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Oscillation theorems for the second order damped nonlinear dynamic equation on time scales

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Abstract. This paper is concerned with the oscillatory behavior of all solutions of nonlinear second order damped dynamic equation

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) + q(t)f(x^{\sigma}(t)) = 0,\ t \in \mathbb{T},$$

where Ψ, f, p, q and r are rd-continuous functions. By using a generalized Riccati transformation and integral averaging technique, we give some new sufficient conditions which ensure that every solution of this equation oscillates.

Key words. Oscillation, dynamic equations, time scales.

1 Introduction

In present paper, we study second order dynamic equation

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) + q(t)f(x^{\sigma}(t)) = 0, \tag{1.1}$$

where Ψ, f, p, q and r are rd-continuous functions.
We will give new oscillation criteria for this equation which has not been previously discussed in the literature.
We assume that:

(H_1) $p,\ q \in C_{rd}(\mathbb{R}, \mathbb{R}^{+})$,

(H_2) $\Psi : \mathbb{T} \to \mathbb{R}$ is such that $\Psi^2(v) \leq \kappa v \Psi(v)$ for $\kappa > 0$, $v \neq 0$,

(H_3) $f : \mathbb{T} \to \mathbb{R}$ is such that $\frac{f(v)}{v} \geq \lambda > 0$, and $vf(v) > 0,\ u \neq 0$,

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(H₄) $r \in C^1_{rd}([t_0,\infty),\mathbb{R}^+),\ \int\limits_{t_0}^{\infty}(\frac{1}{r}e_{\frac{-p}{r}}(t,t_0))\Delta t = \infty.$

In the sequel, we suppose that solutions to equation (1.1) exist for all $t \in [t_0,\infty)_{\mathbb{T}}$ and a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)_{\mathbb{T}}$; otherwise, it is said to be nonoscillatory. Equation (1.1) is termed oscillatory if all its solutions oscillate. The equation itself is called oscillatory if all its solutions are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis (see [1]). Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspects of the new theory, see the paper by Agarwal, Bohner, O'Regan, and Peterson [2], Saker, Agarwal, O'Regan [3]. The book on the subject of time scales, by Bohner and Peterson [4] summarizes and organizes much of time scale calculus.

The increasing interest in oscillation of solutions to different classes of dynamic equations is motivated by their applications in the natural sciences, we refer the reader to [3, 5-11].

In [3], Saker, Agarwal, O'Regan considered the non-linear dynamic equation

$$(a(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta^{\sigma}}(t) + q(t)f(x^{\sigma}(t)) = 0,$$

where $a,\ p,\ q$ are positive functions. They established some sufficient conditions for oscillation. The authors supposed that $uf(u) > 0, f(u)/u \geq K > 0$ and $f'(u) \geq k$ for $u \neq 0$.
Şenel [11] studied the second order damped dynamic equation

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) + q(t)x^{\sigma}(t) = 0.$$

He assumed that $\Psi : \mathbb{T} \to \mathbb{R},\ \frac{\Psi(u)}{|u|} \geq \kappa$ for $\kappa > 0,\ u \neq 0$. In this paper, we have dealt with more general equation. The purpose of this paper is to extend related results reported in [3,11] to a nonlinear dynamic equation (1.1).
Note that, in special case $\mathbb{T} = \mathbb{R},\ \sigma(t) = t,\ \mu(t) = 0,\ x^{\Delta}(t) = x'(t),\ \Psi(x^{\Delta}(t)) = \Psi(x'(t))$. In this case, (1.1) involves the non-linear second order differential equation

$$(r(t)\Psi(x'(t)))' + p(t)\Psi(x'(t)) + q(t)f(x(t)) = 0.$$

2 Main results

Theorem 2.1 *Assume that $(H_1) - (H_4)$ hold and there exists a positive real rd-continuous differentiable function $\rho(t)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\lambda\rho(s)q(s) - \delta(s)\xi^2(s)\right] \Delta s = \infty, \tag{2.1}$$

where $\delta(t) = \frac{\kappa r(s)}{4\rho(s)}$, $\xi(t) = \left[\rho^{\Delta}(t) - \frac{\rho(t)p(t)}{r(t)}\right]$. Then, (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of (1.1) for $t \geq T_1 > t_0$. Now, we assert that $x^{\Delta}(t)$ is either positive or negative sign on the interval $[T_2, \infty)$ for some $T_2 \geq T_1$. From (1.1), since $q(t) > 0$ and $f(x(t)) > 0$ it follows that

