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Special concircular projective curvature collineation in recurrent Finsler space

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A concircular transformation in Riemannian spaces was introduced and studied in series of papers by K. Yano [1][1]. M . Okumura [4] has developed a similar transformation in non- Riemannian symmetric spaces. Affine motion in recurrent Finsler space was discussed by R.S. Sinha [7]. Present author [9,10] has studied curvature collineation in Finsler spaces. The purpose of present paper is to develop special concircular projective curvature collineation in recurrent Finsler space .

1. Preliminaries

We consider an n-dimensional Finsler space F_n with Berwald's connection parameters $G^i_{jk}(x,\dot{x})$[2]. The curvature tensor field H^i_{jkh}, arising from this connection parameter, is homogeneous function of degree zero in $\dot{x}$ and hence we have

(1.1) $$H^i_{jkh}\dot{x}^j = H^i_{kh},$$

(1.2) (a) $$\dot{\partial}_l H^i_{jkh}\dot{x}^j = \dot{\partial}_l H^i_{jkh}\dot{x}^l = 0 \ ^{3)}.$$

The commutation formulae involving the curvature tensor H^i_{jkh} are given by [3]

(1.2) (b) $$2T_{[(h)(k)]} = T_{(h)(k)} - T_{(k)(h)} = -\dot{\partial}_i T H^i_{hk},$$

(1.2) (c) $$2T^i_{j[(h)(k)]} = -\dot{\partial}_r T^i_j H^r_{hk} - T^i_r H^r_{jhk} + T^r_j H^i_{rhk},$$

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1) The numbers in brackets refer to the references at the end of this paper.

2) The line element $(x^i, \dot{x}^i)$ is briefly represented by $(x, \dot{x})$.

3) $\dot{\partial}_l = \partial/\partial\dot{x}^l$, $\partial_l = \partial/\partial x^l$.

where index in round bracket( ) represents covariant differentiation in the sense of Berwald [3].It satisfies the following identities :

(1.3) $$H^i_{jkh} = -H^i_{jhk},$$

(1.4) $$H^i_{jki} = H_{jk},$$

(1.5) $$H^l_{ikh} = 2H_{[hk]},$$

A non-flat Finsler space F_n in which there exists a non-zero vector field, whose components K_m are positively homogeneous functions of degree zero in $\dot{x}^i$, such that the curvature tensor field H^i_{jkh} satisfied the relation

(1.6) $$H^i_{jkh(m)} = K_m H^i_{jkh},$$

is called a recurrent Finsler space [6,8] .We denote such a Finsler space by F_n^*.

Let us consider a point transformation

(1.7) $$\bar{x}^i = x^i + \varepsilon v^i(x),$$

where $v^i(x)$ is a contravariant vector field. Then the Lie-derivative of a tensor $T^i_j(x,\dot{x})$ and the connection coefficients are characterised by [2]

(1.8) $$\pounds T^i_j = v^h T^i_{j(h)} - T^h_j v^i_{(h)} + T^i_h v^h_{(j)} + (\dot{\partial}_h T^i_j) v^h_{(s)} \dot{x}^s$$

and

(1.9) $$\pounds G^i_{jk} = v^i_{(j)(k)} + H^i_{jkh} v^h + G^i_{jkh} v^h_{(r)} \dot{x}^r$$

respectively. The Lie-derivative of the curvature tensor H^i_{jkh} is given by

(1.10) $$\pounds H^i_{jkh} = v^l H^i_{jkh(l)} - H^l_{jkh} v^i_{(l)} + H^i_{lkh} v^l_{(j)} + H^i_{jlh} v^l_{(k)} + H^i_{jkl} v^l_{(h)} + (\dot{\partial}_l H^i_{jkh}) v^l_{(m)} \dot{x}^m.$$

The processes of Lie-differentiation and other differentiations are connected by

(1.11) $$(\pounds T^i_{jk(l)}) - (\pounds T^i_{jk})_{(l)} = (\pounds G^i_{rl}) T^r_{jk} - (\pounds G^r_{jl}) T^i_{rk} - (\pounds G^r_{kl}) T^i_{jr} - (\pounds G^r_{lp}) \dot{x}^p \dot{\partial}_r T^i_{jk},$$

(1.12) $\left(\pounds G^i_{jh}\right)_{(k)} - \left(\pounds G^i_{kh}\right)_{(j)} = \pounds H^i_{hjk} + \left(\pounds G^r_{kl}\right)\dot{x}^l G^i_{rjh}$,

(1.13) $\dot{\partial}_l\left(\pounds T^i_{jk}\right) - \pounds\left(\dot{\partial}_l T^i_{jk}\right) = 0.$

We also consider an infinitesimal transformation similar to that of M. Okumura [5] of the form

(1.14) $\bar{x}^i = x^i + \varepsilon v^i, \quad v^i_{(k)} = \lambda \delta^i_k,$

where $\lambda(x, \dot{x})$ is a scalar function. Such a transformation is called a special concircular transformation.

