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SOME SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS

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Abstract: In the second part of this paper, we define H_k classes of all harmonic functions in $U = \{z : |z| < 1\}$, $f(z) = z + \sum_2^\infty a_n z^n + \sum_1^\infty a_{-n}\bar{z}^n$ such that $\sum_2^\infty n^k(|a_n| + |a_{-n}|) \leq 1 - |a_{-1}|$, $(k \in Z^+, \ |a_{-1}| < 1)$. We also give some theorems on neighborhoods of these classes, distortion theorems, namely, a covering theorem and a theorem on the uniform convergency of a sequence in H_2. In the third part, we give a characterization of locally univalent harmonic functions and univalent harmonic functions in $U = \{z : |z| < 1\}$ by the Hadamard product. Moreover, we prove that two subclasses of close-to-convex functions class are invariant under Hadamard product.

1. INTRODUCTION.

Clunie and Sheil-Small [2] studied the class S_H of all complex harmonic, sense preserving, univalent functions in $U = \{z : |z| < 1\}$, which are normalized by $f(0) = 0$ and $f_z(0) = 1$. These functions can be written as

$$f(z) = h(z) + \bar{g}(z) = z + \sum_2^\infty a_n z^n + \sum_1^\infty a_{-n}\bar{z}^n \qquad (|a_{-1}| < 1) \qquad (1.1)$$

where h, g are analytic in U. S_H^0 is the subclass of S_H with $a_{-1} = 0$. The class of all functions $f(z)$ in (1.1) which satisfy the inequality $\sum_2^\infty n(|a_n| + |a_{-n}|) \leq 1 - |a_{-1}|$, is denoted by HS and the class of all functions $f(z)$ in (1.1), which satisfy $\sum_2^\infty n^2(|a_n| + |a_{-n}|) \leq 1 - |a_{-1}|$, is denoted by HC. The corresponding subclasses of HS and HC with $a_{-1} = 0$ are HS° and HC°, respectively. The neighborhood of a harmonic function $f(z)$, which has the Taylor series (1.1), consists of harmonic functions $F(z) = z + \sum_2^\infty A_n z^n + \sum_1^\infty A_{-n}\bar{z}^n$ such that $\sum_2^\infty (|a_n - A_n| + |a_{-n} - A_{-n}|) + |a_{-1} - A_{-1}| \leq \delta$ and is denoted $N_\delta(f)$.

We shall denote by H_k, the classes of all functions of the form (1.1) that satisfy

$$\sum_2^\infty n^k(|a_n| + |a_{-n}|) \leq 1 - |a_{-1}| \qquad (k \in \mathbf{Z}^+).$$

Proof: Let $f(z) \in H_k$ and $F(z) = z + \sum_2^\infty A_n z^n + \sum_1^\infty A_{-n} \overline{z}^n \in N_\delta(f)$ for $\delta \le \frac{k-1}{k}(1-|a_{-1}|)$. Applying the definition of H_1 to $F(z)$ and using the definition of neighborhood, we obtain

$$|A_{-1}| + \sum_2^\infty n(|A_n| + |A_{-n}|) \le |A_{-1} - a_{-1}|$$

$$+ \sum_2^\infty n(|A_n - a_n| + |A_{-n} - a_{-n}|) + |a_{-1}| + \sum_2^\infty n(|a_n| + |a_{-n}|)$$

$$\le \delta + |a_{-1}| + \sum_2^\infty n(|a_n| + |a_{-n}|) \le \delta + |a_{-1}| + \frac{1}{k}\sum^\infty n^k(|a_n| + |a_{-n}|)$$

$$\le \frac{k-1}{k}(1 - |a_{-1}|) + |a_{-1}| + \frac{(1 - |a_{-1}|)}{k} = 1.$$

Theorem 2.4. If $f(z) \in H_k^0$, then $|f(z)|$ is an increasing function with respect to $|z| = r$.

