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ON THE EXISTENCE OF RELATIVE FIX POINTS

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Abstract We introduce the idea of relative iterations of functions and using this, extend a theorem on fix point of complex function involving exact order.

1. Introduction

A single valued function f(z) of the complex variable z is said to belong to (i) class I if f(z) is entire transcendental, (ii) class II if it is regular in the plane punctured at a, b ($a \neq b$) and has an essential singularity at b and a singularity at a and if f(z) omits the values a and b except possibly at a.

The functions in class II may be normalised by taking $a = 0$ and $b = \infty$. In future we shall consider such normalised functions in class II.

For arbitrary f(z), the iterations are defined inductively by

$f_0(z) = z$ and $f_{n+1}(z) = f(f_n(z))$, $n = 0, 1, 2, \ldots\ldots$.

A point α is called a <u>fix point</u> of f(z) of order n if α is a solution of $f_n(z) = z$. It is said to be of <u>exact order</u> n if α is a solution of $f_j(z) = z$ for $j = n$ but not for $j < n$.

Regarding the existence of a fix point, Baker [1] proved the following theorem.

<u>Theorem A</u>. If f(z) belongs to class I, then f(z) has fix points of exact order n, except for atmost one value of n.

Bhattacharyya [2] extended Theorem A to functions in class II as follows.

<u>Theorem B</u>. If f(z) belongs to class II, then f(z) has an infinity of fix points of exact order n, for every positive integer n.

In this paper we observe that Theorem B may be proved under more general settings by using the concept of relative fix point (defined below).

2. Preliminaries and Definitions

Let f(z) and $\phi(z)$ be functions of the complex variable z. Let

$f_1(z) = f(z)$

$f_2(z) = f(\phi(z)) = f(\phi_1(z))$

$f_3(z) = f(\phi(f(z))) = f(\phi_2(z)) = f(\phi(f_1(z)))$

$f_4(z) = f(\phi(f(\phi(z)))) = f(\phi_3(z)) = f(\phi(f_2(z)))$

....

$f_n(z) = f(\phi(f.......(f(z) \text{ or } \phi(z)..........)))$, according as n is odd or even

$\quad = f(\phi_{n-1}(z)) = f(\phi(f_{n-2}(z)),$

and so

$\phi_1(z) = \phi(z)$

$\phi_2(z) = \phi(f(z)) = \phi(f_1(z))$

$\phi_3(z) = \phi(f_2(z)) = \phi(f(\phi_1(z)))$

....

$\phi_n(z) = \phi(f_{n-1}(z)) = \phi(f(\phi_{n-2}(z))).$

Clearly all $f_n(z)$ and $\phi_n(z)$ are functions in class II, if f(z) and $\phi(z)$ are so.

A point α is called a fix point of f(z) of order n with respect to $\phi(z)$, if $f_n(\alpha) = \alpha$ and a fix point of exact order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$, k = I, 2, ..., n – 1. Such points α are also called relative fix points.

Let $f(z) = z^2 - z$ and $\phi(z) = z^2$. Then $f_2(z) = z^4 - z^2$. So, z = 0 is a fix point of f(z) of order 2 with respect to $\phi(z)$ which is not an exact fix point because z = 0 is a solution of the equation f(z) = z also. It is clear that all the solutions of $z^3-z-1 = 0$ are fix points of f(z) of exact order 2 with respect to $\phi(z)$.

Let $f(z)$ be meromorphic in $r_0 \le |z| < \infty$, $r_0 > 0$. We use the following notations [3] :

$n(t, a, f)$ = number of roots of $f(z) = a$ in $r_0 < |z| \le t$,

$$N(r, a, f) = \int_{r_0}^{r} \frac{n(t, a, f)}{t} dt.$$

If $a = \infty$, then we write $n(t, \infty, f) = n(t, f)$ = the number of poles in $r_0 < |z| \le t$, counted with due regard to multiplicity and $N(r, \infty, f) = N(r, f)$. Also

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,$$

$$m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta.$$

With these notations, Jensen's formula can be written as [3]

$$m(r, f) + N(r, f) = m(r, 1/f) + N(r, 1/f) + O(\log r).$$

Writing $m(r, f) + N(r, f) = T(r, f)$, the above becomes

$$T(r, f) = T(r, 1/f) + O(\log r).$$

In this case the first fundamental theorem takes the form

$$m(r, a, f) + N(r, a, f) = T(r, f) + O(\log r) \tag{1}$$

where the region is always $r_0 \le |z| < \infty$, $r_0 > 0$.

