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An Investigation on P-Adic U Numbers[1]

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ABSTRACT : In this paper, firstly we show that there are infinitely many p-adic numbers γ such that $\gamma \in U_m$ and $P_i(\gamma) \in U_m$ where $k \in \mathrm{N}$, $1 \le i \le k$ and $P_i(x)$ are non–constant polynomials with integer coefficients. Secondly, we prove that the finite linear combination of p–adic algebraic numbers and semi–strong p–adic U–numbers belong to $A \cup U$. Finally, we prove that if γ_1 is a p–adic U–number and γ_2 is a semi-strong p–adic U–number, then both $\gamma_1 + \gamma_2$ and $\gamma_1 . \gamma_2$ numbers belong to $A \cup U$. Moreover, we remark that if γ_2 is taken as a p–adic U–number the last statement fails to be true.

Introduction

Mahler [10] divided the complex numbers into four classes as A, S, T, U. Later, Koksma [7] set up another classification of complex numbers. He divided them into four classes as A*, S*, T*, U*. Wirsing [14] has shown that these two classifications are equivalent.

Let p be a fixed prime number and $| \dots |_p$ denotes the p-adic valuation of the set of rational numbers Q. Furthermore let Q_p denote all the p-adic numbers over Q.

Mahler [11] had a classification of p-adic numbers as follows: Let $P(x)$ be a polynomial with integral coefficients and $H(P)$ be the height of $P(x)$. Suppose that $H, n \in N$ and $\xi \in Q_p$. Mahler lets

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$$w_n(H,\xi) = \min\{|P(\xi)| : \deg P \le n, H(P) \le H, P(\xi) \ne 0\}.$$

It is clear that $0 \le w_n(\xi, H) \le 1$, since, if $P(x) = 1$, then $|P(\alpha)|_p = 1$. Next Mahler lets

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(H,\xi)}{\log H} \quad \text{and} \quad w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

It is clear that $w_n(\xi)$ is nondecreasing as function of *n*. One has, $0 \le w_n(\xi) \le \infty$ and $0 \le w(\xi) \le \infty$. If $w_n(\xi) = \infty$ for some integer *n*, let $\mu(\xi)$ be the smallest such integer; if $w_n(\xi) < \infty$ for every *n*, let $\mu(\xi) = \infty$. Mahler calls the number ξ a

A – number if $w(\xi) = 0$ and $\mu(\xi) = \infty$,
S – number if $0 < w(\xi) < \infty$ and $\mu(\xi) = \infty$,
T – number if $w(\xi) = \infty$ and $\mu(\xi) = \infty$,
U – number if $w(\xi) = \infty$ and $\mu(\xi) < \infty$.

On the other hand, Schlickewei [14] gives a classification of p-adic numbers as follows: Let $\xi \in Q_p$ and

$$w_n^*(H,\xi) = \min\{|\xi - \alpha| : \deg \alpha \le n, H(\alpha) \le H, \xi \ne \alpha\}$$

where *H* and *n* are natural numbers. Let

$$w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(H w_n^*(H,\xi))}{\log H}, \quad \text{and} \quad w^*(\xi) = \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

It is clear that the inequalities $0 \le w^*_n(\xi) \le \infty$ and $0 \le w^*(\xi) \le \infty$ hold. If for any index $w^*_n(\xi) = \infty$, then $\mu^*(\xi)$ is defined as the smallest of them; otherwise, $\mu^*(\xi) = \infty$. So $\mu^*(\xi)$ is uniquely determined and neither $\mu^*(\xi)$ nor $w^*(\xi)$ can be finite. There are the following four possibilities for ξ. The p-adic number ξ is called

A* – number if $w^*(\xi) = 0$ and $\mu^*(\xi) = \infty$
S*– number if $0 < w^*(\xi) < \infty$ and $\mu^*(\xi) = \infty$
T*– number if $w^*(\xi) = \infty$ and $\mu^*(\xi) = \infty$
U*– number if $w^*(\xi) = \infty$ and $\mu^*(\xi) < \infty$.

ξ is called a U*– number of degree m $(m \geq 1)$ if $\mu^*(\xi) = m$. The set of p-adic U*– numbers of degree m is denoted by U_m^*. Thus $U^* = \bigcup_{m=1}^{\infty} U^*_m$ holds. The p-adic set U_1^* is called p-adic Liouville numbers. Long [9] proved that $U_m = U^*_m$. We give some definition and lemmas.

Definition 1. Let $\gamma \in Q_p$ and $m \in N$. The number γ is called p-adic U_m number if for every $w > 0$, there are infinitely many algebraic numbers α of degree m with

$$0 < |\gamma - \alpha|_p < H(\alpha)^{-w}$$

and if there are constants $C, K > 0$ depending only on γ and m such that the relation

$$0 < |\gamma - \beta|_p < C.H(\beta)^{-K}$$

holds for every algebraic number β in Q_p which has degree less than m.

Lemma 1. (Schlickewei) Let α and β are two nonconjugate algebraic numbers of degree t and k, respectively. Then, for $M > \max\{t, k\}$

$$|\alpha - \beta|_p > \frac{c_1}{H(\alpha)^{M-1} H(\beta)^M}$$

where $|\alpha|_p = p^{-h}$, $r = \min\{0, h\}$ and $c_1 = p^{(M-1)r - M(|h|+1)}((2M)!)^{-1}$ (See [13]).

Lemma 2. (J. F. Morrison) Let $\alpha \in Q_p$ and

$$P(x) = a_0 + a_1 x + ... + a_n x^n \in Z[x]$$

such that $P(\alpha) = 0$. Then,

$$|\alpha|_p > H(P)^{-1}. \text{ (See. [12]).}$$

Lemma 3. (O. Ş. İçen) Let $\alpha_1,...,\alpha_k$ $(k \geq 1)$ be algebraic numbers in Q_p with $[Q(\alpha_1,...,\alpha_k):Q] = g$ and let $F(y, x_1,...,x_k)$ be a polynomial with integral coefficients, whose degree in y is at least one. If η is an algebraic number such that $F(\eta,\alpha_1,...,\alpha_k) = 0$, then the degree of $\eta \leq dg$ and

$$H(\eta) \leq 3^{2dg+(l_1+...+l_k)g} H^g H(\alpha_1)^{l_1 g} ... H(\alpha_k)^{l_k g}$$

where $H(\eta)$ is the height of η, $H(\alpha_i)(i = 1,...,k)$ is the height of α_i and, H is the maximum of absolute values of the coefficients of F, $l_i (i = 1,...,k)$ is the degree of F in x_i and d is the degree of F in y. (See. [6]).

Theorem 1. Let $\{\alpha_i\}$ be sequence of algebraic numbers in $\mathrm{Q_p}$ with

(1) $\quad \deg \alpha_i = m_i \leq \ell$ and $\lim_{i\to\infty} H(\alpha_i) = \infty \quad (\ell \in Z^+)$

(2) $\quad |\alpha_{i+1} - \alpha_i| = \dfrac{1}{H(\alpha_i)^{w_i}}$, where $\lim_{i\to\infty} w_i = \infty$

(3) $\quad 0 < |\alpha_{i+1} - \alpha_i| < \dfrac{1}{H(\alpha_{i+1})^{\delta}}$ for $\delta > 0$.

Then, $\lim_{i\to\infty} \alpha_i \in U_m^*$ where $m = \liminf_{i\to\infty} m_i$. (See. [4]).

Definition 2. Let $\gamma \in Q_p$. If there are infinitely many p-adic algebraic numbers $\{\alpha_i\}$ such that

(1) $\deg \alpha_i = m_i \leq \ell$ and $\lim_{i\to\infty} H(\alpha_i) = \infty \quad (\ell \in Z^+)$

(2) $0 < |\alpha_{i+1} - \alpha_i| = \dfrac{1}{H(\alpha_i)^{w_i}}$ where $\lim_{i\to\infty} w_i = \infty$

(3) $0 < |\alpha_{i+1} - \alpha_i| < \dfrac{1}{H(\alpha_{i+1})^{\delta}}$ for some fixed $\delta > 0$.