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) = -q(t)f(x^{\sigma}(t)) < 0,$$

i.e.,

$$(r(t)\Psi(x^{\Delta}(t))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) < 0.$$

Let

$$y(t) = r(t)\Psi(x^{\Delta}(t)),$$

then we have

$$y^{\Delta}(t) + \frac{p(t)y(t)}{r(t)} < 0$$

which implies that

$$\left(y(t)e_{\frac{p}{r}}(t, T_1)\right)^{\Delta} < 0.$$

Then,

$$y(t)e_{\frac{p}{r}}(t, T_1)$$

is decreasing and thus $y(t)$ is eventually negative or positive. Then, $x^{\Delta}(t)$ has fixed sign for all sufficiently large t and we have one of the following:
First, we consider $x^{\Delta}(t) \geq 0$ on $[T_2, \infty)$ for some $T_2 \geq T_1$. From (1.1) we have

$$x(t) > 0, x^{\Delta}(t) \geq 0, (r(t)\Psi(x^{\Delta}(t))^{\Delta} \leq 0, t \geq T_2. \tag{2.2}$$

We now define

$$w(t) := \rho(t)\frac{r(t)\Psi(x^{\Delta}(t))}{x(t)}, t \geq T_2. \tag{2.3}$$

Then, $w(t) > 0$, and satisfies

$$w^{\Delta}(t) = \left[r(t)\Psi(x^{\Delta}(t))\right]^{\sigma}\left[\frac{\rho(t)}{x(t)}\right]^{\Delta} + \frac{\rho(t)}{x(t)}\left[r(t)\Psi(x^{\Delta}(t))\right]^{\Delta}.$$

In view of (1.1) and (2.2), we see that for $t \geq T_3 = \sigma(T_2)$

$$\begin{aligned} w^{\Delta}(t) = & \frac{\rho^{\Delta}(t)x(t) - \rho(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)}\left[r(t)\Psi(x^{\Delta}(t))\right]^{\sigma} \\ - & \frac{\rho(t)}{x(t)}\left[p(t)\Psi(x^{\Delta}(t)) + q(t)f(x^{\sigma}(t))\right]. \end{aligned} \tag{2.4}$$

However, from (2.2) it follows that

$$r(t)\Psi(x^{\Delta}(t)) \geq (r(t)\Psi(x^{\Delta}(t)))^{\sigma}, x^{\sigma}(t) \geq x(t). \tag{2.5}$$

Using (2.5), (H_2) and (H_3) in (2.4), we have

$$\begin{aligned} w^{\Delta}(t) \leq \rho^{\Delta}(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} & - \frac{\rho(t)}{x^{\sigma}(t)}p(t)\Psi(x^{\Delta}(t)) - \rho(t)\frac{q(t)f(x^{\sigma}(t))}{x^{\sigma}(t)} \\ & - \rho(t)\frac{x^{\Delta}(t)}{(x^{\sigma}(t))^2}[r(t)\Psi(x^{\Delta}(t))]^{\sigma} \end{aligned} \tag{2.6}$$

$$\begin{aligned} \leq \rho^{\Delta}(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} & - \frac{\rho(t)}{x^{\sigma}(t)}p(t)\frac{(r(t)\Psi(x^{\Delta}(t)))^{\sigma}}{r(t)} - \lambda\rho(t)q(t) \\ & - \rho(t)\frac{x^{\Delta}(t)(r(t)\Psi(x^{\Delta}(t)))^{\sigma}(w^{\sigma}(t))^2}{((\rho(t)r(t)\Psi(x^{\Delta}(t)))^{\sigma})^2} \end{aligned} \tag{2.7}$$

