

## ON FOUR-DIMENSIONAL SEMI-DECOMPOSABLE RIEMANNIAN SPACES

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**Abstract.** The object of the present paper is to study semi-decomposable semi-symmetric and Weyl-semi-symmetric Riemannian spaces.

1. **Introduction** An  $n$ -dimensional Riemannian space  $V_n$  is said to be semi-symmetric [1] its curvature tensor  $R_{hijk}$  satisfies the condition

$$R_{hijk,lm} - R_{hijk,ml} = 0 \tag{1.1}$$

where comma denotes covariant differentiation in  $V_n$ . Some authors have called such a space an  $s$ -manifold [2]. Further a Riemannian space  $V_n$  is said to be Weyl-semi-symmetric [1] if its conformal curvature tensor  $C_{hijk}$  satisfies

$$C_{hijk,lm} - C_{hijk,ml} = 0 \tag{1.2}$$

where

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij} R_{hk} - g_{ik} R_{hj} + g_{hk} R_{ij} - g_{hj} R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{hk} g_{ij} - g_{hj} g_{ik}), \tag{1.3}$$

$R_{ij}$  is the Ricci tensor and  $R$  denotes the scalar curvature. It follows easily from (1.1) and (1.2) that every semi-symmetric Riemannian space is necessarily Wyle-semi-symmetric, but the converse is not, in general, true. However it is known [3] that if  $n \geq 5$  then (1.1) and (1.2) are equivalent. But when  $n = 4$  this is not true. A suitable example was given in [4] (Lemma 1.1). The existence of a semi-symmetric and Weyl-semi-symmetric Riemannian spaces have been proved in [5].

An  $n$ -dimensional ( $n > 2$ ) Riemannian space  $V_n$  is said to be semi-decomposable [6] if in some coordinates its metric is given by

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + \sigma \bar{g}_{\alpha\beta}^* dx^\alpha dx^\beta \quad (1.4)$$

where  $i, j, k, \dots = 1, 2, \dots, n$ ;  $a, b, c, \dots = 1, 2, \dots, q, (q < n)$ ;  $\alpha, \beta, \gamma, \dots = q+1, q+2, \dots, n$ ;  $\bar{g}_{ab}$  and  $\sigma$  are functions of  $x^1, \dots, x^q$  only and  $\bar{g}_{\alpha\beta}^*$  are functions of  $x^{q+1}, \dots, x^n$  only. The two parts of (1.4) are the metrics of a  $V_q$  and a  $V_{n-q}$  which are called the decomposition spaces of  $V_n$ . Throughout this paper each object denoted by a bar is assumed to be formed from  $\bar{g}_{ab}$  and each object denoted by star, from  $\bar{g}_{\alpha\beta}^*$ . A comma, a dot and a semicolon shall denote covariant differentiation in  $V_n, V_q$  and  $V_{n-q}$  respectively. If in particular  $\sigma = 1$ , then the  $V_n$  reduces to a decomposable space [7].

The present paper deals with semi-decomposable semi-symmetric and Weyl-semi-symmetric Riemannian spaces with non-constant function  $\sigma$ . In section 3 it is shown that for a four-dimensional semi-decomposable semi-symmetric space, the part  $V_q$  is semi-symmetric and the part  $V_{n-q}$  is a space of constant curvature. In the last section it is proved that for a four-dimensional semi-decomposable Weyl-semi-symmetric space the part  $V_q$  is Weyl-semi-symmetric and the part  $V_{n-q}$  is a space of constant curvature.

**2. Preliminaries** Let  $\{b^{\bar{a}}_{\bar{c}}\}, \bar{R}_{abcd}, \bar{R}_{ad}, \bar{R}$  denote the Christoffel symbol, curvature tensor, Ricci tensor and scalar curvature respectively of the part  $V_q$  and  $\{\beta^{\bar{a}}_{\bar{c}}\}, \bar{R}^*_{\alpha\beta\gamma\delta}, \bar{R}^*_{\alpha\delta}, \bar{R}^*$  denote respectively the Christoffel symbol, curvature tensor, Ricci tensor and scalar curvature of the part  $V_{n-q}$  of a semi decomposable space with non-constant function  $\sigma$ . Then ([8], p.16, 17)