Proof: In order to show that $|f(z)|$ is an increasing function with respect to $|z| = r$, it suffices to prove that $\log|f(z)|$ is an increasing with respect to $|z| = r$ or $\frac{\partial}{\partial r}\log|f(re^{i\theta})| = Re\frac{\partial}{\partial r}\log f(re^{i\theta}) > 0$.

$$\frac{\partial}{\partial r}\log|f(re^{i\theta})| = Re\frac{\partial}{\partial r}\log(re^{i\theta} + \sum_2^\infty (a_n r^n e^{in\theta} + a_{-n} r^n e^{-in\theta}))$$

$$= \frac{1}{r}Re[1 + \frac{\sum_2^\infty (n-1)(a_n r^{n-1}e^{in\theta} + a_{-n}r^{n-1}e^{-in\theta})}{e^{i\theta} + \sum_2^\infty (a_n r^{n-1}e^{in\theta} + a_{-n}r^{n-1}e^{-in\theta})}]$$

$$= \frac{1}{rA}Re[|e^{i\theta} + \sum_2^\infty r^{n-1}(a_n e^{in\theta} + a_{-n}e^{-in\theta})|^2 +$$

$$+ \sum_2^\infty (n-1)r^{n-1}(a_n e^{in\theta} + a_{-n}e^{-in\theta})\overline{(e^{i\theta} + \sum_2^\infty r^{n-1}(a_n e^{in\theta} + a_{-n}e^{-in\theta})})]$$

$$= \frac{1}{rA}\{|e^{i\theta} + \sum_2^\infty r^{n-1}(a_n e^{in\theta} + a_{-n}e^{-in\theta}) + \frac{1}{2}\sum_2^\infty (n-1)r^{n-1}(a_n e^{in\theta} + a_{-n}e^{-in\theta})|^2$$

$$-\frac{1}{4}|\sum_2^\infty (n-1)r^{n-1}(a_n e^{in\theta} + a_{-n}e^{-in\theta})|^2\} \qquad (2.3)$$

where

$$A = |e^{i\theta} + \sum_2^\infty (a_n r^{n-1} e^{in\theta} + a_{-n} r^{n-1} e^{-in\theta})|^2 > 0.$$

Thus,

$$|e^{i\theta} + \frac{1}{2}\sum_2^\infty (n+1) r^{n-1}(a_n e^{in\theta} + a_{-n} e^{-in\theta})|$$

$$-\frac{1}{2}|\sum_2^\infty (n-1) r^{n-1}(a_n e^{in\theta} + a_{-n} e^{-in\theta})|$$

$$\geq 1 - \frac{1}{2}\sum_2^\infty (n+1) r^{n-1}(|a_n| + |a_{-n}|) - \frac{1}{2}\sum_2^\infty (n-1) r^{n-1}(|a_n| + |a_{-n}|)$$

$$= 1 - \sum_2^\infty n r^{n-1}(|a_n| + |a_{-n}|)$$

$$\geq 1 - \sum_2^\infty n(|a_n| + |a_{-n}|) > 1 - \sum_2^\infty n^k(|a_n| + |a_{-n}|) \geq 0.$$

This proves that $\frac{\partial}{\partial r}\log|f(re^{i\theta})| > 0$.

Theorem 2.5. If the sequence $\{f_n(z)\} = \{h_n(z) + \overline{g}_n(z)\} \subset H_2$ converges to harmonic function $f(z) = h(z) + \overline{g}(z)$ on uniformly convergence topology, then the sequences $\{h_n(z)\}$ and $\{g_n(z)\}$ converge respectively to $h(z)$ and $g(z)$ on the same topology.

