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A note on the Goursat problem for a multidimensional hyperbolic equation

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Abstract. In the present paper, the Goursat problem for a multidimensional hyperbolic equation is investigated. Uniqueness of the solution and weak solvability of the Goursat problem are established.

Key words. Goursat problem, multidimensional hyperbolic equation, uniqueness, weak solvability.

1 Introduction. Formulation of the problem

The study of well-posedness of the Cauchy problem and boundary value problems for hyperbolic partial differential equations has been studied extensively in a large cycle of papers (see, for example [1–12] and the references therein). In the paper [13], the Goursat problem in a three dimensional space was studied. Uniqueness of the solution and weak solvability of the Goursat problem were established.

In present paper, we consider a multidimensional hyperbolic equation

$$Lu \equiv \sum_{i=1}^{n} k_i(t)\, u_{x_i x_i} - u_{tt} + \sum_{i=1}^{n} a_i u_{x_i} + b u_t + cu = f(x,t)\,, \tag{1.1}$$

where $x = (x_1, \dots, x_n)$, $k_i(t)$, $a_i(x,t)$, $b(x,t)$, $c(x,t)$ and $f(x,t)$ are given functions in a domain G bounded by below with S_0 part of hyperplane $t = 0$ and by above with the characteristic of equation (1.1) involving apex $O(x_0, t_0)(t_0 > 0)$. Let S be a characteristic-cone surface of equation (1.1). Then a domain G bounded by $\Gamma = S_0 \cup S$ surface.

Remark 1.1 *For all $(x,t) \in S$ we have the following identity*

$$\sum_{i=1}^{n} k_i(t)\, v_i^2 - v_{n+1}^2 = 0. \tag{1.2}$$

Here, $v_i = \cos(\overrightarrow{n}, x_i)\,(i = 1, \ldots, n)$, $v_{n+1} = \cos(\overrightarrow{n}, t)$, $\overrightarrow{n}$ *be the outward normal to the boundary* Γ *of the domain* G.

Problem G (Goursat). Obtain the solution $u(x,t)$ of equation (1.1) in the domain G satisfying conditions

$$u(x,t) \in C^{(1)}\left(\overline{G}\right) \cap C^{(2)}(G), \tag{1.3}$$

$$u(x,t)|_S = 0. \tag{1.4}$$

In the present paper, the Goursat problem for a multidimensional hyperbolic equation is investigated. Uniqueness of the solution and weak solvability of the Goursat problem are established.

The paper is organized as follows. Section 1 is introduction where we provide the formulation of the Goursat problem. In Section 2, theorem on uniqueness of the solution of the Goursat problem is established. In Section 3, theorem on weak solvability of the Goursat problem is proved. Finally, Section 4 is conclusion.

2 Uniqueness of the solution of the Goursat problem

Now, we will introduce some notations which are used throughout the paper. Let D_1 be a set of all functions $u(x,t)$ defined on G and satisfying conditions (1.3)-(1.4). Let D_2 be a set of all functions $u(x,t)$ defined on G and satisfying condition (1.3) and initial conditions

$$u(x,0) = 0, u_t(x,0) = 0. \tag{2.1}$$

Theorem 2.1 *Suppose that all coefficients of equation (1.1) are continuously differentiable functions on closed domain* G *and satisfying conditions*

$$k_i'(t) \geq 0, k_i(0) > 0, k_i(t) \geq k_i^0 > 0, \tag{2.2}$$

$$b(x,t) \geq b_0 > 0, \tag{2.3}$$

$$c(x,t) \leq c_0 < 0, c_t'(x,t) \leq 0. \tag{2.4}$$

If there exists a solution of Goursat problem in D_1*, then it is unique.*

Proof. Let λ be any negative number and $u(x,t) \in D_1$ be any function. Then, multiplying both sides of equation (1.1) by $e^{-\lambda t} u_t(x,t)$ and taking the integral over the domain G, we get the following identity

$$\int_G e^{-\lambda t} u_t L(u)\, dG = \int_G e^{-\lambda t} u_t f(x,t)\, dG. \tag{2.5}$$

Integrating by parts and using the Green formula, we obtain

$$\frac{1}{2}\int_G e^{-\lambda t}\left\{\sum_{i=1}^{n}\left[k_i'(t) - \lambda k_i(t)\right] u_{x_i}^2 + [2b-\lambda]\, u_t^2 + [\lambda c - c_t']\, u^2\right\} dG$$
$$+\int_G e^{-\lambda t}\sum_{i=1}^{n} a_i u_t u_{x_i}\, dG + \frac{1}{2}\int_\Gamma e^{-\lambda t}\left\{\sum_{i=1}^{n}\left[2k_i(t)\, u_{x_i} u_t v_i - k_i(t)\, u_{x_i}^2 v_{n+1}\right] - u_t^2 v_{n+1}\right\} ds$$
$$+\frac{1}{2}\int_\Gamma e^{-\lambda t} c(x,t)\, u^2 v_{n+1}\, ds = \int_G e^{-\lambda t} u_t f(x,t)\, dG. \tag{2.6}$$