Then, the number $\lim_{i\to\infty} \alpha_i = \gamma \in Q_p$ is said to be an *irregular semi-strong p-adic U–number.* If $\liminf_{i\to\infty} m_i = \lim_{i\to\infty} m_i$, γ is called a *semi-strong p-adic U–number.* If $\liminf_{i\to\infty} m_i = m$ Theorem 1 proves that $\gamma \in U_m$.

In this paper U_m^s denotes all semi-strong p-adic U_m–numbers and U^s denotes all semi-strong p-adic U–numbers.

Main results of this paper are the following theorems.

Theorem 2. Let $m \in Z^+$ and $P_i(x) \in Z[x]$ where $\deg P_i \geq 1$ $(i = 1,\dots,k)$. Then there are infinitely many $\gamma \in U_m$ such that $P_i(\gamma) \in U_m$ for every $1 \leq i \leq k$.

Proof: Let α be a p-adic algebraic number of degree m and $\alpha^{(1)} = \alpha, \alpha^{(2)}, \dots, \alpha^{(m)}$ denotes the conjugates of α. Consider the equation

$$P_i(\alpha^{(r)} + y) = P_i(\alpha^{(s)} + y) \quad (1 \leq r, s \leq m, r \neq s). \tag{1.1}$$

For fixed r, s, i, (1.1) is equivalent to some polynomial equation

$$c_t y^t + \dots + c_1 y + c_0 = 0$$

where the coefficients c_j are p-adic algebraic numbers. Since $\alpha^{(r)} \neq \alpha^{(s)}$ for $r \neq s$, $c_t \neq 0$ and so (1.1) has only finitely many solutions in $y \in Q_p$. Consider $y = p^n$, then there is a natural number n_0 such that $\deg P_i(\alpha + p^n) = m$ $(i = 1,\ldots,k)$ for $\forall n \geq n_0$.

Let $\{w(i)\}$ be a sequence of positive real numbers with $\lim_{i\to\infty} w_i = \infty$. We define algebraic numbers α_i and integers n_i $(i = 1,2,\ldots)$ as

$$\deg P_i(\alpha + p^{n_1}) = m \quad (i = 1,\ldots,k), \quad \alpha_1 = \alpha + p^{n_1} \tag{1.2}$$

(a) $\deg P_t(\alpha + p^{n_{i+1}}) = m$ $(t = 1,\ldots,k)$

(b) $H(\alpha_i)^{w(i)} < p^{n_{i+1}}$ (1.3)

(c) $n_i^2 < n_{i+1}$ $(i \geq 1)$

$$\alpha_{i+1} = \alpha_i + p^{n_{i+1}} \tag{1.4}$$

From (1.2) and (1.4) we have $\alpha_{i+1} = \alpha + \sum_{j=1}^{i+1} p^{n_j}$. $F(\alpha_{i+1}, \alpha, \sum_{j=1}^{i+1} p^{n_j}) = 0$ holds for the polynomial $F(y, x_1, x_2) = y - x_1 - x_2$. Applying Lemma 3 we find

$$H(\alpha_{i+1}) \leq 3^{4m} H(\alpha)^{2m} H(\sum_{j=1}^{i+1} p^{n_j})^{2m}.$$

Using (1.3)(c) we write

$$H(\sum_{j=1}^{i+1} p^{n_j}) = p^{n_1} + \ldots + p^{n_{i+1}} \leq (i+1) p^{n_{i+1}} \leq p^{2n_{i+1}}$$

Since $\lim_{i\to\infty} p^{n_i} = \infty$, there is a natural number i_1 such that $p^{n+1} \geq 3^{4m} H(\alpha)^{2m}$ for $\forall i \geq i_1$. So, we can write

$$H(\alpha_{i+1}) \le (p^{n_{i+1}})^{2m+1} \quad (\forall \mathrm{i} \ge \mathrm{i}_1). \tag{1.5}$$

A combination of (1.4) and (1.5) gives us

$$\left|\alpha_{i+1} - \alpha_i\right|_p = \left|p^{n_{i+1}}\right|_p = \frac{1}{p^{n_{i+1}}} \le \frac{1}{H(\alpha_{i+1})^{1/(2m+1)}} \quad (\forall \mathrm{i} \ge \mathrm{i}_1).$$

Writing a $\delta = 1/(2m+1)$, we obtain

$$\left|\alpha_{i+1} - \alpha_i\right|_p \le \frac{1}{H(\alpha_{i+1})^{\delta}} \quad (\forall \mathrm{i} \ge \mathrm{i}_1). \tag{1.6}$$

On the other hand, it follows from (1.3)(b) and (1.4) that

$$\left|\alpha_{i+1} - \alpha_i\right|_p \le \frac{1}{H(\alpha_i)^{w(i)}} \quad (\forall \mathrm{i} \ge \mathrm{i}_1). \tag{1.7}$$

Thus, $\{\alpha_i\}$ satisfies the conditions (1), (2) and (3) of Theorem 1 and so we have $\lim\limits_{i\to\infty} \alpha_i = \gamma \in U_m$.

Now we show that $P_t(\gamma) \in U_m$ $(t = 1,\dots,k)$. Put $\beta_i = P_t(\alpha_i)$. Applying Taylor Formula, we have

$$P_t(\alpha_{i+1}) = P_t(\alpha_i) + (\alpha_{i+1} - \alpha_i)\frac{P_t'(\alpha_i)}{1!} + (\alpha_{i+1} - \alpha_i)^2 \frac{P_t''(\alpha_i)}{2!} + \dots$$

It is clear that $P_t^{(j)}(\alpha_i) = 0$ for $\forall j \ge M$ where $M > \max\{\deg P_1(x), \dots, \deg P_k(x)\}$. Thus, taking $|\ |_p$ of both sides we write

$$\left|\beta_{i+1} - \beta_i\right|_p = \left|\alpha_{i+1} - \alpha_i\right|_p \left|P_t'(\alpha_i) + \dots + (\alpha_{i+1} - \alpha_i)^{M-1} \frac{P_t^{(M-1)}(\alpha_i)}{(M-1)!}\right|_p$$

Now, we can determine an upper bound for the value

$$\left|P_t'(\alpha_i)+\ldots+(\alpha_{i+1}-\alpha_i)^{M-1}\frac{P_t^{(M-1)}(\alpha_i)}{(M-1)!}\right|_p .$$

On the other hand, it can be easily proved that there is a natural number i_2 such that, $|\alpha_i|_p=|\alpha_{i+1}|_p$ for $\forall i\geq i_2$. Thus, since $|\alpha_{i+1}-\alpha_i|_p<1$, $\left|P_t^{(i)}(\alpha_i)\right|_p<p^{M|h|}$ and $\left|\frac{1}{j!}\right|_p<p^M$ $(1\leq j<M)$. So, we have

$$\left|P_t'(\alpha_i)+\ldots+(\alpha_{i+1}-\alpha_i)^{M-1}\frac{P_t^{(M-1)}(\alpha_i)}{(M-1)!}\right|_p<p^{M|h|}.$$

