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The approximation of Bitzadze-Samarsky type inverse problem for elliptic equations with Neumann conditions

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Abstract. This paper is devoted to the approximation of inverse Bitzadze-Samarsky problem for abstract elliptic differential equations with Neumann conditions. The presentation uses general approximation scheme and is based on C_0-semigroup theory and a functional analysis approach.

Key words. Abstract differential equations, abstract elliptic problem, analytic C_0-semigroups, Banach spaces, semidiscretization, inverse problem with overdetermination, well-posedness, difference schemes, discrete semigroups.

1 Introduction

In a complex Banach space E we consider the problem of finding a function $u(\cdot) \in C^2([0,T];E) \cap C([0,T];D(A))$ and an element $\varphi \in E$ from the system

$$\begin{cases} u''(t) = Au(t) + \varphi, \quad 0 \le t \le T, \\ u'(0) = x, \\ u'(T) = \sum\limits_{i=1}^{L} k_i u'(\xi_i) + y, \\ u(\theta) = z, \end{cases} \tag{1.1}$$

where $\{\xi_i\}$ is the sequence of the various numbers in the interval $(0,T)$, the number $\theta \in (0,T)$ is fixed and the coefficients $\{k_i\}$ are real, A is a closed linear operator with dense domain $D(A)$

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in the space E, the element $z \in D(A)$ is given. The investigation of Bitzadze-Samarsky type problems in abstract setting were started in [3, 4, 14].

We assume that the differential equation in (1.1) has an elliptic type. This means that A is a positive operator (see [31]) : all non positive numbers belong to its resolvent set, and the estimate

$$||(\lambda I + A)^{-1}|| \leq \frac{M}{1+\lambda} \text{ for any } \lambda \geq 0 \tag{1.2}$$

holds for some $M > 0$.

Any positive operator A possesses the positive root $B = A^{1/2}$ and operator $-A^{1/2}$ generates a strongly continuous (see [1, 19, 23]) analytic C_0-semigroup $V(t) = \exp(-t\sqrt{A})$. Moreover, for any positive values of t the spectral radius of the operator $V(t)$ is less than 1. This fact provides the invertibility of the operator $I - V(t)$ for any $t > 0$ ([5, 9, 10]). The value of semigroup $V(\cdot)$ is determined through the Green's function of the basic boundary value problem.

The problem (1.1) is inverse problem with overdetermination. One can find details of description of such class of problems in [13, 15–21, 24–27].

2 Solution of direct problem

Inverse problem (1.1) is related to the direct problem which were considered in [4, 14]

$$\begin{cases} u''(t) = Au(t) + f(t), \quad 0 \leq t \leq T, \\ u'(0) = x, \\ u'(T) = \sum\limits_{i=1}^{L} k_i u'(\xi_i) + y. \end{cases} \tag{2.1}$$

The solution of problem (2.1) is connected with Neumann problem

$$\begin{cases} u''(t) = Au(t) + f(t), \quad 0 \leq t \leq T, \\ u'(0) = \tilde{u}^0, \\ u'(T) = \tilde{u}^T. \end{cases} \tag{2.2}$$

The Green's function of problem (2.2) has the form

$$G(t,s) = \frac{1}{2} A^{-1/2}(V(2T) - I)^{-1}\Big(V(t+s) + V(|t-s|) + V(2T-t-s) + V(2T-|t-s|)\Big). \tag{2.3}$$

Assume that $\tilde{u}^0, \tilde{u}^T \in D(A^{1/2})$, $f(\cdot) \in C^1([0,T];E)$, then the solution of problem (2.2) exists, is unique and is given by the formula (see [10–12])

$$u(t) = (V(2T) - I)^{-1} \left[(V(t) + V(2T-t))A^{-1/2}\tilde{u}^0 - (V(T-t) + V(T+t))A^{-1/2}\tilde{u}^T\right]$$

$$+ \int_0^T G(t,s)f(s)ds. \tag{2.4}$$

One also has

$$u'(t) = (I - V(2T))^{-1}\Big((V(t) - V(2T-t))\tilde{u}^0 + (V(T-t) - V(T+t))\tilde{u}^T\Big) + \int_0^T G_t(t,s)f(s)ds. \tag{2.5}$$

The solvability of problem (2.1) is closely related to the distribution of zeros of the entire function

$$\eta(w) = 1 - e^{-2Tw} - \sum_{i=1}^{L} k_i \left(e^{-(T-\xi_i)w} - e^{-(T+\xi_i)w}\right). \tag{2.6}$$

Definition 2.1 *Function* (2.6) *is called the characteristic function of problem* (2.1).