Using identity (1.2)

$$\begin{aligned} &\frac{1}{2}\int_\Gamma e^{-\lambda t} c(x,t)\, u^2 v_{n+1}\, ds \\ =\ & \frac{1}{2}\int_S e^{-\lambda t} c(x,t)\, u^2 v_{n+1}\, ds + \frac{1}{2}\int_{S_0} e^{-\lambda t} c(x,t)\, u^2 v_{n+1}\, ds \\ =\ & \frac{1}{2}\int_{S_0} e^{-\lambda t} c(x,t)\, u^2 v_{n+1}\, ds. \end{aligned}$$

Since

$$v_i = 0 \ \ (i = 1, \cdots, n),\, v_{n+1} = -1 \text{ on } S_0, \tag{2.7}$$

we have that

$$\frac{1}{2}\int_\Gamma e^{-\lambda t} c(x,t)\, u^2 v_{n+1}\, ds = -\frac{1}{2}\int_{S_0} e^{-\lambda t} c(x,t)\, u^2\, ds.$$

Then, applying condition (2.4), we get

$$\frac{1}{2}\int_\Gamma e^{-\lambda t} c(x,t)\, u^2 v_{n+1}\, ds \geq 0. \tag{2.8}$$

Since

$$u(x,t) = 0 \text{ on } S,$$

we have that

$$u_{x_i} = u_n v_i \ \ (i = 1, \cdots, n),\, u_t = u_n v_{n+1}. \tag{2.9}$$

Then, using identity (1.2), condition (2.2), (2.7), and (2.9) we get

$$\frac{1}{2}\int\limits_{\Gamma} e^{-\lambda t}\left\{\sum_{i=1}^{n}\left[2k_i(t)\,u_{x_i}u_t v_i - k_i(t)\,u_{x_i}^2 v_{n+1}\right] - u_t^2 v_{n+1}\right\} ds$$

$$= \frac{1}{2}\int\limits_{S} e^{-\lambda t}\left\{\sum_{i=1}^{n}\left[2k_i(t)\,u_{x_i}u_t v_i - k_i(t)\,u_{x_i}^2 v_{n+1}\right] - u_t^2 v_{n+1}\right\} ds$$

$$+\frac{1}{2}\int\limits_{S_0} e^{-\lambda t}\sum_{i=1}^{n} 2k_i(t)\,u_{x_i}u_t v_i ds + \frac{1}{2}\int\limits_{S_0} e^{-\lambda t}\left\{-\sum_{i=1}^{n} k_i(t)\,u_{x_i}^2 v_{n+1} - u_t^2 v_{n+1}\right\} ds$$

$$= \frac{1}{2}\int\limits_{S} e^{-\lambda t}\left\{\sum_{i=1}^{n} k_i(t)\left[2v_i^2 - v_i^2\right] - v_{n+1}^2\right\} v_{n+1}u_n^2 ds$$

$$+\frac{1}{2}\int\limits_{S_0} e^{-\lambda t}\left[\sum_{i=1}^{n} k_i(t)\,u_{x_i}^2 + u_t^2\right] ds + \frac{1}{2}\int\limits_{S_0} e^{-\lambda t}\left[\sum_{i=1}^{n} k_i(t)\,u_{x_i}^2 + u_t^2\right] ds \geq 0. \tag{2.10}$$

Applying (2.6), (2.8), (2.10) we get

$$\frac{1}{2}\int\limits_{G} e^{-\lambda t}\left\{\sum_{i=1}^{n}\left[k_i'(t) - \lambda k_i(t)\right] u_{x_i}^2 + \left[2b-\lambda\right] u_t^2 + \left[\lambda c - c_t'\right] u^2\right\} dG$$

$$+\int\limits_{G} e^{-\lambda t}\sum_{i=1}^{n} a_i u_t u_{x_i} dG \leq \int\limits_{G} e^{-\lambda t} u_t f(x,t)\, dG. \tag{2.11}$$