Hence, we find that

$$|\beta_{i+1}-\beta_i|_p=|\alpha_{i+1}-\alpha_i|_p c_1 \tag{1.8}$$

where $c_1=p^{M(|h|+1)}$. We consider the polynomial $F(y,x)=y-P_t(x)$. Then, $F(\beta_i,\alpha_i)=0$ holds. Applying Lemma 3 gives

$$H(\beta_i)\leq 3^{2m+M}H(\alpha_i)^{mM}.$$

Since $\lim_{i\to\infty}H(\alpha_i)=\infty$, there is a natural number i_3 such that $i_3\geq i_2$ and

$$H(\alpha_i)>3^{2m+M}\quad(\forall i\geq i_3).$$

Hence, putting $p=mM+1$ we have

$$H(\beta_i)\leq H(\alpha_i)^p\quad(\forall i\geq i_3). \qquad (1.9)_i$$

and also using $(1.9)_i$ and (1.8) in (1.6) we write

$$\left|\beta_{i+1}-\beta_i\right|_p \le \frac{c_1}{H(\alpha_i)^{w(i)}} \le \frac{c_1}{H(\beta_i)^{w(i)/p}}.$$

Since $\lim_{i\to\infty} H(\beta_i)=\infty$ there is a natural number $i_4 \ge i_3$ such that

$$H(\beta_i) > c_1 \ (\forall i \ge i_4).$$

Hence, we have

$$\left|\beta_{i+1}-\beta_i\right|_p \le \frac{1}{H(\beta_i)^{(w(i)-p)/p}} (\forall i \ge i_4). \tag{1.10}$$

using (1.8) and $(1.9)_{i+1}$ in (1.6) we obtain

$$\left|\beta_{i+1}-\beta_i\right|_p \le c_1\left|\alpha_{i+1}-\alpha_i\right|_p \le \frac{c_1}{H(\alpha_{i+1})^{\delta}} \le \frac{c_1}{H(\beta_{i+1})^{\delta/p}} \quad (\forall i \ge i_4).$$

Put $\delta_1 = \delta/2p$. Since $\lim_{i\to\infty} H(\beta_i)=\infty$ there is a natural number $i_5 \ge i_4$ such that $H(\beta_i)^{\delta_1} > c_1 (\forall i \ge i_5)$. Hence, we write

$$\left|\beta_{i+1}-\beta_i\right|_p \le \frac{1}{H(\beta_{i+1})^{\delta_1}} (\forall i \ge i_5). \tag{1.11}$$

$\{\beta_i\}$ satisfies the condition (1), (2) and (3) of Theorem 1 by (1.10) and (1.11). Finally, we have

$$\lim_{i\to\infty} \beta_i = P_t(\lim_{i\to\infty} \alpha_i) = P_t(\gamma) \in U_m \quad (t=1,\dots,k).$$

Example 1. Consider the function $y^m = x^n$ where $n, m \in N$.

If we take as $y=t^n, x=t^m$ and we consider the polynomials

$$P_1(t) = t^n \quad \text{and} \quad P_2(t) = t^m ,$$

By Theorem 2, there are infinitely many numbers $\gamma \in U_m$ such that $P_1(\gamma) \in U_m$ and $P_2(\gamma) \in U_m$. Hence, there are infinitely many numbers $x, y \in U_m$ satisfying the condition $y^m = x^n$.

Theorem 3. Let $\alpha_0, \alpha_1, ..., \alpha_k$ be p–adic algebraic numbers and $\gamma_1, ..., \gamma_k$ be semi-strong p-adic U-numbers. Then, the number $\gamma = \alpha_0 + \alpha_1\gamma_1 + ... + \alpha_k\gamma_k$ belongs to $A \cup U$.

Proof : From Definition 2, there are p-adic algebraic sequences $\{\alpha_j^i\}$ which satisfies the following properties such that $\lim_{j\to\infty} \alpha_j^{(i)} = \gamma_i$ for $i = 1, ..., k$.

$$\deg \alpha_j^{(i)} = m_j^{(i)} \le \ell \text{ and } \lim_{j\to\infty} H(\alpha_j^{(i)}) = \infty \quad (\ell \in Z^+) \qquad (2.1)_i$$

$$\left|\gamma_i - \alpha_j^{(i)}\right|_p = \frac{1}{H(\alpha_j^{(i)})^{w_i(j)}} < \frac{1}{H(\alpha_{j+1}^{(i)})^{\delta_i}} \qquad (2.2)_i$$

where $\lim_{j\to\infty} w_i(j) = \infty$ and some fixed numbers $\delta_i > 0 \quad (i = 1, ..., k)$.

On the other hand, as equivalent to $(2.2)_i$ we can write

$$\left|\alpha_{j+1}^{(i)} - \alpha_j^{(i)}\right|_p = \frac{1}{H(\alpha_j^{(i)})^{w_i(j)}} < \frac{1}{H(\alpha_{j+1}^{(i)})^{\delta_i}} \quad (\forall 1 \le i \le k). \quad (2.3)_i$$

Now, we will show that

$$\lim_{j\to\infty} \frac{\log H(\alpha_{j+1}^{(i)})}{\log H(\alpha_j^{(i)})} = \infty \ (\forall 1 \le i \le k).$$

It is given that $\ell \geq \max\left\{\deg \alpha_{j+1}^{(i)}, \deg \alpha_{j}^{(i)}\right\}$. Also, using $(2.3)_i$ in Lemma 1 we write

$$\frac{c_1}{H(\alpha_{j+1}^{(i)})^{\ell-1} H(\alpha_j^{(i)})^{\ell}} < \left|\alpha_{j+1}^{(i)} - \alpha_j^{(i)}\right|_p = \frac{1}{H(\alpha_j^{(i)})^{w_i(j)}}$$

where $\left|\alpha_{j+1}^{(i)}\right|_p = p^{-h}$, $r = \min\{0, h\}$ and

$$c_1 = p^{(\ell-1)r-\ell(|h|+1)}((2\ell)!)^{-1}.$$

Hence, we find

$$H(\alpha_j^{(i)})^{w_i(j)-\ell} < H(\alpha_{j+1}^{(i)})^{\ell-1} c_1^{-1}.$$

So, taking logarithms of both sides of the last inequality we have

$$(w_i(j)-\ell)\log H(\alpha_j^{(i)}) < (\ell-1)(\log H(\alpha_{j+1}^{(i)}) + \log c_1^{-1})$$

or

$$\frac{(w_i(j)-\ell)}{\ell-1} < \frac{\log H(\alpha_{j+1}^{(i)})}{\log H(\alpha_j^{(i)})} + \frac{\log c_1^{-1}}{\log H(\alpha_j^{(i)})}.$$

Since $\lim_{j\to\infty} w_i(j) = \infty$ and $\lim_{j\to\infty} \frac{\log c_1^{-1}}{\log H(\alpha_j^{(i)})} = 0$ holds

$$\lim_{j\to\infty} \frac{\log H(\alpha_{j+1}^{(i)})}{\log H(\alpha_j^{(i)})} = \infty. \qquad (2.4)_i$$

Let H_j be a monotone union of $H(\alpha_j^{(i)})$ $(i = 1, \ldots, k)$. Now, we are in a position to prove that

$$\limsup_{j\to\infty} \frac{\log H_{j+1}}{\log H_j} = \infty .$$

To this end, it is sufficient to find a subsequence $\{H_{j_n}\}$ such that

$$\lim_{n\to\infty} \frac{\log H_{j_n+1}}{\log H_{j_n}} = \infty .$$

Putting $\mu_i(n) = \dfrac{\log H(\alpha_{j+1}^{(i)})}{\log H(\alpha_j^{(i)})}$ where $1 \le i \le k$, $\lim_{n\to\infty} \mu_i(n) = \infty$ holds for $\forall i = 1, \ldots, k$.

For fixed $i \in \{1, \ldots, k\}$, we define subsequence H_{j_n} that every element is selected from the interval $[H(\alpha_n^{(i)}), H(\alpha_{n+1}^{(i)})]$ for $\forall n \in \mathrm{N}$ as in the following: Let $n \in \mathrm{N}$.