$$\left. \begin{aligned} g_{ab} &= \bar{g}_{ab}, g_{\alpha\beta} = \sigma \bar{g}_{\alpha\beta}^*, g^{ab} = \bar{g}^{ab}, g^{\alpha\beta} = \frac{1}{\sigma} \bar{g}^{\alpha\beta} \\ g_{a\alpha} &= 0, g^{a\alpha} = 0 \end{aligned} \right\} \quad (2.1)$$

and the only Christoffel symbols which are not identically zero are as follows :

$$\left. \begin{aligned} \{a^c b\} &= \{a^c \bar{b}\}, \{\beta^\alpha \gamma\} = \{\beta^\alpha \bar{\gamma}\} \\ \{a^\alpha \beta\} &= -\frac{1}{2} g^{ab} \sigma_{,b} \bar{g}_{\alpha\beta}, \{a^\alpha \bar{\beta}\} = \frac{1}{2} \sigma_{,a} \delta^\alpha_\beta \end{aligned} \right\} \quad (2.2)$$

Further

$$R_{\alpha\beta\gamma\delta} = 0 \quad (2.3)$$

$$R_{a\alpha} = 0 \quad (2.4)$$

$$R_{abcd} = \bar{R}_{abcd} \quad (2.5)$$

$$R_{abcd,e} = \bar{R}_{abcd,e} \quad (2.6)$$

$$R_{abcd,ef} = \bar{R}_{abcd,ef} \quad (2.7)$$

$$R_{ab} = \bar{R}_{ab} + \frac{n-q}{2\sigma} T_{ab} \quad (2.8)$$

$$\begin{aligned} R_{ab,cd} = & \bar{R}_{ab,cd} + \frac{n-q}{2\sigma} T_{ab,cd} - \frac{n-q}{2\sigma^2} \sigma_{,cd} T_{ab} \\ & - \frac{n-q}{2\sigma^2} \sigma_{,d} T_{ab,c} - \frac{n-q}{2\sigma^2} \sigma_{,c} T_{ab,d} + \frac{n-q}{\sigma^3} \sigma_{,c} \sigma_{,d} T_{ab} \end{aligned} \quad (2.9)$$

$$\text{where } T_{ab} = \sigma_{,ab} - \frac{1}{2\sigma} \sigma_{,a} \sigma_{,b} \quad (2.10)$$

### 3. Semi-decomposable semi-symmetric space with non-constant function

$\sigma$ .

Let  $V_n$  be a semi-decomposable semi-symmetric space.

From (1.1) and (2.7) we obtain

$$\bar{R}_{abcd,ef} - \bar{R}_{abcd,fe} = 0$$

that is,  $V_q$  is a semi-symmetric space. .... (A)

Now  $R_{\alpha\beta\gamma\delta,c\alpha}$  can be expressed in the form ([8], p. 19)

$$R_{\alpha\beta\gamma\delta,c\alpha} = \frac{3}{4} - \frac{\sigma_{,a} \sigma_{,c}}{\sigma} \bar{R}_{\alpha\beta\gamma\delta} + A_{ac} (\bar{g}_{\beta\gamma}^* \bar{g}_{\alpha\delta}^* - \bar{g}_{\alpha\gamma}^* \bar{g}_{\beta\delta}^*) \quad (3.1)$$

Again, it can be shown that

$$R_{a\beta\gamma\delta,ae} = -\frac{1}{2\sigma} \sigma_{,ae} \overset{*}{R}_{\alpha\beta\gamma\delta} - B_{ae} \left( \overset{*}{g}_{\beta\gamma} \overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\alpha\gamma} \overset{*}{g}_{\beta\delta} \right) \quad (3.2)$$

where  $\sigma_{,b} = \sigma_{,a} g^{\bar{a}b}$ ,  $\Delta_1 \sigma = \sigma_{,a} \sigma_{,a}$ ,  $T_a^b = T_{ae} g^{\bar{e}b}$

$$A_{ae} = \left( \frac{\sigma_{,a} \sigma_{,e}}{2\sigma^2} \Delta_1 \sigma - \frac{\sigma_{,a} (\Delta_1 \sigma)_{,e}}{8\sigma} - \frac{3}{16\sigma} \sigma_{,e} (\Delta_1 \sigma)_{,a} + \frac{1}{4} \sigma_{,a} T_{ae} \right)$$

$$B_{ae} = \frac{1}{4} \left( \frac{1}{2\sigma} (\Delta_1 \sigma) \sigma_{,ae} - \frac{1}{2\sigma^2} \Delta_1 \sigma \sigma_{,a} \sigma_{,e} + \frac{1}{4\sigma} (\Delta_1 \sigma)_{,e} \sigma_{,a} + \sigma_{,b} T_{ab,e} + T_{,a}^b \sigma_{,be} \right)$$