Proof: Since $f_n \in H_2$ for each $n \in \mathbf{N}$, the sequence $\{h_n(z)\}$ is uniformly bounded. Therefore we may select an uniformly convergent subsequence $\{h_{n_k}(z)\}$. Consider the subsequence $\{g_{n_k}(z)\}$. Again, we may select the uniformly convergent subsequence $\{g_{n_{k_l}}(z)\}$. Now, we suppose that $\{h_{n_k}(z)\}$ converges uniformly on compact subsets to $s(z)$ and $\{g_{n_{k_l}}(z)\}$ converges uniformly on compact subsets to $t(z)$. Because of uniformly convergence of $\{h_{n_k}(z)\}$, each subsequences $\{h_{n_{k_l}}(z)\} \subset \{h_{n_k}(z)\}$ converges uniformly to $s(z)$, too. Hence $\{f_{n_{k_l}}(z)\} = \{h_{n_{k_l}}(z)\} + \{\overline{g}_{n_{k_l}}(z)\}$ converges uniformly on compact subsets to $r(z) = s(z) + \overline{t}(z)$. Since $\{f_n(z)\}$ converges unifomly on compact subsets to $f(z) = h(z) + \overline{g}(z)$, $f(z) = r(z)$. Thus, we get $a_n = b_n$, $a_{-n} = b_{-n}$ and $t(z) = g(z)$, $s(z) = h(z)$ by the Taylor series of $f(z)$ and $r(z)$

$$f(z) = h(z) + \overline{g}(z) = z + \sum_2^\infty a_n z^n + \sum_1^\infty a_{-n} \overline{z}^n$$

and

$$r(z) = s(z) + \bar{t}(z) = z + \sum_{2}^{\infty} b_n z^n + \sum_{1}^{\infty} b_{-n} \bar{z}^n$$

This result is satisfied all uniformly convergent subsequences of $\{h_n(z)\}$ and $\{g_n(z)\}$. Therefore, $\{h_n(z)\}$ converges uniformly on compact subsets to $h(z)$ and $\{g_n(z)\}$ converges uniformly on compact subsets to $g(z)$.

3. HADAMARD PRODUCT ON HARMONIC UNIVALENT FUNCTIONS

Theorem 3.1. A harmonic function

$$f(z) = z + \sum_{2}^{\infty} a_n z^n + \sum_{1}^{\infty} a_{-n} \bar{z}^n \qquad (|a_{-1}| < 1, \quad |z| < 1) \tag{3.1}$$

belongs to HLu iff

$$(f * k)(z) \neq 0 \quad \text{for} \quad (0 < |z| < 1) \tag{3.2}$$

$k(z)$ is defined as

$$k(z) = \frac{z}{(1-z)^2} + \varepsilon \overline{\frac{z}{(1-z)^2}}, \quad (|\varepsilon| \leq 1, \quad |z| < 1). \tag{3.3}$$

Proof : Let $f \in HLu$. By the definition of HLu, $|f_{\bar{z}}| < |f_z|$ for all $z \in U$ and $|f_z| \neq 0$. Therefore, the inequalities

$$|(f * k)(z)| = |zf_z + \varepsilon \bar{z} f_{\bar{z}}| = |z||f_z||1 + \varepsilon \frac{\bar{z} f_{\bar{z}}}{z f_z}|$$

$$\geq |z||f_z|[1 - |\varepsilon| \frac{|f_{\bar{z}}|}{|f_z|}]$$

$$> |z||f_z|(1 - |\varepsilon|) \geq 0$$

are satisfied. Thus, we obtain for $(0 < |z| < 1)$ $(f * k)(z) \neq 0$. For the converse, let $(f * k)(z) \neq 0$ for all z $(0 < |z| < 1)$. Therefore, the inequalities

$$zf_z + \varepsilon \bar{z} f_{\bar{z}} \neq 0 \ ,$$

$$zf_z \neq -\varepsilon \bar{z} f_{\bar{z}} \tag{3.4}$$

are satisfied. We shall investigate the last inequality in two different cases:

i) For every z $(0 < |z| < 1)$ which satisfy $f_{\bar{z}}(z) = 0$, (4) becomes $f_z(z) \neq 0$. Thus, the inequality $|f_{\bar{z}}(z)| < |f_z(z)|$ is obtained and this means that $f(z)$ is locally univalent at these points.

ii) Let $f_{\bar{z}}(z) \neq 0$ for some z $(0 < |z| < 1)$. Since (4) is satisfied for each ε $(|\varepsilon| \leq 1)$ and for these points,

$$|f_z(z)| > |f_{\bar{z}}(z)|$$

is obtained. Thus $f(z)$ is locally univalent at these points, too. Moreover, $|f_{\bar{z}}(0)| = |a_{-1}| < |f_z(0)| = 1$ is right by hypothesis. Thus, $f \in HLu$.