Case I: When $H_j = H(\alpha_n^{(i)})$ if $H_{j+1} = H(\alpha_{n+1}^{(i)})$, then, $H_{j_n} = H(\alpha_n^{(i)})$ selection is made. In this case $\dfrac{\log H_{j_n+1}}{\log H_{j_n}} = \mu_i(n)$ holds.

Case II: If there are at most m elements $H(\alpha_r^{(s)})$ between $H(\alpha_n^{(i)})$ and $H(\alpha_{n+1}^{(i)})$ where $m \le k-1$ and $(s \ne i, 1 \le s \le k, r \in \mathrm{N})$, i.e, $H_j = H(\alpha_n^{(i)})$ and $H_{j+m+1} = H(\alpha_{n+1}^{(i)})$, then $\mu_i(n)$ can be written as

$$\mu_i(n) = \frac{\log H(\alpha_{n+1}^{(i)})}{\log H(\alpha_n^{(i)})} = \frac{\log H(\alpha_{n+1}^{(i)})}{\log H_{j+m}} \frac{\log H_{j+m}}{\log H_{j+m-1}} \cdots \frac{\log H_{j+1}}{\log H(\alpha_n^{(i)})}.$$

Let $\frac{\log H_{t+1}}{\log H_t}$ denote the maximum of

$$\frac{\log H(\alpha_{n+1}^{(i)})}{\log H_{j+m}}, \frac{\log H_{j+m}}{\log H_{j+m-1}}, \ldots, \frac{\log H_{j+1}}{\log H(\alpha_n^{(i)})}$$

and define $H_{j_n} = H_t$, then

$$\mu_i(n) \leq \left(\frac{\log H_{j_n+1}}{\log H_{j_n}}\right)^{m+1} \quad \text{or} \quad \frac{\log H_{j_n+1}}{\log H_{j_n}} \geq (\mu_i(n))^{1/(m+1)} \text{ holds.}$$

Case III : If there are more than $k-1$ elements $H(\alpha_r^{(s)})$ between $H(\alpha_n^{(i)})$ and $H(\alpha_{n+1}^{(i)})$ where $(s \neq i, 1 \leq s \leq k, r \in \mathrm{N})$, then, there is some $\alpha_r^{(s)}$ $(s \neq i)$ such that

$$[H(\alpha_r^{(s)}), H(\alpha_{r+1}^{(s)})] \subset [H(\alpha_n^{(i)}), H(\alpha_{n+1}^{(s)})].$$

In this case the element H_{j_n} is selected in the subinterval $[H(\alpha_r^{(s)}), H(\alpha_{r+1}^{(s)})]$.

a) If there are at most $k-2$ elements H_j between $H(\alpha_r^{(s)})$ and $H(\alpha_{r+1}^{(s)})$ then, the element H_{j_n} is selected as in Case II.

b) If there are more than $k-2$ elements H_j between $H(\alpha_r^{(s)})$ and $H(\alpha_{r+1}^{(s)})$ then, there is at least one index ν $(1 \leq \nu \leq k)$ different from i and s such that

$$[H(\alpha_\ell^{(\nu)}),\ H(\alpha_{\ell+1}^{(\nu)})]\subset[H(\alpha_r^{(s)}),\ H(\alpha_{r+1}^{(s)})].$$

Now, the same discussion is considered for the interval $[H(\alpha_\ell^{(\nu)}),\ H(\alpha_{\ell+1}^{(\nu)})]$. This discussion is completed at most finitely step. So, for the selected elements H_{j_n}

$$\frac{\log H_{j_n+1}}{\log H_{j_n}} \geq (\mu_\nu(n))^{1/(m+1)} \quad (m<k, 1\leq \nu \leq k)$$

holds and also, since $\lim_{n\to\infty} \mu_i(n)=\infty$ for $\forall 1\leq i\leq k$ we write

$$\lim_{n\to\infty} \frac{\log H_{j_n+1}}{\log H_{j_n}} = \infty \quad \text{or} \quad \limsup_{j\to\infty} \frac{\log H_{j+1}}{\log H_j} = \infty. \tag{2.5}$$

Let j_0 be a natural number such that

$$H_j \geq \max\left\{H(\alpha_1^{(1)}), \ldots, H(\alpha_1^{(k)})\right\} \text{ for } \forall j\geq j_0.$$

We define the natural numbers $t_i(j)$ and the p-adic algebraic numbers γ_j as

$$t_i(j)=\max\left\{\nu \,\middle|\, H(\alpha_\nu^{(i)})\leq H_j\right\} \qquad (1\leq i\leq k) \tag*{(2.6)$_i$}$$

$$\gamma_j = \alpha_0+\alpha_1\alpha_{t_1(j)}^{(1)}+\ldots+\alpha_k\alpha_{t_k(j)}^{(k)} \quad (1\leq i\leq k). \tag{2.7}$$

Now, for the polynomial

$F(y, x_1, \ldots, x_{2k+1}) = y - x_1 - x_2x_3 - \ldots - x_{2k}x_{2k+1}$

$F(\gamma_j, \alpha_0, \ldots, \alpha_{t_k(j)}^{(k)})=0$ holds. Applying Lemma 3 we write

$$H(\gamma_j) \le 3^{2m\ell^k + (2k+1)m\ell^k} \left[H(\alpha_0)...H(\alpha_k)\right]^{m\ell^k} \left[H(\alpha^{(1)}_{t_1(j)})...H(\alpha^{(k)}_{t_k(j)})\right]^{m\ell^k}$$

where $[Q(\alpha_0,..., \alpha_k):Q] = m$. Putting

$$c_2 = 3^{2m\ell^k + (2k+1)m\ell^k} \left[H(\alpha_0),..., H(\alpha_k)\right]^{m\ell^k}$$

and from $(1.6)_i$ we find $H(\gamma_j) \le c_2 H_j^{mk\ell^k}$. Since $\lim_{j\to\infty} H_j = \infty$ there is a natural number j_1 such that $j_1 \ge j_0$ and $H_j > c_2$ for $\forall j \ge j_1$. Hence, taking $\rho = mk\ell^k + 1$ we have

$$H(\gamma_j) \le H_j^{\rho} \ (\forall j \ge j_1). \qquad (2.8)_i$$

We suppose that $\gamma \notin A$ (If $\gamma \in A$, the theorem holds). We approximate γ with the algebraic numbers γ_j. From $(1.2)_i$ we write

$$0 < \left|\gamma - \gamma_j\right|_p < c_3 \max\left\{\frac{1}{H(\alpha^{(1)}_{t_1(j)+1})^{\delta_1}}, ..., \frac{1}{H(\alpha^{(k)}_{t_k(j)+1})^{\delta_k}}\right\} \qquad (2.9)$$

where $c_3 = \max\left\{\left|\alpha_1\right|_p, ..., \left|\alpha_k\right|_p\right\}$. Using the definition of H_j and $(2.6)_i$ we have

$$H(\alpha^{(i)}_{t_i(j)+1}) \ge H_{j+1} (\forall 1 \le i \le k). \qquad (2.10)$$

With a combination of (2.9) and (2.10) we write

$$0 < \left|\gamma - \gamma_j\right|_p < c_3 \frac{1}{H^{\delta'}_{j+1}}$$

where $\delta' = \min\{\delta_1, ..., \delta_k\}$. On the other hand, since $\lim_{j\to\infty} H_j = \infty$ there is a natural number j_2 such that $j_2 \geq j_1$ and $H_{j+1}^{\delta'/2} > c_3$ for $\forall j \geq j_2$. Thus, putting $\delta = \delta'/2$

$$0 < \left|\gamma - \gamma_j\right|_p < \frac{1}{H_{j+1}^{\delta}} \quad (\forall j \geq j_2) \tag{2.11}$$

holds. Taking logarithms of both sides $(2.8)_i$ we have

$$\frac{\log H(\gamma_j)}{\rho \log H_j} < 1. \tag{2.12}$$

Let us define $w(j) = \dfrac{\delta \log H_{j+1}}{\rho \log H_j}$. From (2.12) we obtain

$$H(\gamma_j)^{w(j)} < H_{j+1}^{\delta} \ (\forall j \geq j_2). \tag{2.13}$$

So, using (2.13) in (2.11) we find that

$$0 < \left|\gamma - \gamma_j\right|_p < \frac{1}{H(\gamma_j)^{w(j)}} \quad (\forall j \geq j_2). \tag{2.14}$$

From (2.5) $\lim_{n\to\infty} w(j_n) = \lim_{n\to\infty} \dfrac{\delta \log H_{j_n+1}}{\rho \log H_{j_n}} = \infty$ and by (2.14)

$$0 < \left|\gamma - \gamma_{j_n}\right|_p < \frac{1}{H(\gamma_{j_n})^{w(j_n)}}. \tag{2.15}$$

Thus, we have $\gamma \in U$.

