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On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions

Mehmet Zeki Sarıkaya, Necmettin Alp and Hakan Bozkurt

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
e-mail: sarikayamz@gmail.com, placenn@gmail.com, insedi@yahoo.com

Abstract. In this paper, we extend some estimates of the left hand side of a Hermite-Hadamard type inequality for nonconvex functions whose derivatives' absolute values are preinvex and log-preinvex.

Key words. Hermite-Hadamard's inequalities, non-convex functions, invex sets, Hölder's inequality.

1 Introduction

The following inequality is well-known in the literature as Hermite-Hadamard inequality: Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if the function f is concave.

Inequalities in (1.1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Recently, Hermite-Hadamard type inequality has been the subject of intensive research. For recent results, refinements, counterparts, generalizations and new Hadamard's-type inequalities, we refer [1, 2, 8–11, 16–21].

In [8], some inequalities of Hermite-Hadamard type for differentiable convex mappings connected with the left part of (1.1) were proved using the following lemma:

Lemma 1.1 *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ (I° is the interior of I) with $a < b$. If $f' \in L([a,b])$, then we have*

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ = \;& (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)dt\right]. \end{aligned} \tag{1.2}$$

One more general result related to (1.2) was established in [9]. The main result in [8] is as follows.

Theorem 1.2 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If the mapping $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right). \tag{1.3}$$

It is well known that convexity plays a key role in mathematical programming, engineering, and optimization theory. The generalization of convexity is one of the most important aspects in mathematical programming and optimization theory. There have been many attempts to weaken the convexity assumptions in the literature (see, [1,2,8–11,16–21]). A significant generalization of convex functions is that of invex functions introduced by Hanson in [12]. Ben-Israel and Mond [14] introduced the concept of preinvex functions, which is a special case of invexity. Pini [15] introduced the concept of prequasiinvex functions as a generalization of invex functions. Noor [5–7] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Barani, Ghazanfari, and Dragomir in [3] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some preinvex functions are involved. His class of nonconvex functions include the classical convex functions and its various classes as special cases. For some recent results related to this nonconvex functions, see the papers [4–7, 12–15].

2 Preliminaries

Let $f : K \to \mathbb{R}^n$, and $\eta(.,.) : K \times K \to \mathbb{R}^n$, where K is a nonempty closed set in $\mathbb{R}^n$ be continuous functions. First of all, we recall the following well known results and concepts given in [4–7, 13] and the references therein.

Definition 2.1 *Let $u,v\in K$. Then the set K is said to be invex at u with respect to $\eta(.,.)$, if*

$$u+t\eta(v,u)\in K,\ \forall u,v\in K,\ t\in[0,1].$$

K is said to be an invex set with respect to η, if K is invex at each $u,v\in K$. The invex set K is also called η-connected set.

Remark 2.1 *We would like to mention that Definition 2.1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points, $u,v\in K$, then $\eta(v,u)=v-u$ and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(v,u)=v-u$, but the converse is not necessarily true, see* [4–7] *and the references therein.*

Definition 2.2 *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u+t\eta(v,u))\leq(1-t)f(u)+tf(v),\ \forall u,v\in K,\ t\in[0,1].$$

The function f is said to be preconcave with respect to η if and only if $-f$ is preinvex. Note that every convex function is an preinvex function, but the converse is not true.

Definition 2.3 *The function f on the invex set K is said to be logarithmic preinvex with respect to η, such that*

$$f(u+t\eta(v,u))\leq(f(u))^{1-t}(f(v))^{t},\ u,v\in K,\ t\in[0,1],$$

where $f(.)>0$.

Now we define a new definition for prequasiinvex functions as follows:

Definition 2.4 *The function f on the invex set K is said to be prequasiinvex with respect to η, if*

$$f(u+t\eta(v,u))\leq\max\{f(u),f(v)\},\ u,v\in K,\ t\in[0,1].$$

From Definition 2.3 and Hölder preliminary inequality, we have

$$\begin{aligned}f(u+t\eta(v,u)) &\leq (f(u))^{1-t}(f(v))^{t}\\ &\leq (1-t)f(u)+tf(v)\\ &\leq \max\{f(u),f(v)\}.\end{aligned}$$

We also need the following assumption regarding the function η which is due to Mohan and Neogy [13].