$$\begin{aligned} \leq \rho^{\Delta}(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} & - \frac{\rho(t)p(t)}{r(t)}\frac{(r(t)\Psi(x^{\Delta}(t)))^{\sigma}}{x^{\sigma}(t)} - \lambda\rho(t)q(t) \\ & - \rho(t)\frac{(w^{\sigma}(t))^2}{\kappa(\rho^{\sigma}(t))^2 r(t)} \end{aligned} \tag{2.8}$$

$$\begin{aligned} \leq \rho^{\Delta}(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} - \frac{\rho(t)p(t)}{r(t)}\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} & - \lambda\rho(t)q(t) \\ & - \rho(t)\frac{(w^{\sigma}(t))^2}{\kappa(\rho^{\sigma}(t))^2 r(t)} \end{aligned} \tag{2.9}$$

$$\begin{aligned} \leq -\lambda\rho(t)q(t) & + \left[\rho^{\Delta}(t) - \frac{\rho(t)p(t)}{r(t)}\right]\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} \\ & - \rho(t)\frac{(w^{\sigma}(t))^2}{\kappa(\rho^{\sigma}(t))^2 r(t)} \end{aligned}$$

$$\leq -\lambda\rho(t)q(t) + \xi(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} - \rho(t)\frac{(w^{\sigma}(t))^2}{\kappa(\rho^{\sigma}(t))^2 r(t)}, \tag{2.10}$$

where

$$\xi(t) = \left[\rho^{\Delta}(t) - \frac{\rho(t)p(t)}{r(t)}\right].$$

Then,

$$w^{\Delta}(t) \quad \leq \quad -\lambda\rho(t)q(t) + \frac{\kappa r(t)\xi^2(t)}{4\rho(t)} - \left[\sqrt{\frac{\rho(t)}{\kappa r(t)}}\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} - \frac{1}{2}\sqrt{\frac{\kappa r(t)}{\rho(t)}}\xi(t)\right]^2,$$

$$w^{\Delta}(t) \leq \lambda\rho(t)q(t) - \delta(t)\xi^2(t).$$

Integrating from T_3 to t, we obtain

$$w(t) - w(T_3) \leq -\int_{T_3}^{t}\left[\lambda\rho(s)q(s) - \delta(s)\xi^2(s)\right]\Delta s$$

which yields

$$\int_{T_3}^{t}\left[\lambda\rho(s)q(s) - \delta(s)\xi^2(s)\right]\Delta s \leq w(T_3) - w(t) < w(T_3), t \geq T_3$$

for all large t. This is contrary to (2.1).

Next, we consider $x^{\Delta}(t) < 0$ for $t \geq T_2 \geq T_1$.

Now we define $z(t) = -r(t)\Psi(x^{\Delta}(t))$. Then, by using (1.1) and $(H_2), (H_4)$, we have

$$z^{\Delta}(t) + \frac{p(t)}{r(t)}z(t) \geq 0 \Rightarrow z(t) \geq z(T_2)e_{\frac{-p}{r}}(t, T_2).$$

Thus,

$$-r(t)\Psi(x^{\Delta}(t)) \geq z(T_2)e_{\frac{-p}{r}}(t, T_2)$$

$$\Psi(x^{\Delta}(t)) \leq -z(T_2)\left(\frac{1}{r(t)}e_{\frac{-p}{r}}(t, T_2)\right).$$

By (H_2) there is a $\kappa > 0$, so that

$$\kappa x^{\Delta}(t) \leq -z(T_2)\left(\frac{1}{r(t)}e_{\frac{-p}{r}}(t, T_2)\right). \tag{2.11}$$

Integrating (2.11) from T_2 to t, we have

$$x(t) - x(T_2) \leq \frac{r(T_2)\Psi(x(T_2))}{\kappa}\int_{T_2}^{t}\left(\frac{1}{r(s)}e_{\frac{-p}{r}}(s, T_2)\right)\Delta s,$$

or

$$x(t) \leq x(T_2) + \frac{r(T_2)\Psi(x(T_2))}{\kappa}\int_{T_2}^{t}\left(\frac{1}{r(s)}e_{\frac{-p}{r}}(s, T_2)\right)\Delta s.$$

Thus, condition (H_4) implies that $x(t)$ is eventually negative. This contradiction completes the proof. ∎

Corollary 2.2 *Suppose that* $(H_1)-(H_4)$ *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\lambda q(s)-\frac{\kappa p^2(s)}{4r(s)}\right]\Delta s=\infty,$$

then (1.1) is oscillatory.