Theorem 4. In Theorem 3, if $\lim_{j\to\infty}\frac{\log H_{j+1}}{\log H_j}=\infty$ and

$r=\liminf_{j\to\infty} \deg\gamma_j$, then $\gamma=\alpha_0+\alpha_1\gamma_1+\ldots+\alpha_k\gamma_k\in A\cup U_r$.

Proof : If $\gamma\in A$ the statement is clear. Let $\gamma\notin A$. We prove that $\gamma\in U_r$. In $(2.8)_j$ we replace $j+1$ for j and write

$$H(\gamma_{j+1})\le H_{j+1}^{\rho}. \tag{2.16}$$

When (2.11) is used

$$0<\left|\gamma-\gamma_j\right|_p<\frac{1}{H(\gamma_{j+1})^{\delta/\rho}}\quad(\forall j\ge j_2) \tag{2.17}$$

holds true. Since $\lim_{j\to\infty}\frac{\log H_{j+1}}{\log H_j}=\infty$ $\lim_{j\to\infty}w(j)=\lim_{n\to\infty}\frac{\delta\log H_{j+1}}{\rho\log H_j}=\infty$ and by

(2.14) $0<\left|\gamma-\gamma_j\right|_p<\frac{1}{H(\gamma_j)^{w(j)}}$ holds.

Now, we shall prove that $\lim_{j\to\infty}H(\gamma_j)=\infty$. From (2.11)

$$\left|\gamma_{j+1}-\gamma_j\right|_p=\left|\gamma_{j+1}-\gamma_j+\gamma-\gamma\right|_p<\max\left\{\left|\gamma_{j+1}-\gamma\right|_p,\left|\gamma-\gamma_j\right|_p\right\}<\frac{1}{H_{j+1}^{\delta}}$$

holds for $\forall j\ge j_2$. A combination of this inequality and Lemma 1 gives

$$\frac{c_2}{H(\gamma_{j+1})^{\ell}H(\gamma_j)^{\ell}}<\left|\gamma_{j+1}-\gamma_j\right|_p<\frac{1}{H_{j+1}^{\delta}}(\forall j\ge j_2) \tag{2.18}$$

or

$$c_2 H_{j+1}^{\delta} < H(\gamma_{j+1})^{\ell} H(\gamma_j)^{\ell} \; (\forall j \geq j_2). \qquad (2.19)$$

Thus, from (2.19) and $(2.8)_j$ we find

$$c_2 H_{j+1}^{\delta} < H(\gamma_{j+1})^{\ell} H(\gamma_j)^{\ell} < H(\gamma_{j+1})^{\ell} H_j^{\ell\rho} \quad (\forall j \geq j_2). \qquad (2.20)$$

Since $\lim_{j\to\infty} \frac{\log H_{j+1}}{\log H_j} = \infty$ there is a natural number j_3 such that $j_3 \geq j_2$ and

$$H_j^{2\ell\rho} < c_2 H_j^{\delta}. \qquad (2.21)$$

From (2.20) and (2.21) we get

$$H_j^{2\ell\rho} < c_2 H_j^{\delta} < H(\gamma_{j+1})^{\ell} H(\gamma_j)^{\ell} < H(\gamma_{j+1})^{\ell} H_j^{\ell\rho}$$

and so we find

$$H_j^{\rho} < H(\gamma_{j+1}) \; (\forall j \geq j_3). \qquad (2.22)$$

On the other hand, since $\lim_{j\to\infty} H_j = \infty$ it is true that

$$\lim_{j\to\infty} H(\gamma_{j+1}) = \lim_{j\to\infty} H(\gamma_j) = \infty.$$

Thus, the conditions (1), (2) and (3) of Theorem 1 are satisfied, then, we have $\gamma \in U_r$ where $r = \liminf_{j\to\infty} \operatorname{der}\gamma_j$.

Theorem 5. If $\gamma_1 \in U$ and $\gamma_2 \in U^s$ in Q_P then $\gamma_1 + \gamma_2, \gamma_1\gamma_2 \in A \cup U$.

Proof : Suppose that the number γ_1 belongs to subclass U_m. From Definition 1 there are infinitely many p-adic algebraic numbers $\{\alpha_i\}$ such that

$$|\gamma_1 - \alpha_i|_p < \frac{1}{H(\alpha_i)^{w_1(i)}}$$

where $\limsup_{i\to\infty} w_1(i) = \infty$ and $\deg \alpha_i = m$. Since $\limsup_{i\to\infty} w_1(i) = \infty$ there is a subsequence $\{w_1(i_k)\}$ of $\{\alpha_i\}$ such that $\lim_{k\to\infty} w_1(i_k) = \infty$ and

$$\left|\gamma_1 - \alpha_{i_k}\right|_p < \frac{1}{H(\alpha_{i_k})^{w_1(i_k)}} \quad . \tag{3.1}$$

On the other hand, since $\gamma_2 \in U^s$ from Definition 2 there are infinitely many p-adic algebraic numbers $\{\beta_k\}$ which satisfy the following properties

$$\deg \beta_k = n_k \le \ell \quad (\ell \in Z^+) \text{ and } \lim_{k\to\infty} H(\beta_k) = \infty \tag{3.2}$$

and

$$|\gamma_2 - \beta_k|_p < \frac{1}{H(\beta_k)^{w_2(k)}} < \frac{1}{H(\beta_{k+1})^{\delta 1}} \tag{3.3}$$

where $\lim_{k\to\infty} w_2(k) = \infty$ and some fixed $\delta_1 > 0$. Also, we can write

$$\begin{aligned} |\beta_{k+1} - \beta_k|_p &= |\beta_{k+1} - \gamma_2 + \gamma_2 - \beta_k|_p \\ &\le \max\left\{|\beta_{k+1} - \gamma_2|_p, |\gamma_2 - \beta_k|_p\right\} \end{aligned}$$

and using (3.3) we find

$$|\beta_{k+1} - \beta_k|_p < \max\left\{\frac{1}{H(\beta_{k+1})^{w_2(k+1)}}, \frac{1}{H(\beta_k)^{w_2(k)}}\right\}.$$

From $H(\beta_k) < H(\beta_{k+1})$

$$\left|\beta_{k+1}-\beta_k\right|_p < \frac{1}{H(\beta_k)^{w(k)}} \tag{3.4}$$

holds where $w(k)=\min\{\, w_2(k),\, w_2(k+1) \,\}$. A combination of (3.4) and Lemma1 gives

$$\frac{c_1}{H(\beta_{k+1})^n H(\beta_k)^n} < \left|\beta_{k+1}-\beta_k\right|_p < \frac{1}{H(\beta_k)^{w(k)}}$$

or

$$H(\beta_k)^{w(k)-n} < \frac{1}{c_1} H(\beta_{k+1})^n .$$

Since $\lim_{k\to\infty} H(\beta_k)=\infty$ there is a natural number k_0 such that $H(\beta_{k+1}) > \frac{1}{c_1}$ for $\forall k \geq k_0$. Hence we can write

$$H(\beta_k)^{w(k)-n} < H(\beta_{k+1})^{n+1}$$

for $\forall k \geq k_0$. Taking the logarithms of both sides of the last inequality we write

$$(w(k)-n) < (n+1)\frac{\log H(\beta_{k+1})}{\log H(\beta_k)} .$$

Thus, since $\lim_{k\to\infty} w(k)=\infty$

$$\lim_{k\to\infty} \frac{\log H(\beta_{k+1})}{\log H(\beta_k)} = \infty \tag{3.5}$$

is valid.