Theorem 2.1 ([14]) *Let* $x, y \in D(A^{1/2})$, $f(\cdot) \in C^1([0,T];E)$. *Assume also that characteristic function* (2.6) *is not vanished in the half-plane* $\operatorname{Re} z > 0$. *Then there is a unique solution of problem* (2.1).

Corollary 2.1 ([14]) *Let* $x, y \in D(A^{1/2})$, $f(\cdot) \in C^1([0,T];E)$, $L = 1$, $|k_1| \leq 1$. *Then there is a unique solution of problem* (2.1).

Corollary 2.2 ([14]) *Let* E *be a Hilbert space, the operator* A *be self-adjoint,* $x, y \in D(A^{1/2})$, $f(\cdot) \in C^1([0,T];E)$. *Assume that characteristic function* (2.6) *is not vanished on the positive semi-axis of the real line. Then there is a unique solution of problem* (2.1).

Theorem 2.2 ([14]) *Let* E *be a Hilbert space, the operator* A *be self-adjoint and positive definite,* $x, y \in D(A^{1/2})$, $f(\cdot) \in C^1([0,T];E)$. *Assume also that a condition*

$$\sum_{i=1}^{L} (|k_i| + k_i) \leq 2 \tag{2.7}$$

is satisfied. Then there is a unique solution of problem (2.1).

3 Solution of inverse problem

Move on to the analysis of inverse problem (1.1). One was considered in the work [14].

Lemma 3.1 ([14]) *For the Green's function* (2.3) *of problem* (2.1) *the following identities are true*

$$\int_0^T G(t,s)ds = -A^{-1}, \quad \int_0^T G_t(t,s)ds = 0. \tag{3.1}$$

Because of Lemma 3.1 formulas (2.4) and (2.5) are converted into the following

$$\begin{aligned} u(t) &= (V(2T)-I)^{-1}\Big((V(t)+V(2T-t))A^{-1/2}x \\ &\quad -(V(T-t)+V(T+t))A^{-1/2}u'(T)\Big) - A^{-1}\varphi, \end{aligned} \tag{3.2}$$

and

$$u'(t) = (I-V(2T))^{-1}\Big((V(t)-V(2T-t))x + (V(T-t)-V(T+t)u'(T))\Big). \tag{3.3}$$

In the problem (1.1) the last two equality leads to the following system to find the unknown elements $u'(T) \in D(A^{1/2})$ and $\varphi \in E$

$$u'(T) = \sum_{i=1}^{L} k_i\Big((I-V(2T))^{-1}((V(\xi_i)-V(2T-\xi_i))x$$

$$+(V(T-\xi_i)-V(T+\xi_i))u'(T))(I-V(2T))^{-1}\Big) + y,$$

$$(V(2T)-I)^{-1}\Big((V(\theta)+V(2T-\theta))A^{-1/2}x$$

$$-(V(T-\theta)+V(T+\theta))A^{-1/2}u'(T)A^{-1/2}\Big) - A^{-1}\varphi = z.$$

Multiplying the first equation by $I-V(2T)$, and the second by A, we arrive at the following system

$$\begin{cases} \Psi u'(T) = h, \\ \varphi = (V(2T)-I)^{-1}\Big((V(\theta)+V(2T-\theta))A^{1/2}x \\ \qquad -(V(T-\theta)+V(T+\theta))A^{1/2}u'(T)\Big) - Az, \end{cases} \tag{3.4}$$

