem

$$X^{(5)} + F(\ddot{X}, \ddot{X})X^{(4)} + \Phi(\ddot{X}, \ddot{X}) + G(\dot{X}) + H(X) + \Psi(X) = 0$$

The first result obtained is comparable in generality with the results of Abou-El-Ela & Sadek [4], Sadek [8] and Tunç [13]. This paper also gives an n -dimensional extension for Abou-El-Ela & Sadek [3] and Chukwu [5]. Furthermore, the present work is the first attempt to obtain sufficient conditions for boundedness of solutions of fifth order vector differential equations.

We need the following notation and definitions :

1. $\lambda_i (A)$, ($i = 1, 2, \dots, n$) are the eigenvalues of the nxn -matrix A .

2. $\langle X, Y \rangle$ corresponding to any pair X, Y of vectors in R^n is the usual scalar product $\sum_{i=1}^n x_i y_i$.

$\langle X, X \rangle = \|X\|^2$ for arbitrary X in R^n .

3. The Jacobian matrices $J(\Phi(Z, W) | Z)$, $J(\Phi(Z, W) | W)$, $J(G(Y, Z) | Y)$, $J(G(Y, Z) | Z)$, $J_H(Y)$ and $J_\Psi(X)$ are given by

$$J(\Phi(Z, W) | Z) = \begin{pmatrix} \frac{\partial \phi_i}{\partial z_j} \end{pmatrix},$$

$$J(\Phi(Z, W) | W) = \begin{pmatrix} \frac{\partial \phi_i}{\partial w_j} \end{pmatrix},$$

$$J(G(Y, Z) | Y) = \begin{pmatrix} \frac{\partial g_i}{\partial y_j} \end{pmatrix},$$

$$J(G(Y, Z) | Z) = \begin{pmatrix} \frac{\partial g_i}{\partial z_j} \end{pmatrix},$$

$$J_H(Y) = \begin{pmatrix} \frac{\partial h_i}{\partial y_j} \end{pmatrix}, \quad J_\Psi(X) = \begin{pmatrix} \frac{\partial \psi_i}{\partial x_j} \end{pmatrix}, \quad (i, j = 1, 2, \dots, n).$$

Moreover, let the Jacobian matrices $J(\Phi(Z, W) | Z)$, $J(\Phi(Z, W) | W)$, $J(G(Y, Z) | Y)$, $J(G(Y, Z) | Z)$, $J_H(Y)$ and $J_\Psi(X)$ exist and are continuous.

In the case $P=0$, we shall prove ;

Theorem 1 Further to the basic assumptions on F, Φ, G, H and Ψ , we assume the existence of arbitrary positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and of sufficiently small positive constants $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ such that

$$(i) \quad \alpha_1 \alpha_2 - \alpha_3 > 0, (\alpha_1 \alpha_2 - \alpha_3) \alpha_4 - (\alpha_1 \alpha_4 - \alpha_5) \alpha_1 > 0, \quad (1.4)$$

$$\delta_0 = (\alpha_1 \alpha_2 - \alpha_3) \alpha_4 - (\alpha_1 \alpha_4 - \alpha_5) \alpha_1 > 0,$$

$$\Delta_1 = \frac{(\alpha_3 \alpha_4 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3)}{(\alpha_1 \alpha_4 - \alpha_5)} - \{\alpha_1 \|J_H(Y)\| - \alpha_5\} > 2\varepsilon \alpha_2 \text{ for all } Y \in R^n, \quad (1.5)$$

$$\Delta_2 = \frac{(\alpha_3 \alpha_4 - \alpha_2 \alpha_5)}{(\alpha_1 \alpha_4 - \alpha_5)} - \frac{(\alpha_1 \alpha_4 - \alpha_5) \Gamma(Y)}{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)} - \frac{\varepsilon}{\alpha_1} > 0 \text{ for all } Y \in R^n, \quad (1.6)$$

where

$$\Gamma(Y) = \int_0^1 J_H(\sigma Y) d\sigma. \quad (1.7)$$

(ii) $F(X, Y, Z, W)$ is symmetric and

$$\varepsilon_0 \leq \lambda_i [F(X, Y, Z, W) - \alpha_1 I] \leq \varepsilon_1 \text{ for all } X, Y, Z, W \in R^n \quad (i=1, 2, \dots, n).$$