The necessary and sufficient condition that the transformation (1.7) be a projective motion is that the Lie-derivative of G^i_{jk} is given by

(1.15) $\pounds G^i_{jk} = 2\delta^i_{(j} p_{k)} + \dot{x}^i p_{jk}, \quad p_k = \dot{\partial}_k p, \quad p_{jk} = \dot{\partial}_j p_k,$

where $p(x, \dot{x})$ is homogeneous scalar function of degree one in $\dot{x}^i$ and (jk) represents symmetric part.

2. Special concircular projective curvature collineation

Definition : In a recurrent Finsler space F^*_n, if the curvature tensor field H^i_{jkh} satisfies the relation

(2.1) $\pounds H^i_{jkh} = 0,$

where $\pounds$ represents Lie-derivative defined by the transformation (1.14), which defines a projective motion, then the transformation (1.14) is called the special concircular projective H-curvature collineation.

If a special concircular transformation defines a projective motion, the equation (1.9) in view of (1.14) and (1.15) yields

(2.2) $\delta^i_k \lambda_{(j)} + H^i_{jkh} v^h = 2\delta^i_{(j} p_{k)} + \dot{x}^i p_{jk},$

since G^i_{jk} is homogeneous function of degree zero in $\dot{x}^i$. Contracting the above equation with respect to the indices i, j and using (1.5) and (1.15), we find

(2.3) $\lambda_{(k)} + 2H_{[hk]} v^h = (n+1) p_k.$

where $[hk]$ represents skew-symmetric part.
Now if we contract (2.2) with respect to indices i,k, we obtain

(2.4) $$n\lambda_{(j)} - H_{jh}v^h = (n+1)p_j,$$
in view of $(1.3),(1.4)$ and (1.15).
Eliminating p_j from the equations (2.3) and (2.4), we get
(2.5) $$H_{hj}v^h + (1-n)\lambda_{(j)} = 0.$$

If λ follows invariance property with respect to Berwald's covariant differentiation, then the projective motion satisfies the following relations:
(2.6)(a) $$H_{hj}v^h = 0, \qquad \text{(b)}\ H_{hj}v^h + (n+1)p_j = 0$$
from the equations (2.4) and (2.5).

Applying the equation (1.15) and the homogeneity property of G^i_{jk} in the equation (1.12), it yields
(2.7) $$£H^i_{hjk} = 2\{\delta^i_h p_{[j(k)]} + \delta^i_{[j} p_{|h|(k)]} + \dot{x}^i p_{[j|h|(k)]}\},$$
where the index between two parallel bars is unaffected when we consider skew-symmetric part.
Contracting the equation (2.7) with respect to indices i and h, we obtain
(2.8) $$£H_{[kj]} = (n+1)p_{[j(k)]}.$$
Since for the infinitesimal transformation (1.7), the vector $v^i(x)$ is Lie-invariant, we have

(2.9) $$£v^i = 0.$$
Transvecting the equation (2.8) by v^k and noting (2.6) (a) and (2.9), we find
(2.10) $$£(H_{jk}v^k) = 2(n+1)p_{[k(j)]}v^k,$$
which yields
(2.11) $$£p_j = 2p_{[j(k)]}v^k$$
in view of (2.6) (b).

Applying (1.8) for p_j and noting (1.14), it gives

(2.12) $$\pounds p_j = p_{j(l)}v^l + \lambda p_j.$$

Hence from the equations (2.11) and (2.12), we get

(2.13) $$p_{k(j)}v^k + \lambda p_j = 0,$$

which immediately reduces to

(2.14) $$\left(p_k v^k\right)_{(j)} = 0$$

in view of (1.14).

