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ON QUASI-CONFORMALLY RECURRENT MANIFOLDS

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Abstract. The object of this paper is to study a Riemannian manifold called quasi-conformally recurrent manifold.

1. INTRODUCTION

In [6] Yano and Sawaki defined and studied a tensor field W on a Riemannian manifold (M, g) of dimension n which includes both the conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$ as special cases.

A manifold (M, g) of dimension n is called quasi-conformally recurrent if the tensor W [6] defined by

$$W(X, Y)Z = -(n-2)b\, C(X, Y)Z + [a + (n-2)b]\, \tilde{C}(X, Y)Z \tag{1.1}$$

satisfies the condition

$$(\nabla_U W)(X, Y)Z = A(U)W(X, Y)Z \tag{1.2}$$

where ∇ denotes covariant differentiation with respect to the metric tensor, C and $\tilde{C}$ are conformal curvature tensor and concircular curvature tensor respectively, a, b are arbitrary constants and A is a non-zero 1-form, ρ is a non-zero vector field such that $g(X, \rho) = A(X)$. Such a manifold will be denoted by QCK_n. This notation is taken because conformally recurrent manifold is denoted by CK_n [2]. In this connection we can mention the work of Amur and Maralabhavi [1] who studied quasi-conformally flat spaces. It is easily seen that a recurrent manifold K_n [5] is a quasi-conformally recurrent manifold QCK_n, but the converse is not necessarily true. In this paper sufficient conditions for a QCK_n to be a K_n is obtained. Also it is shown that a 3-dimensional QCK_n is concircularly recurrent if $a+(n-2)b \neq 0$. In section 3, Einstein QCK_n is studied and it is proved that in an Einstein QCK_n either the associated vector field ρ of the 1-form A is null or the manifold is a space form. In the last section, we consider a QCK_n admitting a recurrent vector field [4].

2. QUASI-CONFORMALLY RECURRENT MANIFOLD

It is known that the conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$ are defined by

$$C(X, Y)Z = R(X, Y)Z + \frac{1}{n-2}[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY$$

$$- g(Y, Z)QX] - \frac{r}{(n-1)(n-2)} r[g(X, Z)Y - g(Y, Z)Y]. \quad (2.1)$$

and $$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \quad (2.2)$$

where R, S and r denotes the Riemannian curvature tensor, Ricci tensor and the scalar curvature respectively; Q is the Ricci operator defined by

$$g(QX, Y) = S(X, Y) \quad (2.3)$$

Using (2.1) and (2.2) in (1.1), we get

$$W(X, Y)Z = aR(X, Y)Z - b[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY$$

$$- g(Y, Z)QX] - \left\{\frac{a + 2b(n-1)}{n(n-1)}\right\} r[g(Y, Z)X - g(X, Z)Y]. \quad (2.4)$$

Theorem 1. A quasi-conformally recurrent manifold is recurrent if it is Ricci recurrent and $a \neq 0$.

Proof : Since the manifold is Ricci recurrent [3], we have

$$(\nabla_U S)(Y, Z) = A(U)S(Y, Z) \quad (2.5)$$

where A is a non-zero 1-form.

From (2.3) and (2.5), we get

$$(\nabla_U Q)(X) = A(U)Q(X). \quad (2.6)$$

Also from (2.6), it follows that

$$\nabla_U r = A(U) r. \tag{2.7}$$

Now from (1.2) and (2.4), we get

$$a(\nabla_U R)(X, Y)Z - b[(\nabla_U S)(X, Z)Y - (\nabla_U S)(Y, Z)X + g(X, Z)(\nabla_U Q)(Y)$$

$$- g(Y, Z)(\nabla_U Q)(X)] - \nabla_U r[\frac{a + 2b(n-1)}{n(n-i)}][g(Y, Z)X - g(X, Z)Y]$$

$$= A(U)\{aR(X, Y)Z - b[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY$$

$$- g(Y, Z)QX] - r[\frac{a + 2b(n-1)}{n(n-1)}][g(Y, Z)X - g(X, Z)Y]. \tag{2.8}$$

Using (2.5), (2.6) and (2.7) in (2.8), we get

$$(\nabla_U R)(X, Y)Z = A(U)R(X, Y)Z, \quad \text{if} \quad a \neq 0,$$

which implies that the manifold is recurrent.