(iii) $\Phi(Z, 0) = 0, J(\Phi(Z, W) | Z)$ is negative-definite, $J(\Phi(Z, W) | W)$ is symmetric and

$$0 \leq \lambda_i \left(\int_0^1 \{J(\Phi(Z, \sigma W) | \sigma W) - \alpha_2 I\} d\sigma \right) \leq \varepsilon_2 \text{ for all } Z, W \in R^n \quad (i=1, 2, \dots, n).$$

(iv) $G(Y, 0) = 0, J(G(Y, Z) | Z)$ is symmetric, $J(G(Y, Z) | Y)$ is negative definite and

$$0 \leq \lambda_i \left(\int_0^1 \{J(G(Y, \sigma Z) | \sigma Z) - \alpha_3 I\} d\sigma \right) \leq \varepsilon_3 \text{ for all } Y, Z \in R^n \quad (i=1, 2, \dots, n).$$

(v) $H(0) = 0, J_H(Y)$ is symmetric, $\lambda_i \left(\int_0^1 J_H(\sigma Y) d\sigma \right) \geq \alpha_4, \|\alpha_4 I - J_H(Y)\| \leq \varepsilon_4$ and

$$\lambda_i \left(J_H(Y) - \int_0^1 J_H(\sigma Y) d\sigma \right) \leq \frac{\alpha_5 \delta_0}{\alpha_4^2 (\alpha_1 \alpha_2 - \alpha_3)} \text{ for all } Y \in R^n \quad (i=1, 2, \dots, n).$$

(vi) $\Psi(0) = 0, J_\Psi(X)$ is symmetric and

$$0 \leq \lambda_i (\alpha_5 I - J_\Psi(X)) \leq \varepsilon_5 \text{ for all } X \in R^n \quad (i=1, 2, \dots, n).$$

(vii) $J_\Psi(X)$ commutes with $J_\Psi(X')$ for all $X, X' \in R^n$,

$$\lambda_i \left(\int_0^1 J_\Psi(\sigma X) d\sigma \right) \geq \alpha_5' > 0 \text{ for all } X \in R^n \quad (i=1, 2, \dots, n).$$

Then every solution $X(t)$ of (1.1) satisfies

$$\|X(t)\| \rightarrow 0, \|\dot{X}(t)\| \rightarrow 0, \|\ddot{X}(t)\| \rightarrow 0, \|\ddot{X}(t)\| \rightarrow 0, \|X^{(4)}(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

For the case $P \neq 0$, we shall prove;

Theorem 2 Let all the conditions of Theorem 1 be satisfied, and in addition, we assume that the vector P satisfies

$$\|P(t, X, Y, Z, W, U)\| \leq [\delta_1 + \delta_2(\|Y\| + \|Z\| + \|W\| + \|U\|)]\theta(t), \quad (1.8)$$

where δ_1, δ_2 are positive constants and $\theta(t)$ is a nonnegative and continuous function of t , and satisfies $\int_0^t \theta(s) ds \leq A < \infty$, for all $t \geq 0$, where A is a positive constant. Then there exists a constant $D > 0$ such that any solution $(X(t), Y(t), Z(t), W(t), U(t))$ of (1.2) determined by $X(0) = X_0, Y(0) = Y_0, Z(0) = Z_0, W(0) = W_0, U(0) = U_0$, satisfies for all $t \geq 0$,

$$\|X(t)\| \leq D, \|Y(t)\| \leq D, \|Z(t)\| \leq D, \|W(t)\| \leq D, \|U(t)\| \leq D.$$

The following lemmas will be applied by the estimations of a Lyapunov function and its time derivative.