Transvecting (2.12) by v^j and using (1.14), (2.9) and (2.14), we obtain

(2.15) $$\pounds\left(p_j v^j\right) = 0,$$

which is a very useful result.

Also in view of (1.6) and (1.14), the equation (1.10) assumes the form

(2.16) $$\pounds H^i_{jkh} = \left(K_l v^l + 2\lambda\right)H^i_{jkh}.$$

Contracting the equation (2.16) with respect to the indices i,h and then transvecting the results by v^k, we find

(2.17) $$\pounds\left(H_{jk}v^k\right) = \left(K_l v^l + 2\lambda\right)H_{jk}v^k$$

in view of (1.14) and (2.9).

When we apply (2.6) (b) in the above equation, it gives

(2.18) $$\pounds p_j = \left(K_l v^l + 2\lambda\right)p_j.$$

Transvecting (2.18) by v^j and using (2.9) and (2.15), we have

(2.19) $$\left(K_l v^l + 2\lambda\right)p_j v^j = 0,$$

which implies either

(2.20) $$K_l v^l + 2\lambda = 0,$$

or

(2.21) $$p_j v^j = 0.$$

In view of (2.20), the equation (2.16) immediately reduces to

$$\pounds\ H^i_{jkh} = 0.$$

Thus we state

Theorem 2.1: In a recurrent Finsler space F_n^*, the special concircular transformation (1.14), which admits projective motion, is the special concircular projective H-curvature collineation.

Contraction of (2.1) with respect to indices i, j yields

$$(2.22) \qquad \pounds H_{[hk]} = 0$$

in view of (1.5).

Applying (2.22) in the equation (2.8), we get the relation

$$(2.23) \qquad p_{h(k)} = p_{k(h)}.$$

Hence we have

Corollary 2.1: In a recurrent Finsler space F_n^*, which admits special concircular projective H-curvature collineation, the vector field p_j behaves like a gradient vector field.

Applying the identity (1.13) for H^i_{jkh} and using (2.1), it yields

$$(2.24) \qquad \pounds\left(\dot{\partial}_l H^i_{jkh}\right) = 0$$

and hence we state

Lemma 2.1: In recurrent Finsler space F_n^*, which admits special concircular projective H-curvature collineation, the partial derivative of the curvature tensor H^i_{jkh} is Lie-invariant.

By virtue of the commutation formula $(1.2)'$ (c) for the curvature tensor H^i_{jkh}, we find

$$(2.25) \quad 2H^i_{jkh[(l)(m)]} = -\dot{\partial}_r H^i_{jkh} H^r_{lm} + H^r_{jkh} H^i_{rlm} - H^i_{rkh} H^r_{jlm} - H^i_{jrh} H^r_{klm} - H^i_{jkr} H^r_{hlm}.$$

Taking Lie-derivative of both sides of the above equation and applying (2.1) and using Lemma 2.1, it reduces to

$$(2.26) \qquad \pounds\left(H^i_{jkh[(l)(m)]}\right) = 0.$$

Accordingly we have

Theorem 2.2: In a recurrent Finsler space F_n^*, which admits special concircular projective H-curvature collineation, the relation (2.26) holds.

The partial differentiation of (2.16) with respect to $\dot{x}^m$ yields

$$(2.27) \qquad \left(v^l \dot{\partial}_m K_l + 2\dot{\partial}_m \lambda\right) H^i_{jkh} + \left(K_l v^l + 2\lambda\right) \dot{\partial}_m H^i_{jkh} = 0$$

in view of lemma 2.1.

Transvecting the equation (2.27) by $\dot{x}^j$ and using (1.2) (a), we obtain

$$(2.28) \qquad \left(v^l \dot{\partial}_m K_l + 2\dot{\partial}_m \lambda\right) H^i_{kh} = 0.$$

Since the space F_n^* is non-flat, the equation (2.28) implies

$$(2.29) \qquad v^l \dot{\partial}_m K_l + 2\dot{\partial}_m \lambda = 0.$$

Transvection of the above equation by $\dot{x}^m$ yields

$$(2.30) \qquad \dot{x}^m \dot{\partial}_m \lambda = 0,$$

since K_l is homogeneous function of degree zero in $\dot{x}^l$.

Thus we state

Theorem 2.3: In a recurrent Finsler space F_n^*, which admits special concircular projective H-curvature collineation, the scalar function λ is homogeneous function of degree zero in $\dot{x}^i$.

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