Let us show that $\gamma_1+\gamma_2 \in \mathrm{A} \cup \mathrm{U}$. If $\gamma_1+\gamma_2 \in A$, then, the statement is clear. We assume that $\gamma_1+\gamma_2 \notin A$. Now, we approximate the number $\gamma_1 + \gamma_2$ with the p-adic algebraic numbers γ_k which is selected in the intervals $[H(\alpha_{i_k}), H(\alpha_{i_k})^{w_1(i_k)}]$ as in the following cases.

Case I: If there is no element $H(\beta_\nu)$ between $H(\alpha_{i_k})$ and $H(\alpha_{i_k})^{w_1(i_k)}$ then the p-adic number is defined as

$$\gamma_k = \alpha_{i_k} + \beta_{t(k)}$$

where $t(k) = \max\left\{ \nu \mid H(\alpha_\nu) \le H(\alpha_{i_k}) \right\}$. In this case

$$H(\beta_{t(k)}) \le H(\alpha_{i_k}) \le H(\alpha_{i_k})^{w_1(i_k)}. \tag{3.6}$$

is valid. From (3.1) and (3.3) we write

$$\left|\gamma_1 + \gamma_2 - \gamma_k\right|_p \le \max\left\{ \frac{1}{H(\alpha_{i_k})^{w_1(i_k)}}, \frac{1}{H(\beta_{t(k)+1})\delta} \right\}$$

and from (3.6) we can write $H(\alpha_{i_k})^{\delta w_1(i_k)} \le H(\beta_{t(k)+1})^{\delta}$. Thus, we have

$$\left|\gamma_1 + \gamma_2 - \gamma_k\right|_p \le \frac{1}{H(\alpha_{i_k})^{\mu(k)}} \tag{3.7}$$

where $\mu(k) = \min\{ w_1(i_k), \delta w_1(i_k) \}$.

It satisfies $F(\gamma_k, \alpha_{i_k}, \beta_{t(k)}) = 0$ for the polynomial

$$F(y, x_1, x_2) = y - x_1 - x_2$$

and applying Lemma 3

$$H(\gamma_k) \le 3^{4\ell^2} H(\alpha_{i_k})^{\ell^2} H(\beta_{t(k)})^{\ell^2}$$

holds where $\ell \ge \max\{ m, n \}$. Using (3.6) in this inequality we have

$$H(\gamma_k) \le 3^{4\ell^2} H(\alpha_{i_k})^{2\ell^2}.$$

Since $\lim_{k\to\infty} H(\alpha_{i_k}) = \infty$ there is a natural number k_1 such that $k_1 \ge k_0$ and $H(\alpha_{i_k}) > 3^{4\ell^2}$ for $\forall k \ge k_1$. Hence, taking as $p = 2\ell^2 + 1$ we have

$$H(\gamma_k) \le H(\alpha_{i_k})^p \ (\forall\, k \ge k_1). \tag{3.8}$$

So, using (3.8) in (3.7) we find

$$|\gamma_1 + \gamma_2 - \gamma_k|_p \le \frac{1}{H(\gamma_k)^{\mu(k)/p}} (\forall\, k \ge k_1). \tag{3.9}$$

Case II: If there is only one element $H(\beta_v)$ between $H(\alpha_{i_k})$ and $H(\alpha_{i_k})^{w_1(i_k)}$, then the number γ_k can be selected in the following manner: Let $H(\beta_{t(k)})$ denote the element between $H(\alpha_{i_k})$ and $H(\alpha_{i_k})^{w_1(i_k)}$. Thus,

$$H(\alpha_{i_k}) < H(\beta_{t(k)}) < H(\alpha_{i_k})^{w_1(i_k)}$$

holds. On the other hand, it can be written that

$$w_1(i_k) = \frac{\log H(\alpha_{i_k})^{w_1(i_k)}}{\log H(\alpha_{i_k})} = \frac{\log H(\alpha_{i_k})^{w_1(i_k)}}{\log H(\beta_{t(k)})} \cdot \frac{\log H(\beta_{t(k)})}{\log H(\alpha_{i_k})}.$$

(i) If $\dfrac{\log H(\alpha_{i_k})^{w_1(i_k)}}{\log H(\beta_{t(k)})} \geq \dfrac{\log H(\beta_{t(k)})}{\log H(\alpha_{i_k})}$, then, it follows that

$$w_1(i_k) \leq \left(\frac{\log H(\alpha_{i_k})^{w_1(i_k)}}{\log H(\beta_{t(k)})} \right)^2 \text{ and so}$$

$$\frac{\log H(\alpha_{i_k})^{w_1(i_k)}}{\log H(\beta_{t(k)})} \geq \sqrt{w_1(i_k)} \tag{3.10}$$

holds. Let be $\gamma_k = \alpha_{i_k} + \beta_{t(k)}$. From (3.1) and (3.3) it follows that

$$\left| \gamma_1 + \gamma_2 - \gamma_k \right|_p \leq \max \left\{ \frac{1}{H(\alpha_{i_k})^{w_1(i_k)}}, \frac{1}{H(\beta_{t(k)})^{w_2(t(k))}} \right\}.$$

Using $\log H(\alpha_{i_k})^{w_1(i_k)} = \log H(\beta_{t(k)})^{w_1(i_k) \frac{\log H(\alpha_{i_k})}{\log H(\beta_{t(k)})}}$ and (3.10) it follows that

$$\left| \gamma_1 + \gamma_2 - \gamma_k \right|_p \leq \max \left\{ \frac{1}{H(\beta_{t(k)})^{\sqrt{w_1(i_k)}}}, \frac{1}{H(\beta_{t(k)})^{w_2(t(k))}} \right\}.$$

Putting $\mu(k) = \min\left\{ \sqrt{w_1(i_k)}, \delta w_2(t(k)) \right\}$ we have

$$\left| \gamma_1 + \gamma_2 - \gamma_k \right|_p \leq \frac{1}{H(\beta_{t(k)})^{\mu(k)}}. \tag{3.11}$$

It satisfies $F(\gamma_k, \alpha_{i_k}, \beta_{t(k)}) = 0$ for the polynomial $F(y, x_1, x_2) = y - x_1 - x_2$ and applying Lemma 3 and using $H(\alpha_{i_k}) < H(\beta_{t(k)})$ it follows that

$$H(\gamma_k) \le 3^{4\ell^2} H(\beta_{t(k)})^{2\ell^2}$$

where $\ell \ge \max\{m, n\}$. Since $\lim_{k\to\infty} H(\beta_{t(k)}) = \infty$ there is a natural number k_2 such that $k_2 \ge k_1$ and $H(\beta_{t(k)}) > 3^{4\ell^2}$ for $\forall k \ge k_2$. Thus, we have

$$H(\gamma_k) \le H(\beta_{t(k)})^p \tag{3.12}$$

for $\forall\, k \ge k_2$ where $p = 2\ell^2 + 1$. Using (3.12) in (3.11) we have

$$|\gamma_1 + \gamma_2 - \gamma_k|_p \le \frac{1}{H(\gamma_k)^{\mu(k)/p}} \ (\forall\, k \ge k_2). \tag{3.13}$$