Lemma 1.1 Let A be a real symmetric $n \times n$ -matrix and $\alpha' \geq \lambda(A) \geq \alpha > 0$ ($i=1, 2, \dots, n$), then

$$\begin{aligned} \alpha' \|X\|^2 &\geq \langle AX, X \rangle \geq \alpha \|X\|^2, \\ \alpha'^2 \|X\|^2 &\geq \langle AX, AX \rangle \geq \alpha^2 \|X\|^2. \end{aligned}$$

Proof See [8].

Lemma 1.2

$$(I) \quad \frac{d}{dt} \int_0^1 \langle \Phi(Z, \sigma W), W \rangle d\sigma \leq \langle \Phi(Z, W), U \rangle.$$

$$(II) \quad \frac{d}{dt} \int_0^1 \langle G(Y, \sigma Z), Z \rangle d\sigma \leq \langle G(Y, Z), W \rangle.$$

$$(III) \quad \frac{d}{dt} \int_0^1 \langle H(\sigma Y), Y \rangle d\sigma = \langle H(Y), Z \rangle.$$

$$(IV) \quad \frac{d}{dt} \int_0^1 \langle \Psi(\sigma X), X \rangle d\sigma = \langle \Psi(X), Y \rangle.$$

Proof (I) See [13].

$$\begin{aligned} (II) \quad \frac{d}{dt} \int_0^1 \langle G(Y, \sigma Z), Z \rangle d\sigma &= \int_0^1 \langle G(Y, \sigma Z), W \rangle d\sigma + \int_0^1 \langle J(G(Y, \sigma Z) | Y) Z, Z \rangle d\sigma \\ &\quad + \int_0^1 \sigma \langle J(G(Y, \sigma Z) | \sigma Z) W, Z \rangle d\sigma. \end{aligned} \quad (1.9)$$

Since $J(G | Z)$ is symmetric from condition (iv) we get

$$\begin{aligned} \int_0^1 \sigma \langle J(G(Y, \sigma Z) | \sigma Z) W, Z \rangle d\sigma &= \int_0^1 \sigma \langle J(G(Y, \sigma Z) | \sigma Z) Z, W \rangle d\sigma = \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle G(Y, \sigma Z), W \rangle d\sigma \\ &= \langle G(Y, Z), W \rangle - \int_0^1 \langle G(Y, \sigma Z), W \rangle d\sigma. \end{aligned} \quad (1.10)$$

From (1.9) and (1.10) we obtain

$$\frac{d}{dt} \int_0^1 \langle G(Y, \sigma Z), Z \rangle d\sigma = \langle G(Y, Z), W \rangle + \int_0^1 \langle J(G(Y, \sigma Z) | Y) Z, Z \rangle d\sigma \leq \langle G(Y, Z), W \rangle,$$

since $J(G|Y)$ is negative-definite from assumption (iv).
 (III) and (IV) can be proved similarly as in (II).

2. The Lyapunov Function $V(X, Y, Z, W, U)$

The proof of the theorems depends on a scalar differentiable comparison function $V(X, Y, Z, W, U)$. This function and its total time derivative satisfy fundamental inequalities. We define V as follows

$$\begin{aligned}
 2V(X, Y, Z, W, U) = & \langle U, U \rangle + 2\alpha_1 \langle U, W \rangle + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \langle U, Z \rangle + 2 \int_0^1 \langle \Phi(Z, \sigma W), W \rangle d\sigma \\
 & + \left[\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right] \langle W, W \rangle + 2 \left\{ \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_4} - \delta \right\} \langle W, Z \rangle \\
 & + 2\langle \Psi(X), W \rangle + 2\alpha_1 \int_0^1 \langle G(Y, \sigma Z), Z \rangle d\sigma + \left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta \right] \langle Z, Z \rangle \\
 & + 2\alpha_2 \delta \langle Y, Z \rangle + 2\alpha_1 \langle Z, H(Y) \rangle - 2\alpha_5 \langle Y, Z \rangle + 2\alpha_1 \langle \Psi(X), Z \rangle + (\delta\alpha_5 - \alpha_1\alpha_5) \langle Y, Y \rangle \\
 & + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \int_0^1 \langle H(\sigma Y), Y \rangle d\sigma + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \langle \Psi(X), Y \rangle + 2\delta \langle Y, U \rangle \\
 & + 2\delta\alpha_1 \langle W, Y \rangle + 2\langle W, H(Y) \rangle + 2\delta \int_0^1 \langle \Psi(\sigma X), X \rangle d\sigma, \tag{2.1}
 \end{aligned}$$