Condition C Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. For any $x, y \in K$ and any $t \in [0,1]$,

$$\begin{aligned} \eta(y, y + t\eta(x,y)) &= -t\eta(x,y), \\ \eta(x, y + t\eta(x,y)) &= (1-t)\,\eta(x,y). \end{aligned}$$

We say that η satisfies Condition C, if for any $x, y \in K$ and $t_1, t_2 \in [0,1]$, we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\,\eta(x,y). \tag{2.1}$$

In [5], Noor proved the Hermite-Hadamard inequality for the preinvex functions as follows.

Theorem 2.2 *Let $f : K = [a, a + \eta(b,a)] \to (0, \infty)$ be an preinvex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + \eta(b,a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a) + f(a + \eta(b,a))}{2} \leq \frac{f(a) + f(b)}{2}. \tag{2.2}$$

In [3], Barani, Gahazanfari and Dragomir proved the following theorems.

Theorem 2.3 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function.Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is prequasiinvex on A, then for every $a, b \in A$ the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(a,b)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \leq\ & \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} \left[\sup\left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right]^{\frac{p}{p-1}}. \end{aligned}$$

Theorem 2.4 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex on A, then for every $a, b \in A$ the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(a,b)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \leq\ & \frac{\eta(b,a)}{4} \max\{|f'(a)|, |f'(b)|\}. \end{aligned}$$

In this article, using functions whose derivatives' absolute values are preinvex and log-preinvex, we obtained new inequalities related to the left side of Hermite-Hadamard inequality for non-convex functions.

3 Hermite-Hadamard type inequalities for preinvex functions

We shall start with the following refinements of the Hermite-Hadamard inequality for preinvex functions. Firstly, we give the following results connected with the left part of (2.2):

Theorem 3.1 *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on K, then for every $a, b \in K$ the following inequality holds:*

$$\left| \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right) \right| \leq \frac{\eta(b,a)}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}}$$

$$\times \left[\left(3\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(\left|f'(a)\right|^{\frac{p}{p-1}} + 3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \right]. \tag{3.1}$$

Proof. Since K is invex with respect to η, for every $t \in [0,1]$, we have $a + t\eta(b,a) \in K$. Integrating by parts implies that

$$\begin{aligned}
& \int_0^{\frac{1}{2}} t f'(a+t\eta(b,a))dt + \int_{\frac{1}{2}}^{1} (t-1) f'(a+t\eta(b,a))dt \\
= & \left[\frac{t f(a+t\eta(b,a))}{\eta(b,a)}\right]_0^{\frac{1}{2}} + \left[\frac{(t-1) f(a+t\eta(b,a))}{\eta(b,a)}\right]_{\frac{1}{2}}^{1} \\
& - \frac{1}{\eta(b,a)} \int_0^1 f(a+t\eta(b,a))dt \\
= & \frac{1}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{[\eta(b,a)]^2} \int_a^{a+\eta(b,a)} f(x)dx. \qquad (3.2)
\end{aligned}$$

From Hölder's inequality and (3.2), we have

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \eta(b,a)\left[\int_0^{\frac{1}{2}} t\left|f'(a+t\eta(b,a))\right|dt + \int_{\frac{1}{2}}^1 (1-t)\left|f'(a+t\eta(b,a))\right|dt\right] \\
\leq\ & \eta(b,a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \left|f'(a+t\eta(b,a))\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^p\, dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 \left|f'(a+t\eta(b,a))\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}\right] \\
\leq\ & \frac{\eta(b,a)}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}} \left[(1-t)\left|f'(a)\right|^{\frac{p}{p-1}} + t\left|f'(b)\right|^{\frac{p}{p-1}}\right]dt\right)^{\frac{p-1}{p}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^1 \left[(1-t)\left|f'(a)\right|^{\frac{p}{p-1}} + t\left|f'(b)\right|^{\frac{p}{p-1}}\right]dt\right)^{\frac{p-1}{p}}\right] \\
=\ & \frac{\eta(b,a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(\left|f'(a)\right|^{\frac{p}{p-1}} + 3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right]
\end{aligned}$$

which completes the proof. ■

Theorem 3.2 *Under the assumptions of Theorem 3.1, for every $a, b \in K$ the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \frac{\eta(b,a)}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].
\end{aligned} \tag{3.3}$$