where

$$\Psi = I - V(2T) - \sum_{i=1}^{n} k_i(V(T-\xi_i)-V(T+\xi_i)),$$

and

$$h = \sum_{i=1}^{n} k_i\left\{(V(\xi_i)-V(2T-\xi_i))x\right\} + (I-V(2T))y.$$

Thus inverse problem (1.1) also is reduced to the equation $\Psi u'(T) = h$. Solving this equation and substituting $u'(T)$ in the second equation of the system (3.4) we find the unknown element

$$\varphi = (V(2T)-I)^{-1}\Big((V(\theta)+V(2T-\theta))A^{1/2}x - (V(T-\theta)+V(T+\theta))A^{1/2}\Psi^{-1}h\Big) - Az. \tag{3.5}$$

The question of invertibility of the operator Ψ was studied in the work [14]. The results of this work leads to the following statements

Theorem 3.1 ([14]) *Let $x, y \in D(A^{1/2})$, $z \in D(A)$. Assume that characteristic function* (2.6) *has zeros only in the half-plane* $\operatorname{Re} z \leq 0$. *Then there is a unique solution of inverse problem* (1.1).

Corollary 3.1 ([14]) *Let $x, y \in D(A^{1/2})$, $z \in D(A)$, $L = 1$ and $|k_1| \leq 1$. Then there is unique solution of problem* (1.1).

Theorem 3.2 ([14]) *Let E be a Hilbert space, the operator A be self-adjoint and positive definite, elements of $x, y \in D(A^{1/2})$, $z \in D(A)$. Assume that condition* (2.7) *is satisfied. Then there is a unique solution of inverse problem* (1.1).

4 Discretization of operators

Here we are following the approach of [20]. Semidiscrete and full discretization schemes will be considered.

4.1 General approximation scheme

The general approximation scheme, due to [7, 28–30] can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n$, $p_n \in B(E, E_n), n \in \mathbb{N} = \{1, 2, \cdots\}$, with the property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E.$$

Definition 4.1 *The sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-convergent to $x \in E$ if and only if $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$ and we write this $x_n \xrightarrow{\mathcal{P}} x$.*

Definition 4.2 *The sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-compact if for any $\mathbb{N}' \subseteq \mathbb{N}$ there exist $\mathbb{N}'' \subseteq \mathbb{N}'$ and $x \in E$ such that $x_n \xrightarrow{\mathcal{P}} x$, as $n \to \infty$ in $\mathbb{N}''$.*

Definition 4.3 *The sequence of bounded linear operators $B_n \in B(E_n), n \in \mathbb{N}$, is said to be $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We write then $B_n \xrightarrow{\mathcal{PP}} B$.*

For analytic C_0-semigroups the following theorem holds.

Theorem 4.1 ([22]) Let operators A and A_n generate analytic C_0-semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1).

(A) Compatibility. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvent converges $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$,

(B_1) Stability. There are some constants $M \geq 1$ and ω such that

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \ Re\lambda > \omega, n \in I\!N,$$

(C_1) Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$ we have

$$\max_{\eta \in \Sigma(\theta,\mu)} \| \exp(\eta A_n) u_n^0 - p_n \exp(\eta A) u^0 \| \to 0$$

as $n \to \infty$ whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$. Here $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$, and $\Sigma(\theta) = \{z \in \mathbb{C} : |arg\, z| \leq \theta\}$.

Normally they assume that conditions (A) and (B_1) for the corresponding C_0-semigroup case are satisfied without any restriction of generality if any discretization processes in time are considered.

Definition 4.4 *A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in I\!N$, is said to be stably convergent to an operator $B \in B(E)$ if and only if $B_n \xrightarrow{\mathcal{PP}} B$ and $\|B_n^{-1}\|_{B(E_n)} = O(1)$, $n \to \infty$. We will write this as: $B_n \xrightarrow{\mathcal{PP}} B$ stably.*

Definition 4.5 *A sequence of operators $\{B_n\}$, $B_n \in B(E_n)$, is called regularly convergent to the operator $B \in B(E)$ if and only if $B_n \xrightarrow{\mathcal{PP}} B$ and the following implication holds:*

$$\|x_n\|_{E_n} = O(1) \ \& \ \{B_n x_n\} \textit{ is } \mathcal{P}\textit{-compact} \Longrightarrow \{x_n\} \textit{ is } \mathcal{P}\textit{-compact.}$$

We write this as: $B_n \xrightarrow{\mathcal{PP}} B$ regularly.