where

$$\delta = \left\{ \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_4} + \varepsilon \right\} \tag{2.2}$$

The properties of the function $V(X, Y, Z, W, U)$ are summarized in Lemma 2.1 and Lemma 2.2.

Lemma 2.1 Let all the conditions of Theorem 1 be satisfied. Then the function V satisfies

$$V(X, Y, Z, W, U) = 0 \text{ at } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 = 0, \tag{2.3}$$

$$V(X, Y, Z, W, U) > 0 \text{ if } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 > 0, \tag{2.4}$$

$$V(X, Y, Z, W, U) \rightarrow \infty \text{ as } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 \rightarrow \infty. \tag{2.5}$$

Proof Clearly $V(0, 0, 0, 0, 0) = 0$, since $\Phi(Z, 0) = G(Y, 0) = H(0) = \Psi(0) = 0$. We now prove (2.4) and (2.5). Since $J_H(Y)$ is a symmetric matrix, then it follows that the matrix Γ defined by (1.7) is symmetric. Further, the eigenvalues of Γ are positive because of (v). Consequently the square root $\Gamma^{\frac{1}{2}}$ exists, and this is again symmetric and nonsingular for all $Y \in R^n$. Therefore the function (2.1) admits the representation

$$\begin{aligned}
 2V = & \|U + \alpha_1 W + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} Z + \delta Y\|^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2} \left\| Z + \frac{\alpha_5}{\alpha_4} Y \right\|^2 \\
 & + \frac{\alpha_4(\alpha_1\alpha_4 - \alpha_5)}{\alpha_1\alpha_2 - \alpha_3} \left\| \frac{(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)} \Gamma^{-\frac{1}{2}} \Psi(X) + \frac{(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)} \Gamma^{\frac{1}{2}} Y + \frac{\alpha_1}{\alpha_4} \Gamma^{\frac{1}{2}} Z + \frac{1}{\alpha_4} \Gamma^{\frac{1}{2}} W \right\|^2
 \end{aligned}$$

$$+\Delta_2 \|W+\alpha, Z\|^2 + \sum_{i=1}^5 V_i \quad (2.6)$$

where

$$\begin{aligned} V_1 &= 2\delta \int_0^1 \langle \Psi(\sigma X), X \rangle d\sigma - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \left\| \Gamma^{-\frac{1}{2}} \Psi(X) \right\|^2, \\ V_2 &= \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \left\{ 2 \int_0^1 \langle H(\sigma Y), Y \rangle d\sigma - \langle H(Y), Y \rangle \right\} \\ &\quad + \left\{ \delta\alpha_3 - \alpha_1\alpha_5 - \frac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)} - \delta^2 \right\} \|Y\|^2, \\ V_3 &= \frac{\varepsilon}{\alpha_1} \|W\|^2 + 2 \int_0^1 \langle \Phi(Z, \sigma W), W \rangle d\sigma - \alpha_2 \|W\|^2, \\ V_4 &= 2\alpha_1 \int_0^1 \langle G(Y, \sigma Z), Z \rangle d\sigma - \alpha_1 \alpha_2 \|Z\|^2, \\ V_5 &= \frac{2\varepsilon(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{(\alpha_1\alpha_4 - \alpha_5)} \langle Y, Z \rangle. \end{aligned} \quad (2.7)$$