Suppose that $f(z)$ is nonconstant. Let $a_1, a_2, \ldots\ldots\ldots, a_q$, $q > 2$, be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_v| \ge \delta$ for $1 \le \mu \le v \le q$. Then

$$m(r, f) + \sum_{v=1}^{q} m(r, a_v, f) \le 2T(r, f) - N_1(r) + S(r) \tag{2}$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f')$$

and $S(r) = m\left(r, \frac{f'}{f}\right) + \sum_{\nu=1}^{q} m\left(r, \frac{f'}{(f - a_\nu)}\right) + O(\log r)$.

The proof of (2) can be carried out following the technique as given in {[4], p. 32} and using the modified form as given in (1).

It has been obtained in [3] that $m\left(r, \frac{f'}{f}\right)$ and hence $m\left(r, \frac{f'}{f-a}\right)$ is $O\{\max(\log^+ T(r, f), \log r)\}$ as $r \to \infty$ outside a set of r intervals of finite measure. So, we have $S(r) = O\{\max(\log^+ T(r, f), \log r)\} + O(\log r)$

$$= O\{\max(\log r, \log^+ T(r, f))\}.$$

Adding $N(r, f) + \sum_{\nu=1}^{q} N(r, a_\nu, f)$ to both sides of (2) and using (1) we obtain

$$(q-1)T(r, f) \le \overline{N}(r, f) + \sum_{\nu=1}^{q} \overline{N}(r, a_\nu, f) + S_1(r) \tag{3}$$

where $S_1(r) = O(\log T(r, f))$.

$$\therefore \sum_{\nu=1}^{q} \overline{N}(r, a_\gamma, f) \ge (q-1)T(r, f) - \overline{N}(r, f) - S_1(r) \tag{4}$$

where $\overline{n}, \overline{N}$ correspond to distinct roots.

Further, because f_n has an essential singularity at ∞, we have {[3], p. 90}, $\frac{\log r}{T(r, f_n)} \to 0$ as $r \to \infty$.

3. Lemma and Theorem

To prove our theorem, we need the following lemma.

Lemma. If n is any positive integer and f and ϕ are functions in class II, then for any $r_0 > 0$ and M_1, a positive constant

$$\frac{T(r, f_{n+p})}{T(r, f_n)} > M_1 \quad \text{or} \quad \frac{T(r, \phi_{n+p})}{T(r, f_n)} > M_1$$

according as p is even or odd, for all large r, except a set of r intervals of total finite length.

Proof. Case I. p is even. In this case we consider the equation

$$f_{n+p}(z) = a, \quad \text{where} \quad a \neq 0, \infty$$

$$\text{i.e.,} \quad f_p(f_n(z)) = a.$$

This is equivalent to

$$f_p(w) = a \quad \text{at } w_1, w_2, \ldots\ldots$$

$$\text{and} \quad f_n(z) = w_i \;(i = 1, 2, \ldots\ldots).$$

We observe that because f_p is transcendental, $f_p(w) = a$ has infinitely many roots for every complex number a with two exceptions $a = 0, \infty$.

From (1)

$$O(\log r) + T(r, f_{n+p}) = m(r, a, f_{n+p}) + N(r, a, f_{n+p})$$

$$\text{i.e.,} \quad T(r, f_{n+p}) = m(r, a, f_{n+p}) + N(r, a, f_{n+p}) + O(\log r)$$

$$\geq N(r, a, f_{n+p}) + O(\log r)$$

$$\geq \overline{N}(r, a, f_{n+p}) + O(\log r)$$

$$\geq \sum_{i=1}^{M} \overline{N}(r, w_i, f_n)$$

for a fixed M, say $> M_1 + 3$.