Applying the well-known inequality

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$$

for $a, b > 0$ and $\varepsilon > 0$, we obtain

$$\frac{1}{2}\int\limits_{G} e^{-\lambda t}\left\{\sum_{i=1}^{n}\left[k_i'(t) - \lambda k_i(t)\right] u_{x_i}^2 + \left[2b-\lambda\right] u_t^2 + \left[\lambda c - c_t'\right] u^2\right\} dG$$

$$-\int\limits_{G} e^{-\lambda t}\sum_{i=1}^{n} |a_i|\,\frac{u_{x_i}^2 + u_t^2}{2} dG \leq \frac{1}{2\varepsilon}\int\limits_{G} e^{-2\lambda t} f^2(x,t)\, dG \tag{2.12}$$

for any negative λ. Since $|a_i|$ is a bounded function, we can choose λ such that there exists $\alpha > 0$ where

$$\alpha\left(\sum_{i=1}^{n} u_{x_i}^2 + u_t^2 + u^2\right) \tag{2.13}$$

$$\leq \left\{\sum_{i=1}^{n}\left[k_i^{'}(t) - \lambda k_i(t)\right] u_{x_i}^2 + \left[2b - \lambda - \varepsilon\right] u_t^2 + \left[\lambda c - c_t^{'}\right] u^2\right\} - \sum_{i=1}^{n} |a_i|\,\frac{u_{x_i}^2 + u_t^2}{2}.$$

Applying estimates (2.12) and (2.13), we can write

$$\int\limits_G e^{-\lambda t}\left\{\sum_{i=1}^{n} u_{x_i}^2 + u_t^2 + u^2\right\} dG \leq \frac{1}{2\varepsilon\alpha}\int\limits_G e^{-2\lambda t} f^2(x,t) dG. \tag{2.14}$$

From estimate (2.14) it follows that if there exists a solution of Goursat problem in D_1, then it is unique. Moreover, from estimate (2.14) it follows the stability of the solution of Goursat problem. Theorem 2.1 is proved. ■

3 Solvability of the solution of the Goursat problem

Let $u(x,t) \in D_1$ and $v(x,t) \in D_2$ be any given functions. Then, we have the following identity

$$\int\limits_G vL(u) dG = \int\limits_G vf dG. \tag{3.1}$$

Integrating by parts and using the Green formula, we obtain

$$-\int\limits_G \left\{\sum_{i=1}^{n} k_i(t) u_{x_i} v_{x_i} - u_t v_t + \frac{1}{2}\sum_{i=1}^{n} a_i \left[u v_{x_i} - u_{x_i} v\right]\right\} dG \tag{3.2}$$

$$-\int\limits_G \left\{\frac{1}{2} b\left[u v_t - u_t v\right] + \frac{1}{2}\left[\sum_{i=1}^{n}\frac{\partial a_i}{\partial x_i} + \frac{\partial b}{\partial t} - 2c\right] uv\right\} dG$$

$$+\int\limits_\Gamma \left\{\sum_{i=1}^{n} k_i(t) u_{x_i} v v_i - u_t v v_{n+1} + \frac{1}{2}\sum_{i=1}^{n} a_i u v v_i + \frac{1}{2} b u v v_{n+1}\right\} ds = \int\limits_G vf dG.$$

Since

$$v(x,t) = 0 \text{ on } S_0,$$

we have that

$$\int\limits_\Gamma \left\{\sum_{i=1}^{n} k_i(t) u_{x_i} v v_i - u_t v v_{n+1} + \frac{1}{2}\sum_{i=1}^{n} a_i u v v_i + \frac{1}{2} b u v v_{n+1}\right\} ds$$

$$= \int\limits_S \left\{\sum_{i=1}^{n} k_i(t) u_{x_i} v v_i - u_t v v_{n+1} + \frac{1}{2}\sum_{i=1}^{n} a_i u v v_i + \frac{1}{2} b u v v_{n+1}\right\} ds.$$

Applying (13), we get

$$\int\limits_\Gamma \left\{\sum_{i=1}^{n} k_i(t) u_{x_i} v v_i - u_t v v_{n+1} + \frac{1}{2}\sum_{i=1}^{n} a_i u v v_i + \frac{1}{2} b u v v_{n+1}\right\} ds \tag{3.3}$$

$$= \int\limits_{\Gamma} \left\{ \sum_{i=1}^{n} k_i(t) u_{x_i} v v_i - u_t v v_{n+1} \right\} ds = \int\limits_{\Gamma} \left\{ \sum_{i=1}^{n} k_i(t) v_i^2 - v_{n+1}^2 \right\} u_n v ds = 0.$$

Applying formulas (3.2) and (3.3), we can write

$$\langle Lu, v \rangle = W \langle u, v \rangle \tag{3.4}$$

$$\equiv \quad -\int\limits_{G} \left\{ \sum_{i=1}^{n} k_i(t) u_{x_i} v_{x_i} - u_t v_t + \frac{1}{2} \sum_{i=1}^{n} a_i \left[u v_{x_i} - u_{x_i} v \right] \right\} dG$$

$$-\int\limits_{G} \left\{ \frac{1}{2} b \left[u v_t - u_t v \right] + \frac{1}{2} \left[\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} + \frac{\partial b}{\partial t} - 2c \right] uv \right\} dG = \int\limits_{G} v f dG.$$