From (1.1) we get

$$R_{a\beta\gamma\delta,ca} - R_{a\beta\gamma\delta,ac} = 0. \quad (3.3)$$

In virtue of (3.1) and (3.2) we obtain from (3.3)

$$\left( \frac{3}{4} \frac{\sigma_{,a} \sigma_{,e}}{\sigma} + \frac{1}{2} \sigma_{,ae} \right) \overset{*}{R}_{\alpha\beta\gamma\delta} + (A_{ae} + B_{ae}) \left( \overset{*}{g}_{\beta\gamma} \overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\alpha\gamma} \overset{*}{g}_{\beta\delta} \right) = 0 \quad (3.4)$$

Transvecting (3.4) with  $\overset{*}{g}^{\alpha\delta}$  we get

$$\left( \frac{3}{4} \frac{\sigma_{,a} \sigma_{,e}}{\sigma} + \frac{1}{2} \sigma_{,ae} \right) \overset{*}{R}_{\beta\gamma} + (n-q-1)(A_{ae} + B_{ae}) \overset{*}{g}_{\beta\gamma} = 0$$

Again transvecting the above expression with  $\overset{*}{g}^{\beta\gamma}$  we obtain

$$\left( \frac{3}{4} \frac{\sigma_{,a} \sigma_{,e}}{\sigma} + \frac{1}{2} \sigma_{,ae} \right) \overset{*}{R} + (n-q)(n-q-1)(A_{ae} + B_{ae}) = 0. \quad (3.5)$$

From (3.4) and (3.5) we obtain

$$f_{ae} \left[ \overset{*}{R}_{\alpha\beta\gamma\delta} - \frac{\overset{*}{R}}{(n-q)(n-q-1)} \left( \overset{*}{g}_{\beta\gamma} \overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\alpha\gamma} \overset{*}{g}_{\beta\delta} \right) \right] \quad (3.6)$$

$$\text{where } f_{ae} = \frac{3}{4} \frac{\sigma_{,a} \sigma_{,e}}{\sigma} + \frac{1}{2} \sigma_{,ae} \quad (3.7)$$

Hence if at least one of the component of  $f_{ac}$  is non-zero, then  $V_{n-q}$  is a space of constant curvature. .... (B)

In view of (A) and (B) we obtain

**Theorem 1.** For a semi-decomposable semi-symmetric space with non-constant function  $\sigma$ , the part  $V_q$  is semi-symmetric and when atleast one component of  $f_{ac}$  defined by (3.7) is non-zero, the part  $V_{n-q}$  is a space of constant curvature.

**4. Semi-decomposable Weyl-semi-symmetric space with non-constant function  $\sigma$ .**

Let  $V_n$  be semi-decomposable Weyl-semi-symmetric space.

By virtue of (1.2)  $C_{abce,ef} - C_{abce,fe} = 0$ . (4.1)

Since  $\sigma_{,ef} = \sigma_{,fe}$  (2.9) reduces to

$$R_{ab,ef} - R_{ab,fe} = \bar{R}_{ab,ef} - \bar{R}_{ab,fe} + \frac{n-q}{2\sigma} (T_{ab,ef} - T_{ab,fe}). \quad (4.2)$$

Again since  $R_{,ef} = R_{,fe}$ , using (1.3), (2.7) and (4.2) in (4.1) we obtain

$$\begin{aligned} & \bar{R}_{abcd,ef} - \bar{R}_{abcd,fe} - \frac{1}{n-2} [ \bar{g}_{bc} \{ ( \bar{R}_{ad,ef} - \bar{R}_{ad,fe} ) + \frac{(n-q)}{2\sigma} ( T_{ad,ef} - T_{ad,fe} ) \} \\ & - \bar{g}_{bd} \{ ( \bar{R}_{ac,ef} - \bar{R}_{ac,fe} ) + \frac{n-q}{2\sigma} ( T_{ac,ef} - T_{ac,fe} ) \} + \bar{g}_{ad} \{ ( \bar{R}_{bc,ef} - \bar{R}_{bc,fe} ) \\ & + \frac{n-q}{2\sigma} ( T_{bc,ef} - T_{bc,fe} ) \} - \bar{g}_{ac} \{ ( \bar{R}_{bd,ef} - \bar{R}_{bd,fe} ) + \frac{n-q}{2\sigma} ( T_{bd,ef} - T_{bd,fe} ) \} ] = 0. \quad (4.3) \end{aligned}$$