The functions V_1 , V_2 and V_3 can be estimated as in [13], in fact the estimates there show that

$$V_1 \geq \varepsilon\alpha_5' \|X\|^2, \quad V_2 \geq \left\{ \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} \right\} \|Y\|^2, \quad V_3 \geq \frac{\varepsilon}{\alpha_1} \|W\|^2. \quad (2.8)$$

Since

$$\frac{\partial}{\partial \sigma} (G(Y, \sigma Z)) = [J(G(Y, \sigma Z) | \sigma Z) Z],$$

then we obtain

$$G(Y, Z) = \int_0^1 [J(G(Y, \sigma Z) | \sigma Z) Z] d\sigma.$$

Therefore

$$\begin{aligned} V_4 &= 2\alpha_1 \int_0^1 \langle G(Y, \sigma Z), Z \rangle d\sigma - \alpha_1 \alpha_2 \|Z\|^2 \\ &= 2\alpha_1 \int_0^1 \int_0^1 \langle \{J[G(Y, \sigma_1, \sigma_2 Z) | \sigma_1, \sigma_2 Z] - \alpha_1 I\} \sigma_2 Z, Z \rangle d\sigma_1 d\sigma_2 \geq 0, \end{aligned} \quad (2.9)$$

by (iv).

On gathering the estimates (2.7), (2.8) and (2.9) into (2.6) we conclude that

$$\begin{aligned} 2V \geq & \|U+\alpha, W + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} Z + \delta Y\|^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2} \left\| Z + \frac{\alpha_5}{\alpha_4} Y \right\|^2 + \Delta_2 \|W+\alpha, Z\|^2 \\ & + \varepsilon\alpha_5' \|X\|^2 + \left\{ \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} \right\} \|Y\|^2 + \frac{\varepsilon}{\alpha_1} \|W\|^2 + \frac{2\varepsilon(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{(\alpha_1\alpha_4 - \alpha_5)} \langle Y, Z \rangle. \end{aligned}$$

It follows from the first six terms of this inequality that there exist positive constants $D_i (i=1,2,3,4,5)$ such that

$$2V \geq D_1 \|X\|^2 + 2D_2 \|Y\|^2 + 2D_3 \|Z\|^2 + D_4 \|W\|^2 + D_5 \|U\|^2 + \frac{2\varepsilon(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{(\alpha_1\alpha_4 - \alpha_5)} \langle Y, Z \rangle. \quad (2.10)$$

The inequality (2.10) is the same as the inequality in [13]. Thus the estimates for V give that

$$2V \geq D_1 \|X\|^2 + D_2 \|Y\|^2 + D_3 \|Z\|^2 + D_4 \|W\|^2 + D_5 \|U\|^2$$

as in [13].

Lemma 2.2 For every solution $(X(t), Y(t), Z(t), W(t), U(t))$ of (1.2) we have

$$\dot{V} \equiv \frac{d}{dt} V(X(t), Y(t), Z(t), W(t), U(t)) \leq 0 \text{ for all } t \geq 0, \text{ and especially} \quad (2.11)$$

$$\frac{d}{dt} V(X(t), Y(t), Z(t), W(t), U(t)) < 0 \text{ for } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 \neq 0. \quad (2.12)$$

