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ON A TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Summary : The concept of semi-symmetric non-metric connection on a Riemannian manifold has been introduced by Agashe and Chafle [1]. The properties of a Riemannian manifold admitting a semi-symmetric non-metric connection with recurrent torsion tensor have been studied in [2].In the present paper we study a type of semi-symmetric non-metric connection $\tilde{\nabla}$which satisfying $\tilde{R}(X, Y). T = 0$ and $\omega(\tilde{R}(X,Y)Z) = 0$, where T is the torsion tensor of the semi-symmetric non-metric connection, $\tilde{R}$ is the curvature tensor corresponding to $\tilde{\nabla}$and ω is the associated 1-form of T.

INTRODUCTION

Let M^n be an n-dimensional Riemannian manifold with a metric tensor g and Levi-Civita connection ∇. A linear connection $\tilde{\nabla}$ on M^n is said to be a semi-symmetric non-metric connection if its torsion tensor T and the metric tensor g of the manifold satisfy the following conditions :

$$T(X, Y) = \omega(Y)X - \omega(X)Y \tag{1}$$

for any two vector fields X, Y where ω is a 1-form associated with the torsion tensor of the connection $\tilde{\nabla}$and

$$(\tilde{\nabla}_X g)(Y, Z) = -\omega(Y)\, g(X, Z) - \omega(Z) g(X, Y) \tag{2}$$

Then we have [1] for any vector fields X, Y, Z

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X \tag{3}$$

and

$$(\tilde{\nabla}_X \omega)(Y) = (\nabla_X \omega)(Y) - \omega(X)\,\omega(Y) \tag{4}$$

Also, we have [1]

$$\tilde{R}(X, Y)Z = R(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X \tag{5}$$

where

$$\alpha(Y, Z) = g(AY, Z) = (\nabla_Y\omega)(Z) - \omega(Y)\,\omega(Z), \tag{6}$$

$\tilde{R}$ and R the respective curvature tensors for the connections $\tilde{\nabla}$ and ∇, A being a (1-1) tensor field.

Now, let us suppose that the connection (1) satisfies the following conditions :

$$\tilde{R}(X, Y).T = 0 \tag{7}$$

and

$$\omega(\tilde{R}(X, Y)Z) = 0 \tag{8}$$

where $\tilde{R}(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y.

1. EXPRESSION FOR THE CURVATURE TENSOR OF THE SEMI-SYMMETRIC NON-METRIC CONNECTION

The condition (7) gives

$$\tilde{R}(X,Y)T\,(U,V) - T(\tilde{R}(X,Y)\,U,V) - T(U, \tilde{R}(X,Y)V) - (\tilde{\nabla}_{T(X,Y)}T)\,(U, V) = 0 \tag{1.1}$$

Now $(\tilde{\nabla}_{T(X,Y)}T)\,(U, V)$

$$= (\tilde{\nabla}_{\omega(Y)X - \omega(X)Y}T)\,(U, V)$$

$$= \omega(Y)\,(\tilde{\nabla}_XT)\,(U, V) - \omega(X)\,(\tilde{\nabla}_YT)\,(U, V)$$

$$= \omega(Y)\,[(\nabla_X\omega)\,(V)\,U - (\nabla_X\omega)\,(U)V]$$

$$- \omega(X)\,[(\nabla_Y\omega)\,(V)U - (\nabla_Y\omega)\,(U)V] \tag{1.2}$$

From (1.1) and (1.2) we get

$$\omega(\tilde{R}(X, Y)U)V - \omega(\tilde{R}(X, Y)V)U - \omega(Y)\,[(\nabla_X\omega)(V)U - (\nabla_X\omega)(U)V]$$

$$+ \omega(X)\,[(\nabla_Y\omega)\,(V)U - (\nabla_Y\omega)\,(U)V] = 0 \tag{1.3}$$

Now using the condition (8) it follows from (1.3)

$$\omega(X)\,[(\nabla_Y\omega)(V)U - (\nabla_Y\omega)(U)V] - \omega(Y)[(\nabla_X\omega)(V)U - (\nabla_X\omega)(U)V] = 0 \qquad (1.4)$$

