İstanbul Üniv. Fen Fak. Mat. Dergisi **61-62** (2002-2003), 15-28

On m-D-Separation Axioms

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Abstract

We introduce the notions of m-D-sets and some lower separation axioms m-D_i $(i = 0, 1, 2)$ on m-structures, which are weaker than topological structures, and obtain a unified theory of separation axioms D_i, s-D_i, p-D_i, θ-D_i, δ-$semiD_i$, δ-$preD_i$ $(i = 0, 1, 2)$ in topological spaces.

AMS Math. Subject Classification (2000): 54D10.
Key words and Phrases: m-open, m-D-set, m-D_0, m-D_1, m-D_2, m-T_0, m-T_1, m-T_2.

1 Introduction

In 1982, Tong [28] introduced the notion of D-sets and used these sets to introduce a separation axiom D_1 which is strictly between T_0 and T_1. In 1975, Maheshwari and Prasad [19] introduced new separation axioms semi-T_0, semi-T_1 and semi-T_2 by using semi-open sets due to Levine [17]. Borşan [4] and Caldas [5] introduced the notions of s-D-sets and a separation axiom s-D_1 which is strictly between semi-T_0 and semi-T_1. In 1990, Kar and Bhattachryya [16] introduced new separation axioms pre-T_0, pre-T_1 and pre-T_2 by using preopen sets due to Mashhour et al. [21]. Recently, Caldas [6] and Jafari [15] introduced independently the notions of p-D-sets and a separation axiom p-D_1 which is strictly between pre-T_0 and pre-T_1. Quite recently, Caldas, Fukutake, Georgiou, Jafari and Noiri introduced the notions of θ-D-sets and a separation axiom θ-D_1 [8], δ-semi$D$-sets and a separation axiom δ-semiD_1 [9], and δ-pre$D$-sets and a separation axiom δ-preD_1 [7].

In this paper, we define an m-space (X, m), where X is a nonempty set and m is a subfamily of the power set of X and also satisfies the following conditions: $\emptyset, X \in m$. By using the m-spaces, we define the notions of m-D-sets and separation axioms m-D_i $(i = 0, 1, 2)$ and establish a unified theory of separation axioms D_i, s-D_i, p-D_i, θ-D_i, δ-$semiD_i$, δ-$preD_i$ $(i = 0, 1, 2)$ in topological spaces.

2 Preliminaries

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces. Let A be a subset of X. We denote the closure and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. The θ-closure (resp. δ-closure) of A, $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$), is defined by the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ (resp. $A \cap \mathrm{Int}(\mathrm{Cl}(U)) \neq \emptyset$) for every open set U containing x. A subset A is said to be *θ-closed* (resp. *δ-closed*) [29] if $A = \mathrm{Cl}_\theta(A)$ (resp. $A = \mathrm{Cl}_\delta(A)$). The complement of a θ-closed (resp. δ-closed) set is said to be *θ-open* (resp. *δ-open*).

Definition 2.1 Let (X, τ) be a topological space. A subset A of X is said to be

(1) *semi-open* [17] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(2) *preopen* [21] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(3) *α-open* [24] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(4) *δ-semiopen* [25] if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$,
(5) *δ-preopen* [27] if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$.

The family of all semi-open (resp. preopen, α-open, δ-semiopen, δ-preopen, θ-open, δ-open) sets in a topological space (X, τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$ or τ^α, $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X)$, τ_θ, τ_δ).

Definition 2.2 The complement of a semi-open (resp. preopen, α-open, δ-semiopen, δ-preopen) set is said to be *semi-closed* [11] (resp. *preclosed* [21], *α-closed* [23]), *δ-semiclosed* [25], *δ-preclosed* [27]).

Definition 2.3 The intersection of all semi-closed (resp. preclosed, α-closed, δ-semiclosed, δ-preclosed) sets of (X, τ) containing a subset A is called the *semi-closure* [11] (resp. *preclosure* [14], *α-closure* [23], *δ-semiclosure* [25], *δ-preclosure* [27]) and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{sCl}_\delta(A)$, $\mathrm{pCl}_\delta(A)$).

Definition 2.4 The union of all semi-open (resp. preopen, α-open, δ-semiopen, δ-preopen) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *δ-semiinterior*, *δ-preinterior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{sInt}_\delta(A)$, $\mathrm{pInt}_\delta(A)$).