Theorem 4.2 ([30]) Let $C_n, Q_n \in B(E_n)$, $C, Q \in B(E)$ and $\mathcal{R}(Q) = E$. Assume also that $C_n \xrightarrow{\mathcal{PP}} C$ compactly and $Q_n \xrightarrow{\mathcal{PP}} Q$ stably. Then $Q_n + C_n \xrightarrow{\mathcal{PP}} Q + C$ converge regularly.

Theorem 4.3 ([30]) For $Q_n \in B(E_n)$ and $Q \in B(E)$ the following conditions are equivalent:

(i) $Q_n \xrightarrow{\mathcal{PP}} Q$ regularly, Q_n are Fredholm operators of index 0 and $\mathcal{N}(Q) = \{0\}$,

(ii) $Q_n \xrightarrow{\mathcal{PP}} Q$ stably and $\mathcal{R}(Q) = E$,

(iii) $Q_n \xrightarrow{\mathcal{PP}} Q$ stably and regularly,

(iv) if one of conditions (i)–(iii) holds, then there exist $Q_n^{-1} \in B(E_n)$, $Q^{-1} \in B(E)$, and $Q_n^{-1} \xrightarrow{\mathcal{PP}} Q^{-1}$ regularly and stably.

Definition 4.6 *The region of stability* $\Delta_s = \Delta_s(\{A_n\})$, $A_n \in \mathcal{C}(B_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \rho(A_n)$ *for almost all* n *and such that the sequence* $\{\|(\lambda I_n - A_n)^{-1}\|\}_{n\in\mathbb{N}}$ *is bounded for almost all* n. *The region of convergence* $\Delta_c = \Delta_c(\{A_n\})$, $A_n \in \mathcal{C}(E_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \Delta_s(\{A_n\})$ *and such that the sequence of operators* $\{(\lambda I_n - A_n)^{-1}\}_{n\in\mathbb{N}}$ *is* $\mathcal{PP}$*-convergent to some operator* $S(\lambda) \in B(E)$.

Definition 4.7 *The region of compact convergence of resolvents,* $\Delta_{cc} = \Delta_{cc}(A_n, A)$, *where* $A_n \in \mathcal{C}(E_n)$ *and* $A \in \mathcal{C}(E)$ *is defined as the set of all* $\lambda \in \Delta_c \cap \rho(A)$ *such that* $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ *compactly.*

Theorem 4.4 ([8]) Assume that $\Delta_{cc} \neq \emptyset$. Then for any $\zeta \in \Delta_s$ the following implication holds:

$$\|x_n\|_{E_n} = O(1) \ \& \ \|(\zeta I_n - A_n)x_n\|_{E_n} = O(1) \Longrightarrow \{x_n\} \,\text{is}\, \mathcal{P}\text{-compact}. \tag{4.1}$$

Conversely, if for some $\zeta \in \Delta_c \cap \rho(A)$ implication (4.1) holds, then $\Delta_{cc} \neq \emptyset$.

4.2 Discretization of abstract elliptic problem

One can consider the Neumann problems in Banach spaces E_n:

$$u_n''(t) = A_n u_n(t) + \varphi_n,\ t \in [0,T],\ u_n'(0) = \tilde{u}_n^0,\ u_n'(T) = \tilde{u}_n^T, \tag{4.2}$$

with strongly positive operators A_n, A_n and A are compatible, $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $u_n^T \xrightarrow{\mathcal{P}} u^T$. We are going to describe here also the discretization of (4.2) in variable t. One of the simplest difference scheme is

$$\begin{cases} \frac{U_n^{k+1} - 2U_n^k + U_n^{k-1}}{\tau_n^2} = A_n U_n^k + \bar{\varphi}_n, \quad k \in \{1, \cdots, [\frac{T}{\tau_n}] - 1\}, \\ U_n^1 - U_n^0 = \tau_n \tilde{u}_n^0,\ U_n^K - U_n^{K-1} = \tau_n \tilde{u}_n^T, \end{cases} \tag{4.3}$$

where $\tau_n = T/K$ ($K \in \mathbb{N}$).