(ii) If $\dfrac{\log H(\beta_{t(k)})}{\log H(\alpha_{i_k})} > \dfrac{\log H(\alpha_{i_k})^{w_1(i_k)}}{\log H(\beta_{t(k)})}$, then it follows that

$$\frac{\log H(\beta_{t(k)})}{\log H(\alpha_{i_k})} \ge \sqrt{w_1(i_k)}\,. \tag{3.14}$$

Let be $\gamma_k = \alpha_{i_k} + \beta_{t(k)-1}$. A combination of (3.1) and (3.3) gives

$$|\gamma_1 + \gamma_2 - \gamma_k|_{p,} \le \max\left\{ \frac{1}{H(\alpha_{i_k})^{w_1(i_k)}}, \frac{1}{H(\beta_{t(k)})^{\delta}} \right\}.$$

Using $\log H(\beta_{t(k)})^{\delta} = \log H(\alpha_{i_k})^{\delta \log H(\beta_{t(k)})/\log H(\alpha_{i_k})}$ and (3.4) it follows that

$$|\gamma_1+\gamma_2-\gamma_k|_p \le \max\left\{\frac{1}{H(\alpha_{i_k})^{w_1(i_k)}}, \frac{1}{H(\alpha_{i_k})^{\delta\sqrt{w_1(i_k)}}}\right\}.$$

Putting $\mu(k)=\min\left\{w_1(i_k), \delta\sqrt{w_1(i_k)}\right\}$, we find

$$|\gamma_1+\gamma_2-\gamma_k|_p \le \frac{1}{H(\alpha_{i_k})^{\mu(k)}}. \tag{3.15}$$

Using (3.12) in (3.15), we have

$$|\gamma_1+\gamma_2-\gamma_k|_p \le \frac{1}{H(\gamma_k)^{\mu(k)/p}} \quad (\forall\, k \ge k_2). \tag{3.16}$$

Case III: If there are at least two elements $H(\beta_v)$ between $H(\alpha_{i_k})$ and $H(\alpha_{i_k})^{w_1(i_k)}$, then we define the number γ_k as

$$\gamma_k = \alpha_{i_k} + \beta_{t(k)}$$

where $t(k)=\min\left\{v \mid H(\alpha_v) \ge H(\alpha_{i_k})\right\}$. In this case, it follows that

$$H(\alpha_{i_k}) \le H(\beta_{t(k)}) < H(\beta_{t(k)+1}) \le H(\alpha_{i_k})^{w_1(i_k)}.$$

A combination of (3.1) and (3.3) gives

$$|\gamma_1+\gamma_2-\gamma_k|_p \le \max\left\{\frac{1}{H(\alpha_{i_k})^{w_1(i_k)}}, \frac{1}{H(\beta_{t(k)})^{w_2(t(k))}}\right\}$$

and since $H(\alpha_{i_k})^{w_1(i_k)} > H(\beta_{t(k)+1})$

$$\left|\gamma_1+\gamma_2-\gamma_k\right|_p \le \max\left\{\frac{1}{H(\beta_{t(k)+1})},\frac{1}{H(\beta_{t(k)})^{w_2(t(k))}}\right\}.$$

can be written. If $H(\beta_{t(k)+1}) = H(\beta_{t(k)})^{\log H(\beta_{t(k)+1})/\log H(\beta_{t(k)})}$ is considered

$$\left|\gamma_1+\gamma_2-\gamma_k\right|_p \le \frac{1}{H(\beta_{t(k)})^{\mu(k)}} \tag{3.17}$$

holds where $\mu(k)=\min\left\{\log H(\beta_{t(k)+1})/\log H(\beta_{t(k)}),\ w_2(i_k)\right\}$. Since the inequality (3.12) holds in Case III and using (3.17) we have

$$\left|\gamma_1+\gamma_2-\gamma_k\right|_p \le \frac{1}{H(\gamma_k)^{\mu(k)/p}}\ (\forall\, k \ge k_2). \tag{3.18}$$

There are infinitely many disjoint intervals

$$[H(\alpha_{i_k}), H(\alpha_{i_k})^{w_1(i_k)}] \text{ since } \lim_{k\to\infty} H(\alpha_{i_k}) = \infty.$$

Thus, for the numbers γ_k which are selected for three cases it holds

$$\left|\gamma_1+\gamma_2-\gamma_k\right|_p \le \frac{1}{H(\gamma_k)^{\mu(k)/p}}$$

where $\lim_{k\to\infty}\mu(k)=\infty$. So that $\gamma_1+\gamma_2 \in U$.

On the other hand, it can be easily show that $\gamma_1\gamma_2 \in A\cup U$ with the same method. In fact, we will only consider the first case.

Case I: If there is no element $H(\beta_v)$ between $H(\alpha_{i_k})$ and $H(\alpha_{i_k})^{w_1(i_k)}$ then, the p-adic number is defined as $\gamma'_k = \alpha_{i_k}\cdot\beta_{t(k)}$

where $t(k) = \max\left\{ \nu \mid H(\alpha_\nu) \le H(\alpha_{i_k}) \right\}$. From (3.1) and (3.3) it follows

$$\left|\gamma_1 \cdot \gamma_2 - \gamma'_k\right|_p = \left|\gamma_1 \cdot \gamma_2 - \alpha_{i_k} \beta_{t(k)} + \gamma_1 \beta_{t(k)} - \gamma_1 \alpha_{i_k}\right|_p$$

$$\le \max\left\{ \frac{\left|\gamma_1\right|_p}{H(\beta_{t(k)+1})^\delta}, \frac{\left|\beta_{t(k)}\right|_p}{H(\alpha_{i_k})^{w_1(i_k)}} \right\}$$

and also, using $H(\alpha_{i_k})^{w_1(i_k)} \le H(\beta_{t(k)+1})$

$$\left|\gamma_1 \cdot \gamma_2 - \gamma'_k\right|_p \le \max\left\{ \frac{\left|\gamma_1\right|_p}{H(\alpha_{i_k})^{\delta w_1(i_k)}}, \frac{\left|\beta_{t(k)}\right|_p}{H(\alpha_{i_k})^{w_1(i_k)}} \right\}.$$

(3.19)

holds. On the other hand, there is a natural number k_0 such that

$$\left|\beta_{t(k)}\right|_p = \left|\beta_{t(k_0)}\right|_p$$

for $\forall k \ge k_0$. Let be $C = \max\{ A, B \}$ where $A = \left|\gamma_1\right|_p$ and $B = \left|\beta_{t(k_0)}\right|_p$. Using these notations in (3.19), we write

$$\left|\gamma_1 \cdot \gamma_2 - \gamma'_k\right|_p \le \frac{C}{H(\alpha_{i_k})^{\mu(k)}}$$

where $\mu(k) = \min\{ \delta w_1(i_k), w_1(i_k) \}$. Since $\lim_{k \to \infty} H(\alpha_{i_k}) = \infty$ there is a natural number k_1 such that $k_1 \ge k_0$ and $H(\alpha_{i_k}) > C$ for $\forall k \ge k_1$. So,

$$\left|\gamma_1 \cdot \gamma_2 - \gamma'_k\right|_p \leq \frac{1}{H(\alpha_{i_k})^{\mu(k)-1}} \quad (\forall\, k \geq k_1) \tag{3.20}$$

holds. It satisfies $F(\gamma'_k, \alpha_{i_k}, \beta_{t(k)}) = 0$ for the polynomial $F(y, x_1, x_2) = y - x_1 x_2$ and applying Lemma 3 and using $H(\beta_{t(k)}) \leq H(\alpha_{i_k})$, it follows that

$$H(\gamma'_k) \leq 3^{4\ell^2} H(\alpha_{i_k})^{2\ell^2}.$$

Since $\lim_{i \to \infty} H(\alpha_{i_k}) = \infty$ there is a natural k_2 such that $k_2 \geq k_1$ and $H(\alpha_{i_k}) > 3^{4\ell^2}$. Thus, putting $p = 2\ell^2 + 1$, we have $H(\gamma'_k) \leq H(\alpha_{i_k})^p$ for $\forall k \geq k_2$. In this inequality using in (3.20) we find

$$\left|\gamma_1 \cdot \gamma_2 - \gamma'_k\right|_p \leq \frac{1}{H(\alpha_{i_k})^{(\mu(k)-1)/p}} \quad (\forall k \geq k_2). \tag{3.21}$$

The other cases can be treated with the same method. Finally, we have $\gamma_1 \gamma_2 \in A \cup U$.