If we replace open sets in the usual definition of T_i ($i = 0, 1, 2$) with semi-open (resp. preopen, θ-open, δ-semiopen, δ-preopen) sets, we obtain separation axioms *s-T_i* [19] (resp. *p-T_i* [16], *θ-T_i* [10], *δ-semiT_i* [9], (δ, p)-T_i [7]) ($i = 0, 1, 2$).

Definition 2.5 Let (X, τ) be a topological space. A subset A of X is called a *D-set* [28] (resp. *s-D-set* [4], [5], *p-D-set* [15], [6], *θ-D-set* [10], *δ-semi$D$-set*[9], *δ-pre$D$-set* [7]) if there exist two open (resp. semi-open, preopen, θ-open, δ-semiopen, δ-preopen) sets U, V in X such that $U \neq X$ and $A = U - V$.

If we replace open sets in the usual definitions of T_0, T_1, T_2 with D-sets (resp. s-D-sets, p-D-sets, θ-D-sets, δ-semi$D$-sets, δ-pre$D$-sets), we obtain the definitions of separation axioms D_i [28] (resp. s-D_i [4], [5], p-D_i [15], [6], θ-D_i [10], *δ-semiD_i* [9], (δ, p)-D_i [7]) for $i = 0, 1, 2$.

We shall begin with the definition of m-spaces in order to establish a unified theory of separation axioms D_i, s-D_i, p-D_i, θ-D_i, *δ-semiD_i*, (δ, p)-D_i for $i = 0, 1, 2$.

Definition 2.6 A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called an *minimal structure* (briefly *m-structure*) on X if m satisfies the following properties: $\emptyset \in m$ and $X \in m$.

We call the pair (X, m) an m-space. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*.

Definition 2.7 An minimal structure m on a nonempty set X is said to have property $(\mathcal{B})$ [20] if the union of any family of subsets belonging to m belongs to m.

Remark 2.1 An m-structure with the property $(\mathcal{B})$ is called a *generalized topology* by Lugojan [18]. Császár [12] called a family m a *generalized topology* if it satisfies $\emptyset \in m$ and has the property $(\mathcal{B})$. Mashhour et al.

[22] called a family m *supratopology* if it satisfies $X \in m$ and has the property $(\mathcal{B})$. In the present paper, we do not always assume the property $(\mathcal{B})$ on m-structures.

Remark 2.2 Let (X, τ) be a topological space. Then the families $\tau_\theta, \tau_\delta, \tau$, $\mathrm{SO}(X)$ $\mathrm{PO}(X)$, $\alpha(X)$, $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X)$ are all m-structures on X with the property $(\mathcal{B})$. It is well-known that $\tau_\theta, \tau_\delta, \alpha(X)$ are topologies for X and the other are not topologies.

Definition 2.8 Let (X, m) be an m-space and A a subset of X. The *m-closure* $\mathrm{mCl}(A)$ of A and *m-interior* $\mathrm{mInt}(A)$ of A are defined in [20] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F \text{ and } X - F \in m\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A \text{ and } U \in m\}$.

Remark 2.3 Let (X, τ) be a topological space and A a subset of X. If $m = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X)$, $\alpha(X)$, τ_θ, τ_δ), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\mathrm{sCl}_\delta(A)$, $\mathrm{pCl}_\delta(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{Cl}_\theta(A)$, $\mathrm{Cl}_\delta(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\mathrm{sInt}_\delta(A)$, $\mathrm{pInt}_\delta(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{Int}_\theta(A)$, $\mathrm{Int}_\delta(A)$).

Lemma 2.1 (Maki [20]) *Let (X, m) be an m-space and A, B subsets of X. Then the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) $\mathrm{mCl}(\emptyset) = \emptyset, \mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(3) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(4) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(5) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 2.2 (Popa and Noiri [26]) *Let (X, m) be an m-space, A a subset of X and $x \in X$. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing x.*

Lemma 2.3 (Popa and Noiri [26]) *Let (X, m) be an m-space and m have the property $(\mathcal{B})$. Then for a subset A of X, the following properties hold:*

(1) $A \in m$ if and only if $A = \mathrm{mInt}(A)$,

(2) A is m-closed if and only if $A = \mathrm{mCl}(A)$,

(3) $\mathrm{mCl}(A)$ *is m-closed and* $\mathrm{mInt}(A)$ *is m-open.*

Definition 2.9 A function $f : (X, m_X) \to (Y, m_Y)$, where (X, m_X) and (Y, m_Y) are m-spaces, is said to be *M-continuous* [26] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.