The solution of (4.3) is given by

$$U_n^k = -A_n^{-1}\bar{\varphi}_n - (I_n - R_n)^{-1}(I_n - R_n^{2K})^{-1}(I_n - R_n^{2K-2})^{-1} \tag{4.4}$$

$$\times\Big(\Big((R_n^k + R_n^{2K-k})(I_n - R_n^{2K-1}) + (R_n^{K-k} + R_n^{K+k})(R_n^{K-1} - R_n^K)\Big)\tau_n \tilde{u}_n^0$$

$$-\Big((R_n^k + R_n^{2K-k})(R_n^{K-1} - R_n^K) + (R_n^{K-k} + R_n^{K+k})(I_n - R_n^{2K-1})\Big)\tau_n \tilde{u}_n^T\Big),$$

where

$$R_n = (I_n + \tau_n B_n)^{-1},\ B_n = \frac{A_n\tau_n + \sqrt{A_n}\sqrt{\tau_n^2 A_n + 4}}{2}. \tag{4.5}$$

Indeed, the general solution of equation in (4.3) can be written as

$$U_n^k = (R_n^k + R_n^{2K-k})a + (R_n^{K-k} + R_n^{K+k})b - A_n^{-1}\bar{\varphi}_n,$$

where the system to find a, b looks like

$$\begin{cases} (R_n - I_n)(I_n - R^{2K-1})a + (I_n - R_n)^2 R_n^{K-1} b = \tau_n \tilde{u}_n^0, \\ -(I_n - R_n)^2 R_n^{K-1}\, a - (R_n - I_n)(I_n - R^{2K-1})b = \tau_n \tilde{u}_n^T. \end{cases} \tag{4.6}$$

The determinant of system (4.6) equals to

$$\Delta = -(I_n - R_n)^2 (I_n - R^{2K})(I_n - R^{2K-2}).$$

Thus

$$a = \Delta^{-1}\Big(-(R_n - I_n)(I_n - R^{2K-1})\tau_n \tilde{u}_n^0 - (I_n - R_n)^2 R_n^{K-1} \tau_n \tilde{u}_n^T, \Big),$$

$$b = \Delta^{-1}\Big((I_n - R_n)^2 R_n^{K-1} \tau_n \tilde{u}_n^0 + (R_n - I_n)(I_n - R^{2K-1})\tau_n \tilde{u}_n^T \Big).$$

So one gets (4.4).

Theorem 4.5 *Assume that conditions* (A) *and* (B_1) *are satisfied, resolvents* $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ *are compact and* $\Delta_{cc} \neq \emptyset$. *Assume also that* $\tilde{u}_n^0 \xrightarrow{\mathcal{P}} \tilde{u}^0, \tilde{u}_n^T \xrightarrow{\mathcal{P}} \tilde{u}^T$ *and* $\varphi_n \xrightarrow{\mathcal{P}} f(\cdot) = \varphi$. *Then solutions of* (4.2) *converge to solution of* (2.2), *i.e.* $u_n(t) \xrightarrow{\mathcal{P}} u(t)$ *uniformly in* $t \in [0, T]$.

Proof. Since $\exp(tB_n) \xrightarrow{\mathcal{PP}} \exp(tB)$ compactly for any $t > 0$, it follows that the solutions of (4.2) converge to solution of (2.4). Theorem is proved.

Theorem 4.6 *Assume that conditions* (A) *and* (B_1) *are satisfied, resolvents* $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ *are compact and* $\Delta_{cc} \neq \emptyset$. *Assume also that* $\tilde{u}_n^0 \xrightarrow{\mathcal{P}} \tilde{u}^0, \tilde{u}_n^T \xrightarrow{\mathcal{P}} \tilde{u}^T$ *and* $\bar{\varphi}_n \xrightarrow{\mathcal{P}} f(\cdot) = \bar{\varphi}$. *Then solutions of* (4.3) *converge to solution of* (2.2), *i.e.* $U_n^k \xrightarrow{\mathcal{P}} u(t)$ *uniformly in* $t = k\tau_n \in [0, T]$.