Transvecting (4.3) with  $\bar{g}_{bc}$  we get,

$$\bar{R}_{ab,ef} - \bar{R}_{ab,fe} - \frac{1}{n-2} [ (q-2)( \bar{R}_{ad,ef} - \bar{R}_{ad,fe} ) + \frac{(q-2)(n-q)}{2\sigma} ( T_{ad,ef} - T_{ad,fe} ) ] = 0.$$

Hence 
$$\bar{R}_{d,ef} - \bar{R}_{ab,fe} = \frac{q-2}{2\sigma} (T_{ab,ef} - T_{ab,fe}). \tag{4.4}$$

Now  $C_{abcd,ef} - C_{abcd,fe}$

$$\begin{aligned} &= \bar{R}_{abcd,e} - \bar{R}_{abcd,fe} - \frac{1}{q-2} [ \bar{g}_{bc}(\bar{R}_{ad,ef} - \bar{R}_{ad,fe}) - \bar{g}_{bd}(\bar{R}_{ac,ef} - \bar{R}_{ac,fe}) \\ &+ \bar{g}_{ad}(\bar{R}_{bc,ef} - \bar{R}_{bc,fe}) - \bar{g}_{ac}(\bar{R}_{bd,ef} - \bar{R}_{bd,fe})] \\ &- \frac{q-n}{n-2} [ \bar{g}_{bc}(\bar{R}_{ad,ef} - \bar{R}_{ad,fe}) \frac{1}{q-2} - \frac{1}{2\sigma} \bar{g}_{bd}(T_{ad,ef} - T_{ad,fe}) \} \\ &- \bar{g}_{bd} \{ -\frac{1}{q-2} (\bar{R}_{ac,ef} - \bar{R}_{ac,fe}) - \frac{1}{2\sigma} (T_{ac,ef} - T_{ac,fe}) \} \\ &+ \bar{g}_{ad} \{ -\frac{1}{q-2} (\bar{R}_{bc,ef} - \bar{R}_{bc,fe}) - \frac{1}{2\sigma} (T_{bc,ef} - T_{bc,fe}) \} \\ &- \bar{g}_{ac} \{ -\frac{1}{q-2} (\bar{R}_{bd,ef} - \bar{R}_{bd,fe}) - \frac{1}{2\sigma} (T_{bd,ef} - T_{bd,fe}) \} \end{aligned} \tag{4.5}$$

By virtue of (4.5) the equation (4.5) reduces to

$$C_{abcde,ef} - C_{abcd,fe} = \bar{C}_{abcd,ef} - \bar{C}_{abcd,fe}$$

Hence from (4.1) we obtain

$$\bar{C}_{abcd,ef} - \bar{C}_{abcd,fe} = 0$$

That is,  $V_q$  is Weyl-semi-symmetric space ..... (C)

Now from (1.1) we have

$$\begin{aligned} C_{\alpha\beta\gamma\delta, \epsilon\alpha} - C_{\alpha\beta\gamma\delta, \alpha\epsilon} &= R_{\alpha\beta\gamma\delta, \epsilon\alpha} - R_{\alpha\beta\gamma\delta, \alpha\epsilon} - \frac{1}{n-2} [(R_{\alpha\delta, \epsilon\alpha} - R_{\alpha\delta, \alpha\epsilon})g_{\beta\gamma} \\ &- (R_{\alpha\gamma, \epsilon\alpha} - R_{\alpha\gamma, \alpha\epsilon})g_{\beta\delta} + (R_{\beta\gamma, \epsilon\alpha} - R_{\beta\gamma, \alpha\epsilon})g_{\alpha\delta} - (R_{\beta\delta, \epsilon\alpha} - R_{\beta\delta, \alpha\epsilon})g_{\alpha\gamma}] = 0 \end{aligned} \tag{4.6}$$

since  $R_{, \alpha\epsilon} = R_{, \epsilon\alpha} = 0$ .