Lemma 2.4 (Popa and Noiri [26]) *Let (X, m_X) be an m-space and m have the property $(\mathcal{B})$. For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is M-continuous;

(2) $f^{-1}(V) \in m_X$ for every $V \in m_Y$;

(3) $f^{-1}(K)$ is m_X-closed for every m_Y-closed set K of Y.

3 m-D-sets

Definition 3.1 An m-space is said to be

(1) m-T_0 if for any pair of distinct points x, y of X, there exists an m-open set of X containing x but not y or an m-open set of X containing y but not x,

(2) m-T_1 if for any pair of distinct points x, y of X, there exist an m-open set of X containing x but not y and an m-open set of X containing y but not x,

(3) m-T_2 [26] if for any pair of distinct points x, y of X, there exist m-open sets U, V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 3.2 A subset A of an m-space (X, m) is called an *m-D-set* if there exist two m-open sets U and V such that $U \neq X$ and $A = U - V$.

Every m-open set different from X is an m-D-set since we can take as follows: $A = U$ and $V = \emptyset$.

Definition 3.3 An m-space is said to be

(1) m-D_0 if for any pair of distinct points x, y of X, there exists an m-D-set of X containing x but not y or an m-D-set of X containing y but not x,

(2) m-D_1 if for any pair of distinct points x, y of X, there exist an m-D-set of X containing x but not y and an m-D-set of X containing y but not x,

(3) m-D_2 if for any pair of distinct points x, y of X, there exist m-D-sets U, V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Remark 3.1 Let (X, τ) be a topological space. If $m = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, τ_θ, $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X)$), then we obtain the definitions of separation axioms D_i [28] (resp. s-D_i [4], [5], p-D_i [15], [6], θ-D_i [10], δ-$semiD_i$ [9], (δ, p)-D_i [7]) for $i = 0, 1, 2$.

Remark 3.2 Let (X, m) be an m-space. By Definitions 3.1 and 3.3, we have the following diagram:

$$\begin{array}{ccccc} m\text{-}T_2 & \Rightarrow & m\text{-}T_1 & \Rightarrow & m\text{-}T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ m\text{-}D_2 & \Rightarrow & m\text{-}D_1 & \Rightarrow & m\text{-}D_0 \end{array}$$

First, we obtain characterizations of m-T_0-spaces and m-T_1-spaces.

Theorem 3.1 *An m-space (X, m) is m-T_0 if and only if for any pair of distinct points $x, y \in X$,* $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$.

Proof. *Necessity.* Let x, y be distinct points of X. There exists (1) an m-open set U such that $x \in U$ and $y \notin U$ or (2) an m-open set V such that $y \in V$ and $x \notin V$. In case (1), $x \notin \mathrm{mCl}(\{y\})$ and hence $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$. In case (2), $y \notin \mathrm{mCl}(\{x\})$ and hence $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$.

Sufficiency. Suppose that $x, y \in X, x \neq y$ and $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$. Let z be a point of X such that $z \in \mathrm{mCl}(\{x\})$ and $z \notin \mathrm{mCl}(\{y\})$. Since $z \notin \mathrm{mCl}(\{y\})$, there exists $V \in m$ such that $z \in V$ and $y \notin V$. Since $z \in \mathrm{mCl}(\{x\})$, we have $x \in V$. In case that z is a point of X such that $z \in \mathrm{mCl}(\{y\})$ and $z \notin \mathrm{mCl}(\{x\})$, we obtain an m-open set U such that $x \notin U$ and $y \in U$. This shows that (X, m) is m-T_0.

Theorem 3.2 *Let (X, m) be an m-space and m have the property ($\mathcal{B}$). Then (X, m) is m-T_1 if and only if for each point $x \in X$, the singleton $\{x\}$ is m-closed.*

Proof. *Necessity.* Suppose that (X, m) is m-T_1 and x be any point of X. For each point $y \in X - \{x\}$, there exists an m-open set V_y such that $y \in V_y$ and $x \notin V_y$; hence $y \in V_y \subset X - \{x\}$. Therefore, we obtain $X - \{x\} = \bigcup_{y \in X - \{x\}} V_y$ and $\bigcup_{y \in X - \{x\}} V_y$ is an m-open set and hence $\{x\}$ is m-closed.