Contracting U in (1.4) we obtain

$$\omega(X)\,(\nabla_Y\omega)(V) - \omega(Y)\,(\nabla_X\omega)(V) = 0 \qquad (1.5)$$

Putting $X = \rho$ we get

$$(\nabla_Y\ \omega)(Z) = \frac{\omega(Y)}{\omega(\rho)}\,(\nabla_\rho\omega)(Z) \qquad (1.6)$$

where we take $V = Z$

From (6) and (1.6) we get

$$\alpha(Y, Z) = \frac{\omega(Y)}{\omega(\rho)}\,(\nabla_\rho\omega)(Z) - \omega(Y)\,\omega(Z) \qquad (1.7)$$

Now putting the value of $\alpha(Y, Z)$ in (5) we get

$$'\tilde{R}(X, Y, Z, U) = {}'R(X, Y, Z, U) + \frac{(\nabla_\rho\omega)(Z)}{\omega(\rho)}\,[\omega(X)\,g(Y, U) - \omega(Y)\,g(X, U)$$

$$- \omega(Z)\,(\omega(X)\,g(Y,U) - \omega(Y)\,g(X, U)) \qquad (1.8)$$

where $'\tilde{R}(X, Y, Z, U) = g(\tilde{R}(X, Y)Z, U)$ and $'R(X, Y, Z, U) = g(R(X, Y)Z, U)$

Thus we can state

Theorem 1. Let a Riemannian manifold admits a semi-symmetric non-metric connection (1) satisfying (7) and (8). Then the curvature tensor of the semi-symmetric non-metric connection has the form (1.8).

If, in particular, $\tilde{R} = 0$, then from (5) we get

$$'R(X, Y, Z, U) = \alpha(Y, Z)\,g(X, U) - \alpha(X, Z)\,g(Y, U).$$

Now putting $X = U = e_i$ in the above expression where $\{e_i\}$ is an orthonormal basis of the tangent space at any point we have by taking the sum for $1 \le i \le n$

$$S(Y, Z) = (n-1)\,\alpha(Y, Z).$$

Since S is symmetric, we get

$$\alpha(Y, Z) = \alpha(Z, Y).$$

Hence from (6) we get

$$(\nabla_Y\omega)(Z) = (\nabla_Z\omega)(Y).$$

Therefore

$$(\nabla_\rho\omega)\,(Y) = (\nabla_Y\omega)\,(\rho). \tag{1.9}$$

From (1.6) we get

$$(\nabla_Y\omega)\,(\rho) = \beta\omega(Y)\ . \tag{1.10}$$

where $\beta = \dfrac{(\nabla_\rho\omega)\,(\rho)}{\omega(\rho)}$

Now taking $\tilde{R}= 0$ and using (1.10) in (1.8) we get

$$'R(X, Y, Z, U) = v\,[\omega(Y)\omega(Z)\, g(X, U) - \omega(X)\,\omega(Z)\, g(Y, U)] \tag{1.11}$$

where $v = \dfrac{(\nabla_\rho\omega)\,(\rho)}{\omega(\rho)\,\omega(\rho)} - 1$

Remarks. The conditions (7) and (8) of our paper are weaker than the conditions of [2], since it is known that in a Riemannian manifold recurrent torsion tensor implies $\tilde{R}(X,Y).T= 0$ and $\tilde{R}= 0$ implies $\omega(\tilde{R}(X,Y)Z) = 0$, but the converse are not necessarily true.

Hence we can state the following :

Theorem 2. Let a Riemannian manifold admits a semi-symmetric non-metric connection satisfying (7) and (8) whose curvature tensor vanishes, then the curvature tensor of the manifold is gevin by (1.11).