Proof From (2.1), (1.2) and Lemma 1.2 we have

$$\begin{aligned} \dot{V} \leq & - \{ \langle F(X, Y, Z, W)U, U \rangle - \alpha_1 \langle U, U \rangle \} \\ & - \left[\alpha_1 \langle \Phi(Z, W), W \rangle - \left\{ \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right\} \langle W, W \rangle \right] \\ & - \left[\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \langle G(Y, Z), Z \rangle - \{ \alpha_2 \delta \langle Z, Z \rangle + \alpha_1 \langle J_H(Y)Z, Z \rangle - \alpha_5 \langle Z, Z \rangle \} \right] \\ & - \{ \delta \langle Y, H(Y) \rangle - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \langle J_\Psi(X)Y, Y \rangle \} - \alpha_1 \langle F(X, Y, Z, W)U, W \rangle \\ & + \alpha_1^2 \langle U, W \rangle - \{ \langle G(Y, Z), U \rangle - \alpha_3 \langle Z, U \rangle \} - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \langle F(X, Y, Z, W)U, Z \rangle \\ & + \left\{ \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right\} \langle U, Z \rangle - \delta \langle F(X, Y, Z, W)Y, U \rangle + \delta \alpha_1 \langle Y, U \rangle \\ & - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \{ \langle \Phi(Z, W), Z \rangle - \alpha_1 \langle W, Z \rangle \} - \{ \alpha_4 \langle W, Z \rangle - \langle W, J_H(Y)Z \rangle \} \\ & - \alpha_5 \langle Y, W \rangle + \langle J_\Psi(X)Y, W \rangle - \delta \{ \langle G(Y, Z), Y \rangle - \alpha_3 \langle Y, Z \rangle \} \\ & - \delta \{ \langle \Phi(Z, W), Y \rangle - \alpha_2 \langle W, Y \rangle \} - \alpha_1 \alpha_5 \langle Y, Z \rangle + \alpha_1 \langle J_\Psi(X)Y, Z \rangle. \quad (2.13) \end{aligned}$$

From (ii) and Lemma 1.1 we have

$$\langle F(X, Y, Z, W)U, U \rangle - \alpha_1 \langle U, U \rangle \geq \varepsilon_0 \|U\|^2. \quad (2.14)$$

It follows (2.2), (iii) and Lemma 1.1 that

$$\alpha_1 \langle \Phi(Z, W), W \rangle - \left\{ \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right\} \langle W, W \rangle$$

$$\begin{aligned}
&= \alpha_1 \left[\int_0^1 \langle J[\Phi(Z, \sigma W) | \sigma W] - \alpha_2 I \rangle W, W \rangle d\sigma \right] \\
&\quad + \left\{ \alpha_1 \alpha_2 - \alpha_3 + \delta - \frac{\alpha_1 \alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \right\} \langle W, W \rangle \geq \varepsilon \|W\|^2. \tag{2.15}
\end{aligned}$$

$$\begin{aligned}
&\frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \langle G(Y, Z), Z \rangle - \{ \alpha_2 \delta \langle Z, Z \rangle + \alpha_1 \langle J_H(Y)Z, Z \rangle - \alpha_5 \langle Z, Z \rangle \} \\
&= \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \int_0^1 \langle J(G(Y, \sigma Z) | \sigma Z)Z, Z \rangle d\sigma - \{ \alpha_2 \delta \langle Z, Z \rangle + \alpha_1 \langle J_H(Y)Z, Z \rangle - \alpha_5 \langle Z, Z \rangle \} \\
&\geq \frac{(\alpha_2 \alpha_4 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \langle Z, Z \rangle - \{ \alpha_1 \|J_H(Y)\| - \alpha_5 \} \langle Z, Z \rangle - \varepsilon \alpha_2 \langle Z, Z \rangle \geq \varepsilon \alpha_2 \|Z\|^2
\end{aligned}$$

by (iv), (2.2), (1.5) and Lemma 1.1.

Similarly, by using (2.2), (v), (vi) and Lemma 1.1 we have

$$\begin{aligned}
&\delta \langle Y, H(Y) \rangle - \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \langle J_\Psi(X)Y, Y \rangle \\
&= \delta \int_0^1 \langle Y, J_H(\sigma Y)Y \rangle d\sigma - \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \langle J_\Psi(X)Y, Y \rangle \geq \varepsilon \alpha_4 \|Y\|^2.
\end{aligned}$$

Thus the first four terms in (2.13) are majorizable by

$$-(\varepsilon \alpha_4 \|Y\|^2 + \varepsilon \alpha_2 \|Z\|^2 + \varepsilon \|W\|^2 + \varepsilon_0 \|U\|^2). \tag{2.16}$$

Now let $R(X, Y, Z, W, U)$ denote the sum of the remaining terms in (2.13). By the hypotheses of Theorem 1 and Schwarz's inequality we obtain

$$|R(X, Y, Z, W, U)| \leq D_7 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) (\|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2) \tag{2.17}$$

for some constant $D_7 > 0$.