Sufficiency. Let x, y be any distinct points of X. Then $y \in X - \{x\}$ and $X - \{x\}$ is an m-open set containing y. Similarly, $X - \{y\}$ is an m-open set containing x. This shows that (X, m) is m-T_1.

Definition 3.4 An m-space (X, m) is said to be *m-symmetric* if for points x, y of X, $x \in \mathrm{mCl}(\{y\})$ implies that $y \in \mathrm{mCl}(\{x\})$.

Theorem 3.3 *Let (X, m) be an m-space and m have the property $(\mathcal{B})$. For an m-space (X, m), the following properties are equivalent:*
(1) (X, m) is m-symmetric and m-T_0;
(2) (X, m) is m-T_1.

Proof. (2) $\Rightarrow$ (1): This is obvious by Theorem 3.2.
(1) $\Rightarrow$ (2): Let x, y be distinct points of X. Since (X, m) is m-T_0, we may assume that $x \in U$ and $y \notin U$ for some $U \in m$. Then $x \notin \mathrm{mCl}(\{y\})$ and hence $y \notin \mathrm{mCl}(\{x\})$. Therefore, there exists $V \in m$ such that $y \in V$ and $x \notin V$. This shows that (X, m) is m-T_1.

Theorem 3.4 *An m-space (X, m) is m-D_0 if and only if it is m-T_0.*

Proof. By the definitions, it is obvious that m-T_0 implies m-D_0. We shall show that m-D_0 implies m-T_0. Suppose that (X, m) is m-D_0 and x and y are distinct points of X. Then at least one of these points, say x, belongs to an m-D-open set A and $y \notin A$. Let $A = U - V$, where $U \neq X$ and V, U are m-open sets. Since $y \notin A$, we have two cases: (1) $y \notin U$; (2) $y \in U$ and $y \in V$.
In case (1), we have $x \in U$, $y \notin U$ and $U \in m$.
In case (2), we have $y \in V$, $x \notin V$ and $V \in m$.
This shows that (X, m) is m-T_0.

Remark 3.3 (1) If (X, τ) is a topological space and $m = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$), then the result established in Theorem 1 of [28] (resp. Theorem 2.1 of [4] and Theorem 2.5 of [5], Theorem 2.2 of [6] and Theorem 3.1 of [15]) is obtained by Theorem 3.4.
(2) It follows from examples in [28], [4] and [6] that the separation axiom m-D_1 is strictly between m-T_0 and m-T_1.

Theorem 3.5 *Let (X, m) be an m-space and m have the property $(\mathcal{B})$. Then (X, m) is m-D_1 if and only if it is m-D_2.*

Proof. *Necessity.* Suppose that (X, m) is m-D_1 and x and y are distinct points of X. There exist two m-D-sets A_1 and A_2 such that

$x \in A_1, y \notin A_1$ and $y \in A_2, x \notin A_2$. Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$, where $U_i, V_i \in m$ for $i = 1, 2$, $U_1 \neq X$ and $U_2 \neq X$. Since $x \notin A_2$, there are two cases: (1) $x \notin U_2$ or (2) $x \in U_2$ and $x \in V_2$.

(1) $x \notin U_2$.

Since $y \notin A_1$, there are two cases: $y \notin U_1$ or $y \in U_1$ and $y \in V_1$.

(1_a) $y \notin U_1$.

Since $x \in A_1 = U_1 - V_1$ and $x \notin U_2$, we have $x \in U_1 - (V_1 \cup U_2)$. Since $y \notin U_1$ and $y \in A_2 = U_2 - V_2$, we have $y \in U_2 - (V_2 \cup U_1)$. Since m has the property ($\mathcal{B}$), $V_1 \cup U_2$ and $V_2 \cup U_1$ are m-open sets. Moreover, $U_1 \neq X$ and $U_2 \neq X$ and hence $U_1 - (V_1 \cup U_2)$ and $U_2 - (V_2 \cup U_1)$ are m-D-sets and they are disjoint.

(1_b) $y \in U_1$ and $y \in V_1$.

We have $x \in A_1 = U_1 - V_1, y \in V_1$ and $(U_1 - V_1) \cap V_1 = \emptyset$. Moreover, $(U_1 - V_1)$ and V_1 are m-D-sets.