Proof. We consider the inequality (3.1), i.e.,

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \frac{\eta(b,a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(\left|f'(a)\right|^{\frac{p}{p-1}} + 3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right].
\end{aligned}$$

Let $a_1 = 3|f'(a)|^{\frac{p}{p-1}}$, $b_1 = |f'(b)|^{\frac{p}{p-1}}$, $a_2 = |f'(a)|^{\frac{p}{p-1}}$, $b_2 = 3|f'(b)|^{\frac{p}{p-1}}$. Here $0 < (p-1)/p < 1$, for $p > 1$. Using the fact

$$\sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s, \tag{3.4}$$

where $0 \leq s < 1$, $a_1, a_2, \cdots, a_n \geq 0$ and $b_1, b_2, \cdots, b_n \geq 0$, we obtain

$$\begin{aligned}
&\frac{\eta(b,a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(|f'(a)|^{\frac{p}{p-1}} + 3|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right] \\
\leq\ & \frac{\eta(b,a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(3^{\frac{p-1}{p}} + 1\right)\left[|f'(a)| + |f'(b)|\right] \\
\leq\ & \frac{\eta(b,a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}} 4\left[|f'(a)| + |f'(b)|\right]
\end{aligned}$$

which completes the proof. ■

Theorem 3.3 *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $q \in \mathbb{R}$ with $q \geq 1$. If $|f'|^q$ is preinvex on K then, for every $a, b \in K$ the following inequality holds*

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \frac{\eta(b,a)}{8}\left[\left(\frac{2|f'(a)|^q + |f'(b)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 2|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.5}$$

Proof. Firstly, we suppose that $q = 1$. By the preinvexity of the function $|f'|$ and (3.2), we have

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \eta(b,a)\left[\int_0^{\frac{1}{2}} t\,|f'(a+t\eta(b,a))|\,dt + \int_{\frac{1}{2}}^{1} (1-t)\,|f'(a+t\eta(b,a))|\,dt\right] \\
\leq\ & \eta(b,a)\left[\int_0^{\frac{1}{2}} t\left[(1-t)|f'(a)| + t|f'(b)|\right]dt + \int_{\frac{1}{2}}^{1} (1-t)\left[(1-t)|f'(a)| + t|f'(b)|\right]dt\right] \\
\leq\ & \eta(b,a)\left[\frac{|f'(a)| + |f'(b)|}{8}\right].
\end{aligned}$$

Secondly, we suppose that $q > 1$. Using the well known power mean inequality and (3.2) in the proof of Theorem 3.1, we have

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \eta(b,a)\left[\int_0^{\frac{1}{2}} t\left|f'(a+t\eta(b,a))\right|dt + \int_{\frac{1}{2}}^{1}(1-t)\left|f'(a+t\eta(b,a))\right|dt\right] \\
\leq\ & \eta(b,a)\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} t\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(1-t)\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\eta(b,a)}{8^{\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}} t\left[(1-t)\left|f'(a)\right|^q + t\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)\left[(1-t)\left|f'(a)\right|^q + t\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \\
=\ & \frac{\eta(b,a)}{8}\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. The proof is completed. ■

Theorem 3.4 *Under the assumptions of Theorem 3.3, the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \leq \frac{\eta(b,a)}{8}\left(\frac{2^{\frac{1}{q}}+1}{3^{\frac{1}{q}}}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{3.6}$$

Proof. We consider inequality (3.5), i.e.,

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \frac{\eta(b,a)}{8}\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Let $a_1 = 2|f'(a)|^q/3$, $b_1 = |f'(b)|^q/3$, $a_2 = |f'(a)|^q/3$, $b_2 = 2|f'(b)|^q/3$. Here $0 < 1/q < 1$, for $q \geq 1$. Using the fact (2.2), we obtain

$$\begin{aligned}
&\frac{\eta(b,a)}{8}\left[\left(\frac{2|f'(a)|^q + |f'(b)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 2|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\eta(b,a)}{8}\left(\frac{2^{\frac{1}{q}}+1}{3^{\frac{1}{q}}}\right)\left[|f'(a)| + |f'(b)|\right].
\end{aligned}$$