Definition 3.1 *A function $u_0(x,t) \in D_1$ is called a weak solution of the Goursat problem if it satisfies identity (3.4) for all $v(x,t) \in D_2$.*

For $k = 1, 2$, let us introduce a Euclidean space E_k of all elements $u \in D_k$ with the inner product

$$\langle u, v \rangle = \int\limits_{G} \left\{ \sum_{i=1}^{n} u_{x_i} v_{x_i} + u_t v_t + uv \right\} dG.$$

If we complete E_k in the norm $\|u\|_{E_k} = \sqrt{\langle u, u \rangle}$, then we obtain a Hilbert space H_k.

Theorem 3.1 *Suppose that for all coefficients of equation (1.1), the assumptions of Theorem 2.1 hold. Then, there exists a unique weak solution of Goursat problem in H_1.*

Proof. Note that for all $u(x,t) \in H_1$ and $v(x,t) \in H_2$ identity (3.4) holds. Using formula

$$W \langle u, v \rangle = -\int\limits_{G} \left\{ \sum_{i=1}^{n} k_i(t) u_{x_i} v_{x_i} - u_t v_t + \frac{1}{2} \sum_{i=1}^{n} a_i \left[u v_{x_i} - u_{x_i} v \right] \right\} dG$$

$$-\int\limits_{G} \left\{ \frac{1}{2} b \left[u v_t - u_t v \right] + \frac{1}{2} \left[\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} + \frac{\partial b}{\partial t} - 2c \right] uv \right\} dG \tag{3.5}$$

and Cauchy-Schwarz inequality, we get

$$|W \langle u, v \rangle| \leq N \|u\|_{H_1} \|v\|_{H_2} . \tag{3.6}$$

Here, N is a positive constant does not depend on u and v. From inequality (3.6) it follows boundness of linearly expression $W \langle u, v \rangle$ with respect to u and v. Therefore, the isomorphism of Hilbert spaces H_1 and H_2 is based on identity (3.5). Moreover, for the fixed function $u_0(x,t) \in H_1$ the expression $W \langle u_0, v \rangle$ is a linearly bounded functional with respect $v \in H_2$. Exactly same manner for the fixed function $v_0 \in H_2$ the expression $W\langle u, v_0\rangle$ is the linearly bounded functional with respect to $u \in H_1$. From the isomorphism of Hilbert spaces H_1 and H_2 it follows one-to-one relation

$$u(x,t) \longleftrightarrow v(x,t)$$

and $\|u\|_{H_1} = \|v\|_{H_2}$. Moreover, if $u_1(x,t) \longleftrightarrow v_1(x,t)$ and $u_2(x,t) \longleftrightarrow v_2(x,t)$, then for all α and β numbers it follows that $\alpha u_1(x,t) + \beta u_2(x,t) \leftrightarrow \alpha v_1(x,t) \longleftrightarrow \beta v_2(x,t)$. For the fixed function $u_0(x,t) \in H_1$ identity (3.4) is a linearly bounded functional in H_2. Since $f(x,t)$ is the given function, we have that the right side expression

$$\int\limits_G vf dG$$

of identity (3.4) is a linearly bounded functional with respect to $v \in H_2$. Therefore, by the Riesz theorem [14] there exists unique function $v_0 \in H_2$ such that the following identity

$$W\langle u, v\rangle = \int\limits_G vf dG = \langle v_0, v\rangle$$

holds. That means $W\langle u, v\rangle = \langle v_0, v\rangle$ From the isomorphism of Hilbert spaces H_1 and H_2 it follows that for such $v_0(x,t)$ function there exists unique function $u_0(x,t)$ and

$$v_0(x,t) \longleftrightarrow u_0(x,t).$$

From that it follows identity

$$W\langle u, v\rangle = \int\limits_G vf dG$$

or

$$\langle L(u_0, v)\rangle = W\langle u_0, v\rangle = \int\limits_G vf dG.$$

So, by the definition of a weak solution of the Goursat problem it follows that the function $u_0(x,t) \in H_1$ is the unique weak solution of the Goursat problem. Theorem 3.1 is proved. ■

4 Conclusion

In this paper we investigated the Goursat problem for a multidimensional hyperbolic equation. Uniqueness of the solution and weak solvability of the Goursat problem are established. Of course, the strong solvability of the Goursat problem can been established under the smooth assumptions for $f(x,t)$ and for all coefficients of equation (1.1).

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