Corollary 2.3 *Suppose that* $(H_1)-(H_4)$ *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[s^{\gamma}\lambda q(s)-\frac{\kappa(r(s)(s^{\gamma})^{\Delta}-s^{\gamma}p(s))^2}{4r(s)}\kappa s^{-\gamma}\right]\Delta s=\infty, \tag{2.12}$$

then (1.1) is oscillatory.

Example 2.1 *Consider a second order non-linear dynamic equation*

$$\left(\frac{1}{t^2}\left(\frac{x^{\Delta}(t)}{1+(x^{\Delta}(t))^2}\right)\right)^{\Delta}+\frac{1}{t^2}\left(\frac{x^{\Delta}(t)}{1+(x^{\Delta}(t))^2}\right)+\frac{1}{t}\left(\frac{1}{x^{\sigma}(t)}\right)=0,\quad t>0,$$

where $r(t)=\frac{1}{t^2},\ p(t)=\frac{1}{t^2},\ q(t)=\frac{1}{t},\ \Psi(x^{\Delta})=\frac{x^{\Delta}(t)}{1+(x^{\Delta}(t))^2}$. *All conditions of Corollary 2.2 are satisfied. Hence, it is oscillatory.*

Corollary 2.4 *Assume that* $(H_1)-(H_4)$ *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Z(s,t_0)\lambda q(s)-\frac{\kappa r(s)}{4Z(s,t_0)}\left((Z(s,t_0))^{\Delta}-\frac{Z(s,t_0)p(s)}{r(s)}\right)^2\right]\Delta s=\infty,$$

where $Z(t,t_0)=\int_{t_0}^{t}\frac{1}{r(s)}\Delta s$, *then every solution of (1.1) is oscillatory.*

Now, let us introduce the class of functions $\mathfrak{R}$ which will be extensively used in the sequel. Let $\mathbb{D}_0\equiv\{(t,s)\in\mathbb{T}^2: t>s\geq t_0\}$ and $\mathbb{D}\equiv\{(t,s)\in\mathbb{T}^2: t\geq s\geq t_0\}$. The function $H\in C_{rd}(\mathbb{D},\mathbb{R})$ belongs to the class $\mathfrak{R}$, if

(i) $H(t,t)=0,\ t\geq t_0,\ H(t,s)>0$, on $\mathbb{D}_0$,

(ii) H has a continuous Δ-partial derivative $H_s^{\Delta}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.

(H is rd-continuous function if H is rd-continuous function in t and s.)

Theorem 2.5 *Assume that* $(H_1)-(H_4)$ *hold. Let* $\rho(t)$ *be a positive real rd-continuous differentiable function and let* $H:\mathbb{D}\to\mathbb{R}$ *be rd-continuous function such that* H *belongs to the class* $\mathfrak{R}$ *where*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\lambda\rho(s)q(s)-\frac{\delta(s)(\varphi(t,s))^2}{H(t,s)}\right]\Delta s=\infty, \tag{2.13}$$

where

$$\delta(t)=\frac{\kappa r(s)}{4\rho(s)},\ \varphi(t,s)=\rho^{\sigma}(s)H_s^{\Delta}(t,s)+H(t,s)\xi(s).$$

Then, (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of (1.1) for $t \geq T_1 > t_0$. In view of Theorem 2.1, we see that $x^{\Delta}(t)$ is positive or negative sign. If $x^{\Delta}(t) < 0$, from the second case of Theorem 2.1, we get a contradiction. If $x^{\Delta}(t)$ is eventually positive, there exists $T_2 \geq T_1$ such that $x^{\Delta}(t) \geq 0$ and proceed as in the proof of first part of Theorem 2.1 and get (2.10). From (2.10) it follows that

$$w^{\Delta}(t) \leq -\lambda\rho(t)q(t) + \xi(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} - \rho(t)\frac{(w^{\sigma}(t))^2}{\kappa(\rho^{\sigma}(t))^2 r(t)}. \tag{2.14}$$