■

4 Hermite-Hadamard type inequalities for log-preinvex function

In this section, we shall continue with the following refinements of the Hermite-Hadamard inequality for log-preinvex functions and we give some results connected with the left part of (2.2):

Theorem 4.1 *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is log-preinvex on K, then for every $a, b \in K$ the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \leq \eta(b,a)\left(\frac{|f'(b)|^{\frac{1}{2}} - |f'(a)|^{\frac{1}{2}}}{\log|f'(b)| - \log|f'(a)|}\right)^2.$$

Proof. By assumption and (3.2) in the proof of Theorem 3.1, integrating by parts implies that

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \\
\leq\ & \eta(b,a)\left[\int_0^{\frac{1}{2}} t\,|f'(a+t\eta(b,a))|\,dt + \int_{\frac{1}{2}}^1 (1-t)\,|f'(a+t\eta(b,a))|\,dt\right] \\
\leq\ & \eta(b,a)\left[\int_0^{\frac{1}{2}} t\,|f'(a)|^{1-t}\,|f'(b)|^t\,dt + \int_{\frac{1}{2}}^1 (1-t)\,|f'(a)|^{1-t}\,|f'(b)|^t\,dt\right] \\
=\ & \eta(b,a)\left[\int_0^{\frac{1}{2}} |f'(a)|\,t\left(\frac{|f'(b)|}{|f'(a)|}\right)^t dt + \int_{\frac{1}{2}}^1 (1-t)\,|f'(b)|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{1-t} dt\right]
\end{aligned}$$

$$
\begin{aligned}
&= \eta(b,a)\left[\frac{|f'(a)|}{\log|f'(b)|-\log|f'(a)|}\left[-\frac{1}{\log|f'(b)|-\log|f'(a)|}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}\right]_{0}^{\frac{1}{2}}\right.\\
&\quad\left.+\left[\frac{1}{\log|f'(b)|-\log|f'(a)|}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}\right]_{\frac{1}{2}}^{1}\right]\\
&= \eta(b,a)\left[\frac{-2|f'(a)|^{\frac{1}{2}}|f'(b)|^{\frac{1}{2}}}{(\log|f'(b)|-\log|f'(a)|)^2}+\frac{|f'(a)|}{(\log|f'(b)|-\log|f'(a)|)^2}\right.\\
&\quad\left.+\frac{|f'(a)|}{(\log|f'(b)|-\log|f'(a)|)^2}\right]\\
&= \eta(b,a)\left[\frac{|f'(b)|^{\frac{1}{2}}-|f'(a)|^{\frac{1}{2}}}{\log|f'(b)|-\log|f'(a)|}\right]^2
\end{aligned}
$$

which completes the proof. ■

Theorem 4.2 *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $q \in \mathbb{R}$ with $q \geq 1$. If $|f'|^q$ is log-preinvex on K, then for every $a, b \in K$ the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|\\
&\leq \eta(b,a)\left[\frac{|f'(a)|^{\frac{1}{2}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}}\left(\frac{|f'(b)|^{\frac{q}{2}}-|f'(a)|^{\frac{q}{2}}}{\log|f'(b)|-\log|f'(a)|}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. By Hölder inequality and (3.2) in the proof of Theorem 3.1, we have

$$
\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|\\
&\leq \eta(b,a)\left[\int_0^{\frac{1}{2}} t\,|f'(a+t\eta(b,a))|\,dt+\int_{\frac{1}{2}}^{1}(1-t)\,|f'(a+t\eta(b,a))|\,dt\right]\\
&\leq \eta(b,a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(a+t\eta(b,a))|^q\right)^{\frac{1}{q}} dt\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$\begin{aligned}
&\le \quad \eta(b,a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(|f'(a)|^{1-t}|f'(b)|^t\right)^q dt\right)^{\frac{1}{q}}\right.\\
&\qquad \left.+\left(\int_{\frac{1}{2}}^{1}(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(|f'(a)|^{1-t}|f'(b)|^t\right)^q dt\right)^{\frac{1}{q}}\right]\\
&= \quad \eta(b,a)\left[\frac{|f'(a)|^{\frac{1}{2}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}}\left(\frac{|f'(b)|^{\frac{q}{2}}-|f'(a)|^{\frac{q}{2}}}{\log|f'(b)|-\log|f'(a)|}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. ∎

Now, we give the following results connected with the left part of (1.1) for classical log-convex functions.