Proof. Compact convergence $U_n^k \xrightarrow{\mathcal{PP}} \exp(tB)$ for any $k\tau_n = t > 0$ implies convergence of solutions of (4.4) to solution of (2.4). Indeed, one can rewrite (4.4) in the form

$$U_n^k = -A_n^{-1}\bar{\varphi}_n -$$

$$-(I_n - R_n)^{-1}(I_n - R_n^{2K-2})^{-1}\Big((R_n^k + R_n^{2K-k-1})\tau_n \tilde{u}_n^0 - (R_n^{K-k} + R_n^{K+k-1})\tau_n \tilde{u}_n^T \Big)$$

which is completely correspond to (3.2). The term $(I_n - R_n)^{-1}$ gives us $A_n^{-1/2}$. It is clear that $I_n - R_n^{2K-2} \xrightarrow{\mathcal{PP}} I - V(2T)$ stably. Theorem is proved.

5 Convergence of solutions

In the Banach spaces E_n, let us consider the problem of finding the functions $u_n(\cdot) \in C^2([0;T];E_n)\cap C([0;T];D(A_n))$ and an element $\varphi_n \in E$ from the system of equations

$$\begin{cases} u_n''(t) = A_n u_n(t) + \varphi_n, \quad 0 \le t \le T, \\ u_n'(0) = x_n, \\ u_n'(T) = \sum\limits_{i=1}^{L} k_i u_n'(\xi_i) + y_n, \\ u_n(\theta) = z_n, \end{cases} \tag{5.1}$$

where the points $\{\xi_i\} \subseteq (0,T), \theta \in (0,T), \{k_i\}$ are chosen in the same way as in (1.1), the operators A_n are satisfied conditions $(A), (B_1)$ from Theorem 4.1, and the elements $x_n, y_n, z_n \in D(A_n)$ are such that $x_n \xrightarrow{\mathcal{P}} x, y_n \xrightarrow{\mathcal{P}} y, z_n \xrightarrow{\mathcal{P}} z$ and $A_n x_n \xrightarrow{\mathcal{P}} Ax, A_n y_n \xrightarrow{\mathcal{P}} Ay, A_n z_n \xrightarrow{\mathcal{P}} Az$.

Theorem 5.1 *Assume that resolvents $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ are compact, (B_1) and (1.2) are satisfied and $\Delta_{cc} \neq \emptyset$. Assume also that characteristic function (2.6) has zeros only in a half-plane $\operatorname{Re} z \le 0$. Then solutions of problem (5.1) exist for almost all n and they converge to solution of problem (1.1), i.e. $u_n(t) \xrightarrow{\mathcal{P}} u(t)$ uniformly in $t \in [0,T]$ and $\varphi_n \xrightarrow{\mathcal{P}} \varphi$ as $n \in \mathbb{N}$, whenever $A_n u_n^\theta \xrightarrow{\mathcal{P}} A u^\theta$.*

Proof. We note that one can write the solutions of approximated problem following (3.4) in the form

$$\begin{cases} \Psi_n u_n'(T) = h_n, \\ \varphi_n = (V_n(2T) - I_n)^{-1}\Big((V_n(\theta) + V_n(2T-\theta))A_n^{-1/2} x_n \\ \qquad -(V_n(T-\theta) + V_n(T+\theta))A_n^{-1/2} u_n'(T)\Big) - A_n z_n, \end{cases} \tag{5.2}$$