Theorem 3.2. Let $f = h + \overline{g} \in HLu$. If

$$(f * k_\varepsilon)(z) \neq 0 \qquad 0 < |z| < 1 \tag{3.5}$$

where $k_\varepsilon = k_0 + \varepsilon \overline{k}_0$ $(|\varepsilon| \leq 1)$ and

$$k_0(z) = \frac{z}{(1-xz)(1-yz)} \quad |x| \leq 1,\ |y| \leq 1,\ x \neq y$$

then $f \in S_H$.

Proof : For $0 < |z| < 1$

$$(f * k_\varepsilon)(z) = \frac{1}{x-y}\{[h(xz) - h(yz)] + \varepsilon[\overline{g}(xz) - \overline{g}(yz)]\} \neq 0. \tag{3.6}$$

We consider two cases:

i) For all x and y which satisfy the equation $\overline{g}(xz) - \overline{g}(yz) = 0$ and the conditions $|x| \leq 1, |y| \leq 1$, from (3.6),

$$f(xz) - f(yz) = h(xz) - h(yz) \neq 0.$$

ii) For all x and y which satisfy the relation $\overline{g}(xz) - \overline{g}(yz) \neq 0$ and the conditions $|x| \leq 1, |y| \leq 1$, (3.6) becomes

$$\frac{h(xz) - h(yz)}{\overline{g}(xz) - \overline{g}(yz)} \neq -\varepsilon \quad \text{for each} \quad \varepsilon\ (|\varepsilon| \leq 1)$$

after that, the inequality

$$\frac{|h(xz) - h(yz)|}{|g(xz) - g(yz)|} > 1$$

is satisfied. Therefore,

$$|f(xz) - f(yz)| \geq |h(xz) - h(yz)| - |g(xz) - g(yz)| > 0$$

that is $f(xz) \neq f(yz)$ for these points. Thus, $f(z)$ is a univalent harmonic function in U.

Theorem 3.3. $f(z) = h(z) + \overline{g}(z) \in C_H$ has properties $|g'(0)| < |h'(0)|$ and $h + \varepsilon g \in C$ for all ε $(|\varepsilon| = 1)$. For an analytic convex function $\varphi(z)$, the function

$$(\varphi + \alpha\overline{\varphi})(z) * f(z) \qquad |\alpha| = 1 \tag{3.7}$$

is a close-to-convex function which satisfies properties of $f(z)$.

Proof : Let $F(z) = H(z) + \overline{G}(z) = (\varphi + \alpha\overline{\varphi})(z) * f(z)(\varphi * h)(z) + \overline{\overline{\alpha}(\varphi * g)}(z)$. We apply the conditions of Theorem C to $F(z)$:

i) $f(z) = z + \sum_2^\infty a_n z^n + \sum_1^\infty a_{-n} z^n$ and $\varphi(z) = z + \sum_2^\infty A_n z^n$. Then,

$$|G'(0)| = |\alpha a_{-1}| < |\alpha| = |H'(0)|$$

ii) $H(z) + \varepsilon G(z) = \varphi * (h + \varepsilon\overline{\alpha} g)(z)$ is a close-to-convex function because of hypothesis of theorem and Theorem B. As result, $F \in C_H$ by the Theorem C.

Theorem 3.4. Let $f(z) = h(z) + \overline{g}(z) \in C_H$ such that $h + \varepsilon g \in K$ for some ε, $(|\varepsilon| \leq 1)$ and $\varphi \in K$. Then the function

$$(2Re\varphi * f)(z) \tag{3.8}$$

is a close-to-convex harmonic function and has property of $f(z)$.

Proof : Let $F(z) = H(z) + \overline{G}(z) = (2Re\varphi * f)(z) = (\varphi * h)(z) + \overline{(\varphi * g)}(z)$. Applying the condition of Theorem D, we obtain that the function $(H + \varepsilon G)(z) = (\varphi * (h + \varepsilon g))(z)$ is a convex function by the hypothesis of theorem and Polya-Schoenberg Conjecture. Thus, $F \in C_H$.

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