Corollary 4.3 *Under the assumptions of Theorem 4.1 with $\eta(b,a)=b-a$, the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le(b-a)\left(\frac{|f'(b)|^{\frac{1}{2}}-|f'(a)|^{\frac{1}{2}}}{\log|f'(b)|-\log|f'(a)|}\right)^2.$$

Corollary 4.4 *Under the assumptions of Theorem 4.2 with $\eta(b,a)=b-a$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\\
&\le \quad (b-a)\left[\frac{|f'(a)|^{\frac{1}{2}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}}\left(\frac{|f'(b)|^{\frac{q}{2}}-|f'(a)|^{\frac{q}{2}}}{\log|f'(b)|-\log|f'(a)|}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

5 An extension to several variables functions

In this section, we shall extend the Corollary 4.3 and Corollary 4.4 to functions of several variables defined on invex subsets of $\mathbb{R}^n$.

Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:K\times K\to\mathbb{R}^n$. For every $x,y\in K$ the η-path P_{xv} joining the points x and $v:=x+\eta(y,x)$ is defined as follows

$$P_{xv}=\{z:z=x+t\eta(y,x):t\in[0,1]\}.$$

Proposition 5.1 *Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:K\times K\to\mathbb{R}^n$ and $f:K\to\mathbb{R}$ is a function. Suppose that η satisfies Condition C on K. Then for every $x,y\in K$ the function*

f is log-preinvex with respect to η on η-path P_{xv} if and only if the function $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(t) := f(x + t\eta(y,x))$$

is log-convex on $[0,1]$.

Proof. Suppose that φ is log-convex on $[0,1]$ and $z_1 := x + t_1\eta(y,x) \in P_{xv}$, $z_2 := x + t_2\eta(y,x) \in P_{xv}$. Fix $\lambda \in [0,1]$. By (2.1), we have

$$\begin{aligned} f(z_1 + \lambda\eta(z_2, z_1)) &= f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x)) \\ &= \varphi((1-\lambda)t_1 + \lambda t_2) \\ &\leq [\varphi(t_1)]^{(1-\lambda)}[\varphi(t_2)]^{\lambda} \\ &= [f(z_1)]^{(1-\lambda)}[f(z_2)]^{\lambda}. \end{aligned}$$

Hence, f is log-preinvex with respect to η on η-path P_{xv}.

Conversely, let $x, y \in K$ and the function f be log-preinvex with respect to η on η-path P_{xv}. Suppose that $t_1, t_2 \in [0,1]$. Then, for every $\lambda \in [0,1]$ we have

$$\begin{aligned} \varphi((1-\lambda)t_1 + \lambda t_2) &= f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x)) \\ &= f(x + t_1\eta(y,x) + \lambda\eta(x + t_2\eta(y,x), x + t_1\eta(y,x))) \\ &\leq [f(x + t_1\eta(y,x))]^{(1-\lambda)}[f(x + t_2\eta(y,x))]^{\lambda} \\ &= [\varphi(t_1)]^{(1-\lambda)}[\varphi(t_2)]^{\lambda}. \end{aligned}$$

Therefore, φ is log-convex on $[0,1]$. ■

The following theorem is a generalization of Corollary 4.3.