Combining inequalities (2.14) - (2.17) in (2.13), we obtain

$$\begin{aligned}
\dot{V} &\leq -(\varepsilon \alpha_4 \|Y\|^2 + \varepsilon \alpha_2 \|Z\|^2 + \varepsilon \|W\|^2 + \varepsilon_0 \|U\|^2) \\
&\quad + D_7 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) (\|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2) \\
&\leq -\frac{1}{2} \min \{ \varepsilon \alpha_4, \varepsilon \alpha_2, \varepsilon, \varepsilon_0 \} (\|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2)
\end{aligned}$$

if $D_7 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) \leq \frac{1}{2} \min \{ \varepsilon \alpha_4, \varepsilon \alpha_2, \varepsilon, \varepsilon_0 \}$.

Hence, we can conclude that

$\dot{V}(t) \leq 0$ for all $t \geq 0$, and especially

$$\frac{d}{dt} V(X(t), Y(t), Z(t), W(t), U(t)) < 0 \text{ for } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 > 0$$

which proves the lemma.

3. Proof of Theorem 1

The basic properties of $V(X, Y, Z, W, U)$, which we have proved in Lemma 2.1 and Lemma 2.2, justify that the zero solution of (1.2) is asymptotic stability in the large [7].

The system of equations (1.2) is equivalent to the differential equation (1.1). There follows, thus, the original statement of Theorem 1.

4. Proof of Theorem 2

Consider the function V defined as above. Then under the conditions of Theorem 2 the conclusion of Lemma 2.1 can be obtained, that is,

$$2V \geq D_1 \|X\|^2 + D_2 \|Y\|^2 + D_3 \|Z\|^2 + D_4 \|W\|^2 + D_5 \|U\|^2 \quad (4.1)$$

and since $P(t, X, Y, Z, W, U) \neq 0$ the conclusion Lemma 2.2 can be revised as follows

$$\dot{V} \leq \langle U + \alpha, W + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} Z + \delta Y, P(t, X, Y, Z, W, U) \rangle.$$

Since $P(t, X, Y, Z, W, U)$ satisfies (1.8), by Schwarz's inequality we obtain

$$\begin{aligned} \dot{V} &\leq (\|U\| + \alpha, \|W\| + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \|Z\| + \delta \|Y\|) \|P(t, X, Y, Z, W, U)\| \\ &\leq D_8 (\|U\| + \|W\| + \|Z\| + \|Y\|) [\delta_1 + \delta_2 (\|Y\| + \|Z\| + \|W\| + \|U\|)] \theta(t), \end{aligned}$$

where $D_8 = \max \left\{ 1, \alpha_1, \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}, \delta \right\}$.

Hence we can easily obtain

$$\dot{V} \leq [D_9 + D_{10} (\|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2)] \theta(t), \quad (4.2)$$

where $D_9 = 4\delta_1 D_8$, $D_{10} = \delta_1 D_8 + 4\delta_2 D_8$.

Let $D_{11} = \min \left\{ \frac{D_2}{2}, \frac{D_3}{2}, \frac{D_4}{2}, \frac{D_5}{2} \right\}$. Then it follows from (4.1) that

$$V \geq D_{11} (\|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2). \quad (4.3)$$

From (4.2) and (4.3) we have

$$\dot{V} \leq D_9 \theta(t) + D_{12} V \theta(t),$$

where $D_{12} = \frac{D_{10}}{D_{11}}$.

Gronwall - Bellman inequality yields

$$V \leq D_{13} \exp \left(\int_0^t D_{12} \theta(s) ds \right),$$

where $D_{13} = V(0) + D_9 A$

The proof of Theorem 2 is complete.

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