From (4) taking $a_\nu = w_i$, $f = f_n$ and $q = M$, we obtain

$$\sum_{i=1}^{M} \overline{N}(r, w_i, f_n) \geq (M-1)T(r, f_n) - \overline{N}(r, f_n) - S_1(r) \tag{5}$$

where $S_1(r) = O(\log T(r, f_n))$ and so for all large r

$$S_1(r) \leq T(r, f_n). \tag{6}$$

In view of (6) and using $\overline{N}(r, f_n) \leq T(r, f_n)$ we have from (5)

$$\sum_{i=1}^{M} \overline{N}(r, w_i, f_n) \geq (M-3)T(r, f_n).$$

$$\therefore \quad T(r, f_{n+p}) \geq (M-3)T(r, f_n)$$

outside a set of r intervals of total finite length.

Case II. p is odd. In this case we consider the equation

$$\phi_{n+p}(z) = a, \ a \neq 0, \infty$$

$$\text{i.e.,} \quad \phi_p(f_n(z)) = a.$$

This is equivalent to

$$\phi_p(w') = a \quad \text{at } w'_1, w'_2, \ldots\ldots\ldots\ldots$$

and $\quad f_n(z) = w'_i \qquad (i = 1, 2, \ldots\ldots\ldots\ldots).$

From (1) $\quad O(\log r) + T(r, \phi_{n+p}) = m(r, a, \phi_{n+p}) + N(r, a, \phi_{n+p})$

i.e., $$T(r, \phi_{n+p}) = m(r, a, \phi_{n+p}) + N(r, a, \phi_{n+p}) + O(\log r)$$

$$\geq N(r, a, \phi_{n+p}) + O(\log r)$$

$$> \overline{N}(r, a, \phi_{n+p}) + O(\log r)$$

$$\geq \sum_{i=1}^{M} \overline{N}(r, w_i', f_n)$$

for a fixed M, say $> M_1 + 3$.

Now we have (as in (5))

$$\sum_{i=1}^{M} \overline{N}(r, w_i, f_n) \geq (M-1)T(r, f_n) - \overline{N}(r, f_n) - T(r, f_n)$$

$$T(r, \phi_{n+p}) > (M-3)T(r, f_n)$$

outside a set of r intervals of total finite length and the lemma is proved.

Theorem. If f(z) and ϕ(z) belong to class II, then f(z) has an infinity of relative fix points of exact order n for every positive integer n, provided $\frac{T(r, \phi_n)}{T(r, f_n)}$ is bounded.

Proof. We may assume that $n > 1$, because if $n = 1$, the theorem follows from Theorem B. For a positive integer n (> 1), we consider the function

$$g(z) = \frac{f_n(z)}{z}, \qquad r_0 < |z| < \infty$$

then $T(r, g) = T(r, f_n) + O(\log r)$. (7)

Assume that f(z) has only a finite number of relative fix points of exact order n. Using (3) and then putting $q = 2$, $a_1 = 0$, $a_2 = 1$, we obtain for g ,

$$T(r, g) \leq \overline{N}(r, 0, g) + \overline{N}(r, \infty, g) + \overline{N}(r, 1, g) + S_1(r, g)$$

where $S_1(r, g) = O(\log T(r, g))$ outside a set of r intervals of finite total length {cf. [4], p. 47}.

Now we calculate $\overline{N}(r,0,g)$ and $\overline{N}(r,\infty,g)$. We have $\overline{N}(r,0,g) = \int_{r_0}^{r} \frac{\overline{n}(t,0,g)}{t} dt$,

where $\overline{n}(t,0,g)$ is the number of distinct roots of $g(z) = 0$ in $r_0 < |z| \le t$ counted singly. The distinct roots of $g(z) = 0$ in $r_0 < |z| \le t$ are the roots of $f_n(z) = 0$ in $r_0 < |z| \le t$. By the definition of functions in class II, $f_n(z)$ has a singularity at $z = 0$ and essential singularity at $z = \infty$ and $f_n(z)$ omits the values 0 and ∞ except possibly at 0. So $\overline{n}(t,0) = 0$. Consequently $\overline{N}(r,0,g) = 0$. Arguing similarly we can say that $\overline{N}(r,\infty,g) = 0$. So, $T(r, g) \le \overline{N}(r,1,g) + S_1(r,g)$. We now calculate $\overline{N}(r,1,g)$. If $g(z) = 1$, then $f_n(z) = z$.