Now it can be shown that

$$R_{\alpha\delta,\epsilon\alpha} = \frac{3}{4} \sigma^2 \sigma_{,\epsilon} \sigma_{,\alpha} \dot{R}_{\alpha\delta}^* + D_{ae} \dot{g}_{\alpha\delta}^* \quad (4.7)$$

$$R_{\alpha\delta,\alpha\epsilon} = \left( \frac{\sigma_{,\alpha} \sigma_{,\epsilon}}{\sigma^2} - \frac{1}{2\sigma} \sigma_{,\alpha\epsilon} \right) \dot{R}_{\alpha\delta}^* + E_{ae} \dot{g}_{\alpha\delta}^* \quad (4.8)$$

where  $T = T_{ab} g^{ab}$ ,  $P = \frac{n-q-1}{2\sigma} \Delta_1 \sigma$

$$D_{ae} = \frac{3}{8} \frac{\sigma_{,\alpha} \sigma_{,\epsilon}}{\sigma^2} P - \frac{1}{4} \frac{\sigma_{,\alpha} \sigma_{,\epsilon}}{\sigma} + \frac{1}{2} \left\{ \bar{R}_{am,\epsilon} + \frac{n-q}{2\sigma} T_{am,\epsilon} - \frac{\sigma_{,\epsilon}}{\sigma} T_{am} \right\} \sigma^{,m} \\ - \frac{\sigma_{,\epsilon}}{4\sigma} \left( \bar{R}_{am} + \frac{1}{2} \frac{q}{\sigma} T_{am} \right) \sigma^{,m}$$

and  $E_{ae} = \frac{1}{2} \left[ (\sigma_{,b} \bar{R}^b_{\alpha}{}_{\epsilon}) - \frac{1}{\sigma} \sigma_{,\epsilon} \sigma_{,b} \bar{R}^b_{\alpha} + \frac{n-q}{2\sigma} ((\sigma_{,b} T^b_{\alpha})_{,\epsilon} - \frac{\sigma_{,\epsilon}}{\sigma} \sigma_b T^b_{\alpha}) \right. \\ \left. - \frac{P}{2\sigma} \sigma_{,\alpha\epsilon} + \frac{P}{\sigma^2} \sigma_{,\epsilon} - \frac{1}{2} P_{,\alpha} \sigma_{,\epsilon} \right]$

From (4.7) and (4.8), we get

$$R_{\alpha\delta,\epsilon\alpha} - R_{\alpha\delta,\alpha\epsilon} = \frac{1}{2\sigma} (\sigma_{,\alpha\epsilon} - \frac{1}{2\sigma} \sigma_{,\alpha} \sigma_{,\epsilon}) \dot{R}_{\alpha\delta}^* + (D_{ae} - E_{ae}) \dot{g}_{\alpha\delta}^* \quad (4.9)$$

Also, it can be shown that

$$R_{\beta\gamma,\alpha\epsilon} = R_{\beta\gamma,\epsilon\alpha} = -\frac{3}{2\sigma} \sigma_{,\epsilon} \dot{R}_{\alpha\delta\beta\gamma}^* \quad (4.10)$$

Now let us suppose that at least one component of  $T_{ae}$  is zero such that  $\sigma_{,\epsilon}$  and  $\sigma_{,\epsilon}$  are non-zero. By virtue of (4.9), (4.10), (3.1) and (3.2) the equation (4.6) takes the form

$$\frac{\sigma_{,\alpha} \sigma_{,\epsilon}}{\sigma} \dot{R}_{\alpha\beta\gamma\delta}^* + M_{ae} (\dot{g}_{\beta\gamma}^* \dot{g}_{\alpha\delta}^* - \dot{g}_{\beta\delta}^* \dot{g}_{\alpha\gamma}^*) = 0$$

where 
$$M_{ac} = A_{ac} + B_{ac} - \frac{\sigma}{n-2} (D_{ac} - E_{ac})$$

Now proceeding similarly as in section 3 we obtain

$$\check{R}_{\alpha\beta\gamma\delta} = \frac{\check{R}}{(n-q)(n-q-1)} (\check{g}_{\beta\gamma} \check{g}_{\alpha\delta} - \check{g}_{\beta\delta} \check{g}_{\alpha\gamma})$$

That is,  $V_{n-q}$  is the space of constant curvature . (D)

In view of (C) and (D) we get

**Theorem 2.** For a semi-decomposable Weyl-semi-symmetric space with non constant function  $\sigma$  the part  $V_q$  is Weyl-semi-symmetric and the part  $V_{n-q}$  is a space of constant curvature.

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