In Theorem 5 if we take $\gamma_2 \in U$ instead of $\gamma_2 \in U^s$, the theorem fails to be true. So that if $\gamma_1, \gamma_2 \in U$, then, the number $\gamma_1 + \gamma_2$ does not necessarily belong to $A \cup U$. Now, to prove this, we first prove the following theorem in Q_p which is proved for real numbers by Erdös [6].

Theorem 6. Let x a p-adic number. Then, there are some Liouville numbers γ_1, γ_2 such that $x = \gamma_1 + \gamma_2$.

Proof : If x is a rational number then the statement is clear. In fact, for any Liouville number γ_1, the number $\gamma_2 = x - \gamma_1$ is a Liouville number and $x = \gamma_1 + \gamma_2$ holds.

We assume that x be non-rational p-adic number and $x = \sum_{k=0}^{\infty} a_k p^k$ where $a_k = 0, 1, ..., p-1$. We define the numbers γ_1, γ_2 as

$$\gamma_1 = \sum_{k=0}^{\infty} b_k p^k \text{ and } \gamma_2 = \sum_{k=0}^{\infty} c_k p^k$$

where for $n! \leq k < (n+1)!$

$$b_k = a_k \text{ and } c_k = 0 \quad (n = 1, 3, 5, ...)$$

$$b_k = 0 \text{ and } c_k = a_k \quad (n = 0, 2, 4, ...)$$

i.e.,

$$\gamma_1 = 0 + a_1 p^1 + 0p^2 + ... + 0p^5 + a_6 p^6 + ... + a_{23} p^{23} + 0p^{24} + ...$$

$$\gamma_2 = a_0 + 0p^1 + a_2 p^2 + .. + a_5 p^5 + 0p^6 + ... + 0p^{23} + a_{24} p^{24} + ...$$

a) If there are at most finitely many numbers b_k distinct from 0, then, $\gamma_1 \in Q$ and $\gamma_2 \in U_1$ will be infinitely many numbers c_k distinct from 0. Thus, $\gamma_1 \in Q$ and $x = \gamma_1 + \gamma_2 \in U_1$. Moreover, $\frac{x}{2} \in U_1$ and $x = \frac{x}{2} + \frac{x}{2}$.

b) If there are infinitely many numbers b_k and c_k distinct from 0, then the numbers γ_1 and γ_2 are Liouville numbers. Now, we shall prove this.

Put $s_n = \sum_{k=0}^{n} b_k p^k$. We shall approximate γ_1 by algebraic numbers $s_{(2n)!-1}$. It follows that

$$\left|\gamma_1 - s_{(2n)!-1}\right|_p = \left|a_{(2n+1)!}p^{(2n+1)!} + a_{(2n+1)!+1}p^{(2n+1)!+1} + \ldots\right|_p$$
$$= \left|p^{(2n+1)!}\right|_p \left|a_{(2n+1)!} + a_{(2n+1)!+1}p + \ldots\right|_p$$

and so we have

$$\left|\gamma_1 - s_{(2n)!-1}\right|_p \leq \left(\frac{1}{p^{(2n)!}}\right)^{2n+1} = \frac{1}{H(s_{(2n)!-1})^{2n+1}}.$$

Since $\lim_{n\to\infty}(2n+1) = \infty$ the number γ_1 is a p-adic Liouville number.

With the same method, putting $t_n = \sum_{k=0}^{n} c_k p^k$ it is possible to approximate γ_2 by $t_{(2n-1)!-1}$. It follows that

$$\left|\gamma_1 - t_{(2n-1)!-1}\right|_p \leq \left(\frac{1}{p^{(2n-1)!}-1}\right)^{2n} = \frac{1}{H(t_{(2n-1)!-1})^{2n}}$$

and so the number γ_2 is a p–adic Liouville number.

Let be $x = p^{\alpha} \sum_{k=0}^{\infty} a_k p^k$ where $\alpha \in Z$. From the first part of the proof there are p-adic Liouville numbers γ_1, γ_2 such that $x = p^{\alpha}(\gamma_1 + \gamma_2)$. Then, $p^{\alpha}\gamma_1$ and $p^{\alpha}\gamma_2$ are Liouville numbers that satisfy $x = p^{\alpha}\gamma_1 + p^{\alpha}\gamma_2$.

Finally, for every $x \in Q_p$ there are $\gamma_1, \gamma_2 \in U_1$ such that $x = \gamma_1 + \gamma_2$.

Hence, in the Theorem 5 if we replace the condition $\gamma_2 \in U^s$ with the condition $\gamma_2 \in U$, the theorem fails to be true. Since we know that there are p-adic numbers not belonging to the classes A and U by [13] , for any number $x \notin A \cup U\,(x \in Q_p)$ there are numbers $\gamma_1, \gamma_2 \in U_1$ such that $x = \gamma_1 + \gamma_2$ by Theorem 6. If the Theorem 5 would be true, the number x would have belonged to $\mathrm{A} \cup \mathrm{U}$. But this is impossible since $x \notin A \cup U$.

We can give a result for Theorem 5.

Corollary 1. Let $\xi \in U$, $\gamma_1, \ldots, \gamma_n \in U^s$ and $n \in \mathrm{N}$. Then,

a) $\xi + \sum_{k=1}^{n} \gamma_k \in A \cup U$,

b) $\xi . \prod_{k=1}^{n} \gamma_k \in A \cup U$.

Proof : We shall prove this result with Mathematical . Induction.

For $n = 1$ from Theorem 5 it holds $\xi + \gamma_1$, $\xi . \gamma_1 \in A \cup U$.

Let the statement be true for any number n, i.e.;

$$\xi + \sum_{k=1}^{n} \gamma_k \in A \cup U \quad \text{and} \quad \xi . \prod_{k=1}^{n} \gamma_k \in A \cup U .$$

Now we shall prove the statement for $n+1$.

If $\xi + \sum_{k=1}^{n} \gamma_k \in A$ and $\xi . \prod_{k=1}^{n} \gamma_k \in A$ it is clear that

$$\xi + \sum_{k=1}^{n} \gamma_k + \gamma_{n+1} = \xi + \sum_{k=1}^{n+1} \gamma_k \in U^S \subset U$$

for $\gamma_{n+1} \in U^S$ and $\xi . \sum_{k=1}^{n} \gamma_k . \gamma_{n+1} = \xi . \sum_{k=1}^{n+1} \gamma_k \in U^S \subset U$ holds..

We assume that $\xi + \sum_{k=1}^{n} \gamma_k \in U$ and $\xi . \prod_{k=1}^{n} \gamma_k \in U$. From Theorem 5 we obtain

$$\xi + \sum_{k=1}^{n} \gamma_k + \gamma_{n+1} = \xi + \sum_{k=1}^{n+1} \gamma_k \in A \cup U$$

and

$$\xi . \sum_{k=1}^{n} \gamma_k . \gamma_{n+1} = \xi . \sum_{k=1}^{n+1} \gamma_k \in A \cup U .$$

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