Theorem 5.2 *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}^+$ is a function. Suppose that η satisfies Condition C on K. Then for every $x, y \in K$ the function f is log-preinvex with respect to η on η-path P_{xv}. Then, for every $a, b \in (0,1)$ with $a < b$ the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b \left(\int_0^s f(x + t\eta(y,x))\,dt \right) ds - \int_0^{\frac{a+b}{2}} f(x + s\eta(y,x))\,ds \right| \\ \leq\ & (b-a) \left[\frac{[f(x + b\eta(y,x))]^{\frac{1}{2}} - [f(x + a\eta(y,x))]^{\frac{1}{2}}}{\log f(x + b\eta(y,x)) - \log f(x + a\eta(y,x))} \right]^2. \end{aligned} \tag{5.1}$$

Proof. Let $x, y \in K$ and $a, b \in (0,1)$ with $a < b$. Since f is log-preinvex with respect to η on η-path P_{xv} by Proposition 5.1 the function $\varphi : [0,1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f(x + t\eta(y,x)),$$

is log-convex on $[0,1]$. Now, we define the function $\phi : [0,1] \to \mathbb{R}^+$ as follows

$$\phi(t) := \int_0^t \varphi(s)\, ds = \int_0^t f(x + s\eta(y,x))\, ds.$$

Obviously, for every $t \in (0,1)$ we have

$$\phi'(t) = \varphi(t) = f(x + t\eta(y,x)) \geq 0.$$

Hence, $|\phi'(t)| = \phi'(t)$. Applying Corollary 4.3 to the function ϕ implies that

$$\left| \frac{1}{b-a} \int_a^b \phi(t)\, dt - \phi\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left(\frac{|\phi'(b)|^{\frac{1}{2}} - |\phi'(a)|^{\frac{1}{2}}}{\log|\phi'(b)| - \log|\phi'(a)|} \right)^2$$

and we deduce that (5.1) holds. ■

Remark 5.3 *Let $\varphi(t) : [0,1] \to \mathbb{R}^+$ be a function and q a positive real number, then φ is log-convex if and only if the function $\varphi^q(t) : [0,1] \to \mathbb{R}^+$ is log-convex. Indeed for every $x, y \in [0,1]$, it is easy to see that*

$$\left[[\varphi(x)]^{1-t} [\varphi(y)]^t \right]^q = [\varphi^q(x)]^{1-t} [\varphi^q(y)]^t.$$

Therefore, if $t \in [0,1]$, we have

$$\varphi(tx + (1-t)y) \leq [\varphi(x)]^{1-t} [\varphi(y)]^t$$

if and only if

$$\varphi^q(tx + (1-t)y) \leq [\varphi^q(x)]^{1-t} [\varphi^q(y)]^t.$$

The following theorem is a generalization Corollary 4.4 to functions several variables.

Theorem 5.4 *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}^+$ is a function. Suppose that η satisfies condition C on K. Then for every $x, y \in K$, the function f is log-preinvex with respect to η on η-path P_{xv}. Then, for every $p > 1$ and $a, b \in (0,1)$ with $a < b$ the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b \left(\int_0^s f(x + t\eta(y,x))\, dt \right) ds - \int_0^{\frac{a+b}{2}} f(x + s\eta(y,x))\, ds \right| \\ \leq\ & (b-a) \left[\frac{[f(x + a\eta(y,x))]^{\frac{1}{2}}}{2^{\frac{1}{p}} (p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left(\frac{[f(x + b\eta(y,x))]^{\frac{q}{2}} - [f(x + a\eta(y,x))]^{\frac{q}{2}}}{\log f(x + b\eta(y,x)) - \log f(x + a\eta(y,x))} \right)^{\frac{1}{q}} \right], \quad (5.2) \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $x, y \in K$ and $a, b \in (0,1)$ with $a < b$. Suppose that ϕ and φ are the functions which are defined in the Theorem 5.2. Since $|\phi'| : [0,1] \to \mathbb{R}^+$ is log-convex on $[0,1]$, by Remark 5.3 the function $|\phi'|^q$ is also is log-convex on $[0,1]$. Now, by applying Corollary 4.4 to function ϕ we get

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \phi(x)dx - \phi\left(\frac{a+b}{2}\right) \right| \\ \leq \quad & (b-a) \left[\frac{|\phi'(a)|^{\frac{1}{2}}}{2^{\frac{1}{p}} (p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left(\frac{|\phi'(b)|^{\frac{q}{2}} - |\phi'(a)|^{\frac{q}{2}}}{\log|\phi'(b)| - \log|\phi'(a)|} \right)^{\frac{1}{q}} \right] \end{aligned}$$

and we deduce that (5.2) holds. The proof is complete. ■

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