where $\Psi_n = I_n - V_n(2T) - \sum\limits_{i=1}^{L} k_i (V_n(T-\xi_i) - V_n(T+\xi_i))$ and $h_n = \sum\limits_{i=1}^{L} k_i (V_n(\xi_i) - V_n(2T-\xi_i)) x_n + y_n$. Since $V_n(t) \xrightarrow{\mathcal{PP}} V(t)$ compactly for any $t > 0$ we get from Theorems 4.2 - 4.3 that $\Psi_n \xrightarrow{\mathcal{PP}} \Psi$ and $I_n - V_n(2T) \xrightarrow{\mathcal{PP}} I - V(2T)$ stably, i.e. $\Psi_n^{-1} \xrightarrow{\mathcal{PP}} \Psi^{-1}, (I_n - V_n(2T))^{-1} \xrightarrow{\mathcal{PP}} (I - V(2T))^{-1}$. It follows that $\varphi_n \xrightarrow{\mathcal{P}} \varphi$. Theorem is proved.

Following (4.3) one can consider approximation of (5.1) in the form

$$\begin{cases} \frac{U_n^{k+1} - 2U_n^k + U_n^{k-1}}{\tau_n^2} = A_n U_n^k + \bar{\varphi}_n, \quad k \in \{1, ..., [\frac{T}{\tau_n}] - 1\}, \\ U_n^1 - U_n^0 = \tau_n x_n, \\ U_n^K - U_n^{K-1} = \sum\limits_{i=1}^{L} k_i (U_n^{[\xi_i/\tau_n]+1} - U_n^{[\xi_i/\tau_n]}) + \tau_n y_n, \\ U_n^{[\theta/\tau_n]} = z_n, \end{cases} \tag{5.3}$$

Remark 5.1 *We have in* (5.3) *the approximation of order* $O(\tau_n)$, *but differential equation in* (5.3) *is approximated with order* $O(\tau_n^2)$. *If we change the scheme and approximate* (5.3) *with order* $O(\tau_n^2)$ *the stability will follow the same way as in our case and we get a convergence. Complexity of calculations in this case is dramatically increased.*

Theorem 5.2 *Assume that resolvents* $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ *are compact,* (B_1) *and* (1.2) *are satisfied and* $\Delta_{cc} \neq \emptyset$. *Assume also that characteristic function* (2.6) *has zeros only in a half-plane* $\operatorname{Re} z \leq 0$. *Then solutions of problem* (5.3) *exist for almost all* n *and they converge to solution of problem* (1.1), *i.e.* $U_n^k \xrightarrow{\mathcal{P}} u(t)$ *uniformly in* $t = k\tau_n \in [0, T]$ *and* $\bar{\varphi}_n \xrightarrow{\mathcal{P}} \varphi$ *as* $n \in I\!N$, *whenever* $A_n u_n^\theta \xrightarrow{\mathcal{P}} A u^\theta$.

Proof. To find the function U_n^k which satisfies system (5.3), we start from the discrete solution of Neumann problem. One can find that the solution of (4.3) with some $U_n^1 - U_n^0 = \tau_n x_n$ and $U_n^K - U_n^{K-1} = \tau_n(\partial_{\tau_n} U_n)^{K-1}$ is given by formula (4.4). Now, substituting this function into conditions of (5.3) with $k = [\xi_i/\tau_n], k = [\xi_i/\tau_n] + 1$ and $k = [\theta/\tau_n]$ one gets the system to find φ_n and $\tau_n(\partial_{\tau_n} U_n)^{K-1}$:

$$\overline{\Psi}_n \tau_n (\partial_{\tau_n} U_n)^{K-1} = \overline{h}_n, \tag{5.4}$$

$$\begin{aligned}\varphi_n = -A_n(I_n - R_n)^{-1}(I_n - R_n^{2K})^{-1}(I_n - R_n^{2K-2})^{-1}\Bigg(\Big((R_n^{[\theta/\tau_n]} + R_n^{2K-[\theta/\tau_n]})(I_n - R_n^{2K-1})\\ +(R_n^{K-[\theta/\tau_n]} + R_n^{K+[\theta/\tau_n]})(R_n^{K-1} - R_n^K)\Big)\tau_n x_n - \Big((R_n^{[\theta/\tau_n]} + R_n^{2K-[\theta/\tau_n]})\\ \times(R_n^{K-1} - R_n^K) + (R_n^{K-[\theta/\tau_n]} + R_n^{K+[\theta/\tau_n]})(I_n - R_n^{2K-1})\Big)\tau_n(\partial_{\tau_n} U_n)^{K-1}\Bigg) - A_n z_n,\end{aligned}$$