From (1.8) it can be easily seen that $'\tilde{R}$ satisfies

$$'\tilde{R}(X, Y, Z, U) = -\,'\tilde{R}(Y, X, Z, U).$$

Also we get

$$'\tilde{R}(X, Y, Z, U) + {}'\tilde{R}(Y, Z, X, U) + {}'\tilde{R}(Z, X, Y, U) = 0 \tag{1.12}$$

if and only if

$$\omega(Y)\,(\nabla_\rho\omega)\,(Z) = \omega(Z)\,(\nabla_\rho\omega)\,(Y)\,. \tag{1.13}$$

2. SYMMETRY CONDITION OF THE RICCI TENSOR OF $\tilde{\nabla}$

In this section a necessary and sufficient condition for the symmetry of the Ricci tensor of the semi-symmetric non-metric connection is obtained by proving the following theorem :

Theorem 3. A necessary and sufficient condition that the Ricci-tensor of the semi-symmetric non-metric connection $\tilde{\nabla}$ to be symmetric is that the (0, 4) curvature tensor $'\tilde{R}$ of the connection $\tilde{\nabla}$ satisfies

$$'\tilde{R}(X,Y,Z,U) + {}'\tilde{R}(Y,Z,X,U) + {}'\tilde{R}(Z,X,Y,U) = 0.$$

Proof : Let S and $\tilde{S}$ denote the Ricci tensors of the Levi-Civita connection and the semi-symmetric non-metric connection respectively.

Now putting $X = U = e_i$ in (1.8) we get

$$\tilde{S}(Y,Z) = S(Y,Z) - \frac{(n-1)}{\omega(\rho)}\omega(Y)\,(\nabla_\rho\omega)(Z) + (n-1)\omega(Y)\omega(Z) \tag{2.1}$$

From (2.1) it follows that

$$\tilde{S}(Y,Z) = \tilde{S}(Z,Y) \text{ if and only if } \omega(Y)\,(\nabla_\rho\omega)\,(Z) = \omega(Z)\,(\nabla_\rho\omega)\,(Y)$$

But from (1.12) and (1.13) we see that (1.12) holds if and only if (1.13) holds. Hence $\tilde{S}$ is symmetric if and only if the condition (1.12) holds.

This completes the proof.

3. EXISTENCE OF A GRADIENT VECTOR FIELD

In this section we consider a Riemannian manifold M^n that admits a semi-symmetric non-metric connection $\tilde{\nabla}$ whose Ricci tensor is symmetric and satisfies the conditions (7) and (8). It is shown that if a Riemannian manifold admits such a connection, then the manifold admits a gradient vector field.

If the connection (1) satisfies the conditions (7) and (8), then we get from (1.6)

$$(\nabla_X \omega)(Y) = \frac{\omega(X)}{\omega(\rho)} (\nabla_\rho \omega)(Y)$$

Since $\tilde{S}$ symmetric, we get from theorem 3 and (1.13)

$$\omega(Y)(\nabla_\rho \omega)(X) = \omega(X)(\nabla_\rho \omega)(Y) \tag{3.2}$$

Putting $Y = \rho$ in (3.2) we get

$$(\nabla_\rho \omega)(X) = \beta\omega(X) \tag{3.3}$$

where $\beta = \dfrac{(\nabla_\rho \omega)(\rho)}{\omega(\rho)}$

Using (3.3) in (3.1) we obtain

$$(\nabla_X \omega)(Y) = a\omega(X)\,\omega(Y) \tag{3.4}$$

where $a = \dfrac{(\nabla_\rho \omega)(\rho)}{\omega(\rho)\,\omega(\rho)}$.

From (3.4) it follows that the 1-form ω is closed. That is, the associated vector field ρ is a gradient vector field . Hence we can state the following.

Theorem 4. If a Riemannian manifold admits a semi-symmetric non-metric connection satisfying (7) and (8) with symmetric Ricci tensor, then the manifold admits a gradient vector field.

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