This completes the proof.

Theorem 2. If the Ricci tensor vanishes and $a \neq 0$, then a quasi-conformally recurrent manifold is a recurrent manifold.

Proof : We suppose that $S(X, Y) = 0$. Then the scalar curvature $r = 0$. Hence (2.4) reduces to

$$W(X, Y)Z = a\, R(X, Y)Z \tag{2.9}$$

Using (2.9) in (1.2), we get

$$(\nabla_U R)(X, Y)Z = A(U)R(X, Y)Z, \quad \text{if} \quad a \neq 0.$$

This completes the proof.

Theorem 3. A 3-dimensional quasi-conformally recurrent manifold is concircularly recurrent if $a + (n-2)\,b \neq 0$.

Proof : It is known that in a 3-dimensional Riemannian manifold conformal curvature tensor vanishes. Then (1.1) reduces to

$$W(X, Y)Z = [a + (n-2)b]\, \tilde{C}(X, Y)Z. \qquad (2.10)$$

Since the manifold is QCK_n, we have (1.2). So by virtue of (2.10) and (1.2), we get

$$(\nabla_U \tilde{C})(X, Y)Z = A(U)\, \tilde{C}(X, Y)Z, \quad \text{if} \quad a + (n-2)b \neq 0.$$

Thus the manifold is concircularly recurrent if $a + (n-2)b \neq 0$. This completes the proof.

3. EINSTEIN QUASI-CONFORMALLY RECURRENT MANIFOLD

In this section, we suppose that a QCK_n is an Einstein manifold. Then the Ricci tensor satisfies

$$S(X, Y) = \frac{r}{n}\, g(X, Y) \qquad (3.1)$$

from which follows

$$\nabla_U r = 0 \quad \text{and} \quad (\nabla_U S)(X, Y) = 0. \qquad (3.2)$$

Since the manifold is QCK_n, we get from (1.2) and (2.4) by using (3.1) and (3.2),

$$(\nabla_U R)(X, Y)Z = A(U)\left[R(X, Y)Z - \frac{r}{n(n-1)}\,(g(Y, Z)X - g(X, Z)Y)\right], \qquad (3.3)$$

if $a \neq 0$.

Also (3.3) can be written as

$$(\nabla_U R)(X, Y, Z, V) = A(U)\Big[R(X, Y, Z, V) -$$

$$\frac{r}{n(n-1)}\,\{g(Y,X)\, g(X,V) - g(X,Z) g(Y,V)\}\Big] \qquad (3.4)$$

where $R(X, Y, Z, V) = g(R(X, Y)Z, V)$.

Now from (3.2) and the Bianchi identity

$$(\nabla_U R)(X, Y, Z, V) + (\nabla_Y R)(U, X, Z, V) + (\nabla_X R)(Y, U, Z, V) = 0, \tag{3.5}$$

we get
$$\text{div.}R = 0 \tag{3.6}$$
where 'div' denotes divergence.

Using (3.4) in (3.5) and putting $U = \rho$, we get

$$A(\rho)\, R(X, Y, Z, V) + A(Y)\, R(Z, V, \rho, X) + A(X)\, R(Z, V, Y, \rho) - \frac{r}{n(n-1)} [A(\rho)\{g(Y, Z)\, g(X, V) - g(X, Z)\, g(Y, V)\} + A(Y)\, \{g(X, Z)\, A(V) - g(X, V)A(Z)\} + A(X)\{A(Z)\, g(Y, V) - g(Y, Z)\, A(V)\}] = 0, \tag{3.7}$$

where ρ is a vector field defined by

$$g(X, \rho) = A(X). \tag{3.8}$$

Contracting U in (3.3) and using (3.6), we get

$$R(X, Y, Z, \rho) = \frac{r}{n(n-1)} [g(Y, Z)\, A(X) - g(X, Z)\, A(Y)] \tag{3.9}$$