or

$$\begin{aligned}\varphi_n = -A_n(I_n - R_n)^{-1}(I_n - R_n^{2K-2})^{-1}\Bigg(\Big(R_n^{[\theta/\tau_n]} + R_n^{2K-[\theta/\tau_n]-1}\Big)\tau_n x_n\\ - \Big(R_n^{K-[\theta/\tau_n]} + R_n^{K+[\theta/\tau_n]-1}\Big)\tau_n(\partial_{\tau_n} U_n)^{K-1}\Bigg) - A_n z_n,\end{aligned} \tag{5.5}$$

where

$$\overline{\Psi}_n = I_n - \delta\Sigma_{i=1}^L k_i\Big(C_{\xi_i+1} + D_{\xi_i+1} - C_{\xi_i} - D_{\xi_i}\Big),$$

$$\overline{h}_n = \delta\sum_{i=1}^L k_i\Big(- A_{\xi_i+1} - B_{\xi_i+1} + A_{\xi_i} + B_{\xi_i}\Big)\tau_n x_n + \tau_n y_n,$$

$$\delta = (I_n - R_n)^{-1}(I_n - R_n^{2K})^{-1}(I_n - R_n^{2K-2})^{-1}, \quad A_{\xi_i} = (R_n^{[\xi_i/\tau_n]} + R_n^{2K-[\xi_i/\tau_n]})(I_n - R_n^{2K-1}),$$

$$B_{\xi_i} = (R_n^{K-[\xi_i/\tau_n]} + R_n^{K+[\xi_i/\tau_n]})(R_n^{K-1} - R_n^K), \quad C_{\xi_i} = (R_n^{[\xi_i/\tau_n]} + R_n^{2K-[\xi_i/\tau_n]})(R_n^{K-1} - R_n^K),$$

$$D_{\xi_i} = (R_n^{K-[\xi_i/\tau_n]} + R_n^{K+[\xi_i/\tau_n]})(I_n - R_n^{2K-1}).$$

Tiresome calculations give us

$$\overline{\Psi}_n = (I_n - R_n^{2K-2})^{-1}\Big(I_n - R_n^{2K-2} - \sum_{i=1}^{L} k_i(R_n^{K-[\xi_i/\tau_n]-1} + R_n^{K+[\xi_i/\tau_n]-1})\Big),$$

$$\overline{h}_n = (I_n - R_n^{2K-2})^{-1}\Big(\sum_{i=1}^{L} k_i(R_n^{[\xi_i/\tau_n]} - R_n^{2K-[\xi_i/\tau_n]-2})\tau_n x_n\Big) + \tau_n y_n.$$

Using the same way as in the proof of Theorem 5.1, we can show that $\overline{\Psi}_n = (I_n - R_n^{2K-2})^{-1}\Big(I_n - R_n^{2K-2} - \Sigma_{i=1}^{L} k_i(R_n^{K-[\xi_i/\tau_n]-1} + R_n^{K+[\xi_i/\tau_n]-1})\Big)\xrightarrow{\mathcal{PP}}(I - V(2T))^{-1}\Psi$ stably, i.e. $\overline{\Psi}_n^{-1}\xrightarrow{\mathcal{PP}}(I - V(2T))\Psi^{-1}$. Indeed, $R_n^{2K-2}\xrightarrow{\mathcal{PP}}V(2T)$ compactly, since $\|k\tau_n B_n(I_n + \tau_n B_n)^{-k}\| \le$ *constant* (see [2]). Hence, $I_n - R_n^{2K-2}\xrightarrow{\mathcal{PP}}I - V(2T)$ stably and $(I_n - R_n^{2T-1})\overline{h}_n\xrightarrow{\mathcal{P}}h$. Thus $\bar{\varphi}_n\xrightarrow{\mathcal{P}}\varphi$. Theorem is proved.

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