So,

$$\overline{N}(r,1,g) \le \sum_{j=1}^{n-1} \overline{N}(r,0,f_j - z) + O(\log r).$$

The term $O(\text{iog } r)$ arises due to the assumption that f(z) has only a finite number of relative fix points of exact order n.

$$\begin{aligned}
\therefore\ T(r,g) &\le \sum_{j=1}^{n-1} \overline{N}(r,0,f_j - z) + S_1(r,g) + O(\log r) \\
&\le \sum_{j=1}^{n-1} [T(r,f_j - z) + O(\log r)] + S_1(r,g) + O(\log r) \\
&= \sum_{j=1}^{n-1} T(r,f_j - z) + S_1(r,g) + O(\log r) \\
&= \sum_{j=1}^{n-1} T(r,f_j) + O(\log T(r,g)) + O(\log r)
\end{aligned}$$

$$= T(r, f_n)\left[\sum_{j=1}^{n-1}\frac{T(r, f_j)}{T(r, f_n)} + \frac{O(\log T(r, g))}{T(r, f_n)} + \frac{O(\log r)}{T(r, f_n)}\right]$$

$$= T(r, f_n)\left[\frac{T(r, f_{i1})}{T(r, f_n)} + \frac{T(r, f_{i2})}{T(r, f_n)} + \dots\dots\dots\dots + \frac{T(r, f_{ip})}{T(r, f_n)} + \left\{\frac{T(r, f_{j1})}{T(r, \phi_n)} + \dots\dots + \frac{T(r, f_{jq})}{T(r, \phi_n)}\right\}\frac{T(r, \phi_n)}{T(r, f_n)} + \frac{O(\log\{T(r, f_n) + O(\log r)\})}{T(r, f_n)} + \frac{O(\log r)}{T(r, f_n)}\right], \quad \text{by (7)}$$

where $i_1, i_2, \dots\dots, i_p, j_1, j_2, \dots\dots, j_q$ are (n-1) distinct index together exhausting the set $\{1, 2, \dots\dots, n-1\}$ such that $(n-i_p)$'s are even and $(n-j_q)$'s are odd,

$$= T(r, f_n)\left[\frac{T(r, f_{i1})}{T(r, f_n)} + \dots\dots\dots\dots + \frac{T(r, f_{ip})}{T(r, f_n)} + \left\{\frac{T(r, f_{j1})}{T(r, \phi_n)} + \dots\dots + \frac{T(r, f_{jq})}{T(r, \phi_n)}\right\}\frac{T(r, \phi_n)}{T(r, f_n)} + \frac{O\left(\log\left\{T(r, f_n)\left(1 + \frac{O(\log r)}{T(r, f_n)}\right)\right\}\right)}{T(r, f_n)} + \frac{O(\log r)}{T(r, f_n)}\right]$$

$< T(r, f_n)\left[\frac{n-1}{4n} + \frac{n+1}{4n}\right]$ for all large r, by the Lemma and since $\frac{T(r, \phi_n)}{T(r, f_n)}$ is bounded,

$$= \frac{1}{2} T(r, f_n).$$

$\therefore$ $T(r, g) < \frac{1}{2} T(r, f_n)$ for all large r. This contradicts (7). Hence f(z) has infinitely many relative fix points of exact order n (> 1).

This proves the theorem.

Note. If $\phi(z) = f(z)$ then $\frac{T(r, \phi_n)}{T(r, f_n)}$ is necessarily bounded and the theorem coincides with Theorem B.

References

1. Baker, I. N., The Existence of fix points of entire functions. Math. Z. 73 (1960), 280–284.
2. Bhattacharyya, P., An extension of a theorem of Baker, Publicationes Mathematicae, Debrecen, 27 (1980), 273 – 277.
3. Bieberbach, L. (1953) Theorie der Gewöhnlichen Differentialgleichungen, Berlin.
4. Hayman, W. K. (1964), Meromorphic Functions, Oxford University Press.

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