Using (3.9) in (3.7), we get

$$A(\rho)\, [R(X, Y, Z, V) - \frac{r}{n(n-1)} \{g(Y, Z)\, g(X, V) - g(X, Z)\, g(Y, V)\}] = 0.$$

Then either $A(\rho) = 0$

i.e., $g(\rho, \rho) = 0$,

or, the manifold is of constant curvature. Hence we can state the following theorem:

Theorem 4. If a quasi-conformally recurrent manifold with $a \neq 0$ is an Einstein one, then either the associated vector field ρ is null or the manifold is a space form.

4. QCK_n ADMITTING A RECURRENT VECTOR FIELD

In this section, we assume that the QCK_n admits a recurrent vector field V defined by

$$\nabla_X V = \omega(X)V$$

where ω is a non-zero 1-form such that

$$g(X, V) = \omega(X). \tag{4.1}$$

So we find from (4.1) that

$$g(\nabla_X V, Y) = g(\omega(X)V, Y)$$

i.e., $(\nabla_X \omega)(Y) = \omega(X)\, g(V, Y) = \omega(X)\, \omega(Y)$

Therefore, $(\nabla_X \omega)(Y) - (V_Y \omega)(X) = 0$

i.e., $$(d\omega)(X, Y) = 0 \tag{4.2}$$

where d is the exterior differential.

Also from Ricci identity, we have

$$V_X \nabla_Y V - V_Y \nabla_X V - \nabla_{[X, Y]} V = R(X, Y)V.$$

Using (4.1), we find that

$$R(X, Y)V = (d\omega)(X, Y)V$$

So from (4.2), we find that

$$R(X, Y)V = 0 \tag{4.3}$$

Now from (4.3), we have

$$(\nabla_U R)(X, Y)V = 0. \tag{4.4}$$

Applying Bianchi identity on (4.4), we get

$$(\nabla_V R)(X, Y)U = 0. \tag{4.5}$$

From (4.5), we get

$$(\nabla_V S)(Y, U) = 0. \tag{4.6}$$

From (2.3) and (4.6), we get

$$(\nabla_V Q)(Y) = 0. \tag{4.7}$$

Also from (4.6), we get

$$\nabla_V r = 0. \tag{4.8}$$

Now from (1.2) and (2.4) and using (4.5), (4.6), (4.7) and (4.8), it follows that

$$A(V)W(X, Y)Z = 0$$

Then either $A(V) = 0$ or, $W(X, Y)Z = 0$

i.e., either $g(V, \rho) = 0$ or, $W(X, Y)Z = 0$,

which implies that either V is orthogonal to the associated vector field ρ or the manifold is quasi-conformally flat.

Hence we can state the following theorem :

Theorem 5. If a quasi-conformally recurrent manifold admits a recurrent vector field V, then either V is orthogonal to the associated vector field ρ or the manifold is quasi-conformally flat.

REFERENCES

[1] AMUR, K. and MARALABHAVI, Y. B. :On quasi-conformally flat spacs, Tensor, N. S. Vol. **31**(1977), 194-198.

[2] ADATI, T. and MIYAZAWA, T. :On a Riemannian space with recurrent conformal curvature, Tensor, N.S. **18**(1967), 348-354.

[3] PATTERSON, E. M. : Some theorems on Ricci-recurrent spaces, J. London. Math. Soc. 1952, Vol. **27**, 287-295.

[4] SCHOUTEN, J. A. : Ricci Calculas (2nd edition) Springer-verlag, 1954.

[5] WALKER, A. G. : On Ruse's Spaces of recurrent curvature, Proc. Lond. Math. Soc. (2) **52**(1950), 36-64.

[6] YANO, K. and SAWAKI, S. : Riemannian manifolds admitting a conformal transformation group, J. Diff. Geom. **2**(1968) 161-184.

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