(2) $x \in U_2$ and $x \in V_2$.

Then we have $x \in V_2, y \in A_2 = U_2 - V_2$. Moreover, V_2 and $U_2 - V_2$ are disjoint m-D-sets. Consequently, (X, m) is m-D_2.

Sufficiency. This is obvious by the definitions.

Remark 3.4 If (X, τ) is a topological space and $m = \tau$ (resp. SO(X), PO(X)), then the result established in Theorem 2 of [28] (resp. Theorem 3.3 of [4] and Theorem 2.5 of [5], Theorem 2.2 of [6] and Theorem 3.1 of [15]) is obtained by Theorem 3.5.

Corollary 3.1 *Let (X, m) be a m-symmetric m-space and m have the property ($\mathcal{B}$). For an m-space (X, m), the following axioms are equivalent: m-T_1, m-T_0, m-D_2, m-D_1 and m-D_0.*

Proof. This is an immediate consequence of Theorems 3.3, 3.4 and 3.5.

4 Properties of axiom m-D_1

Definition 4.1 Let (X, m) be an m-space. A point x of X is called an *mcc* point if X is the unique m-open set that contains x.

Remark 4.1 If (X, τ) is a topological space and $m = \tau$ (resp. SO(X), PO(X)), then an *mcc* point is called a c.c. point [28] (resp. s-c.c point [4] and sc.c point [5], pcc point [15] and pc.c point [6]).

Theorem 4.1 *If an m-space (X, m) is m-T_0, then there exists at most one mcc point.*

Proof. Suppose that (X, m) is m-T_0 and distinct points x, y are *mcc* points. Since (X, m) is m-T_0, there exists an m-open set U such that U contains one but not contains the other, say, $x \in U$ and $y \notin U$. Then $y \notin U$ and $U \neq X$. This shows that x is not an *mcc* point.

Theorem 4.2 *An m-T_0 m-space (X, m) is m-D_1 if and only if it does not have any mcc point.*

Proof. *Necessity.* Suppose that (X, m) is m-D_1. Each point $x \in X$ belongs to an m-D-set $A = U - V$, where U is an m-open set and $U \neq X$. Therefore, x is not an *mcc* point.

Sufficiency. Suppose that (X, m) is m-T_0 and it does not have any *mcc* point. Let x, y be any pair of distinct points. Then we may assume that there exists an m-open set U such that $x \in U$ and $y \notin U$. Since y is not an *mcc* point, there exists an m-open set V such that $y \in V$ and $V \neq X$. Now set $B = V - U$, then B is an m-D- set such that $y \in B$ and $x \notin B$. Therefore, (X, m) is m-D_1.

Remark 4.2 If (X, τ) is a topological space and $m = \tau$ (resp. SO(X), PO(X)), then the result established in Theorem 4 of [28] (resp. Theorem 3.2 of [4] and Theorem 2.10 of [5], Theorem 2.5 of [6] and Theorem 3.2 of [15]) is obtained by Theorem 4.2.

Theorem 4.3 *Let $f : (X, m_X) \to (Y, m_Y)$ be an M-continuous surjection and m_X satisfy the property ($\mathcal{B}$). If B is an m-D-set of (Y, m_Y), then $f^{-1}(B)$ is an m-D-set of (X, m_X).*

Proof. Let B be an m-D-set of (Y, m_Y). Then there exist m_Y-open sets U and V such that $B = U - V$ and $U \neq Y$. Since f is M-continuous, by Lemma 2.4 $f^{-1}(U)$ and $f^{-1}(V)$ are m_X-open sets. Since f is surjective, $f^{-1}(U) \neq X$ and $f^{-1}(B) = f^{-1}(U) - f^{-1}(V)$. Therefore, $f^{-1}(B)$ is an m-D-set of (X, m_X).

Remark 4.3 If (X, τ) is a topological space and $m =$ SO(X) (resp. PO(X)), then the result established in Theorem 2.18 of [5] (resp. Theorem 2.11 of [6] and Theorem 5.1 of [15]) is obtained by Theorem 4.3.

Theorem 4.4 *Let $f : (X, m_X) \to (Y, m_Y)$ be an M-continuous bijection and m_X satisfy the property ($\mathcal{B}$). If (Y, m_Y) is m-D_1, then (X, m_X) is m-D_1.*

Proof. Suppose that (Y, m_Y) is m-D_1. Let x and y be any pair of distinct points of X. Since f is injective and (Y, m_Y) is m-D_1, there exist m-D-sets B_x and B_y containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin B_x$ and $f(x) \notin B_y$. By Theorem 4.3, $f^{-1}(B_x)$ and $f^{-1}(B_y)$ are m-D-sets containing x and y, respectively, such that $y \notin f^{-1}(B_x)$ and $x \notin f^{-1}(B_y)$. This shows that (X, m_X) is m-D_1.

Remark 4.4 If (X, τ) is a topological space and $m = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$), then the result established in Theorem 2.19 of [5] (resp. Theorem 2.12 of [6] and Theorem 5.3 of [15]) is obtained by Theorem 4.4.

Theorem 4.5 *An m-space (X, m_X), where m_X satisfies the property ($\mathcal{B}$), is m-D_1 if and only if, for each pair of distinct points x, y of X, there exists an M-continuous surjection f of (X, m_X) onto an m-D_1 m-space (Y, m_Y) such that $f(x) \neq f(y)$.*

Proof. *Necessity.* For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points of X. By hypothesis, there exists an M-continuous surjection f of (X, m_X) onto an m-D_1 m-space (Y, m_Y) such that $f(x) \neq f(y)$. Since (Y, m_Y) is m-D_1, there exist m-D-sets B_x and B_y containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin B_x$ and $f(x) \notin B_y$. By Theorem 4.3, $f^{-1}(B_x)$ and $f^{-1}(B_y)$ are m-D-sets containing x and y, respectively, such that $y \notin f^{-1}(B_x)$ and $x \notin f^{-1}(B_y)$. This shows that (X, m_X) is m-D_1.

Remark 4.5 If (X, τ) is a topological space and $m = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$), then the result established in Theorem 2.20 of [5] (resp. Theorem 2.13 of [6] and Theorem 5.4 of [15]) is obtained by Theorem 4.5.

5 Applications

We recall some modified open sets: semi-θ-open, b-open, β-open and semi-preopen.

Let A be a subset of a topological space (X, τ). The semi-θ-closure of A, $\mathrm{sCl}_\theta(A)$, is defined by the set of the points such that $A \cap \mathrm{sCl}(U) \neq \emptyset$ for every semi-open set U containing x. A subset A is said to be *semi-θ-closed* [13] if $A = \mathrm{sCl}_\theta(A)$. The complement of a semi-θ-closed set is said to be *semi-θ-open*.

Definition 5.1 Let (X, τ) be a topological space. A subset A is said to be

(1) *β-open* [1] or *semi-preopen* [2] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(2) *b-open* [3] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

The family of all β-open (resp. b-open, semi-θ-open) sets in a topological space (X, τ) is denoted by $\beta(X)$ (resp. $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$).

Definition 5.2 The complement of β-open (resp. b-open) sets in a topological space (X, τ) is said to be *β-closed* [1] (resp. *b-closed* [3],).

Definition 5.3 The intersection of all β-closed (resp. b-closed) sets of (X, τ) containing a subset A is called the *β-closure* [1] (resp. *b-closure* [3]) and is denoted by $\beta\mathrm{Cl}(A)$ (resp. $\mathrm{bCl}(A)$).

Definition 5.4 The union of all β-open (resp. b-open, semi-θ-open) sets contained in a subset A of a topological space (X, τ) is called the *β-interior* (resp. *b-interior*, *semi-θ-interior*) of A and is denoted by $\beta\mathrm{Int}(A)$ (resp. $\mathrm{bInt}(A)$, $\mathrm{sInt}_\theta(A)$).

Let (X, τ) be a topological space. The families $\beta(X)$, $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$, τ_δ, τ^α are all m-structures with the property $(\mathcal{B})$. If we put $m = \beta(X)$ (resp. $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$, τ_δ, τ^α) then the pair (X, m) is an m-space, where m has the property $(\mathcal{B})$. Therefore, for each family $\beta(X)$, $\mathrm{BO}(X)$, $\mathrm{S\theta O}(X)$, τ_δ, τ^α, we can apply the results established in Sections 3-5.

Remark 5.1 For τ_θ, $\delta\mathrm{PO}(X)$ and $\delta\mathrm{SO}(X)$, the analogous results to ones established in Sections 3-5 are obtained in [10], [7] and [9], respectively.

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