Multiplying (2.14) by $H(t,s)$, we get

$$\begin{aligned} H(t,s)w^{\Delta}(t) \leq -H(t,s)\lambda\rho(t)q(t) \quad &+ \quad H(t,s)\xi(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} \\ &- \quad H(t,s)\rho(t)\frac{(w^{\sigma}(t))^2}{\kappa(\rho^{\sigma}(t))^2 r(t)}, \end{aligned}$$

or

$$\begin{aligned} H(t,s)\lambda\rho(t)q(t) \leq -H(t,s)w^{\Delta}(t) \quad &+ \quad H(t,s)\xi(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)} \\ &- \quad H(t,s)\rho(t)\frac{(w^{\sigma}(t))^2}{\kappa(\rho^{\sigma}(t))^2 r(t)}, \end{aligned}$$

Using the integration by parts formula, we have

$$\begin{aligned} \int_{T_2}^{t} H(t,s)\lambda\rho(s)q(s)\Delta s \leq \quad &- \quad H(t,t)w(t) + H(t,T_2)w(T_2) + \int_{T_2}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s \\ &+ \int_{T_2}^{t} H(t,s)\xi(s)\frac{w^{\sigma}(s)}{\rho^{\sigma}(s)}\Delta s \\ &- \int_{T_2}^{t} H(t,s)\rho(s)\frac{((w^{\sigma}(s))^2}{\kappa(\rho^{\sigma}(s))^2 r(s)}\Delta s. \end{aligned}$$

Since $H(t,t) = 0$, we obtain

$$\begin{aligned} \int_{T_2}^{t} H(t,s)\lambda\rho(s)q(s)\Delta s \quad &\leq \quad H(t,T_2)w(T_2) \\ &\quad + \int_{T_2}^{t} \left[\rho^{\sigma}(s)H_s^{\Delta}(t,s) + H(t,s)\xi(s)\right]\frac{w^{\sigma}(s)}{\rho^{\sigma}(s)}\Delta s \\ &\quad - \int_{T_2}^{t} H(t,s)\rho(s)\frac{((w^{\sigma}(s))^2}{\kappa(\rho^{\sigma}(s))^2 r(s)}\Delta s \\ &\leq \quad H(t,T_2)w(T_2) + \int_{T_2}^{t} \varphi(t,s)\frac{w^{\sigma}(s)}{\rho^{\sigma}(s)}\Delta s \\ &\quad - \int_{T_2}^{t} H(t,s)\rho(s)\frac{((w^{\sigma}(s))^2}{\kappa(\rho^{\sigma}(s))^2 r(s)}\Delta s. \end{aligned}$$

Therefore, from the proof of Theorem 2.1, we obtain

$$\begin{aligned}\int_{T_2}^{t} H(t,s)\lambda\rho(s)q(s)\Delta s \quad &\leq \quad H(t,T_2)w(T_2)+\int_{T_2}^{t}\frac{\kappa r(s)}{4\rho(s)H(t,s)}\varphi^2(t,s)\Delta s\\ &-\int_{T_2}^{t}\left[\sqrt{\frac{H(t,s)\rho(s)}{\kappa r(s)}}\frac{w^\sigma(s)}{\rho^\sigma(s)}-\frac{1}{2}\sqrt{\frac{\kappa r(s)}{\rho(s)H(t,s)}}\ \varphi(t,s)\right]^2\Delta s.\end{aligned}$$

Hence, we obtain

$$\int_{T_2}^{t} H(t,s)\lambda\rho(s)q(s)\Delta s \leq H(t,T_2)w(T_2)+\int_{T_2}^{t}\frac{\delta(s)}{H(t,s)}\varphi^2(t,s)\Delta s,$$

where $\delta(t)=\frac{\kappa r(t)}{4\rho(t)}$. Then, for all $t\geq T_2$, we have

$$\int_{T_2}^{t}\left[H(t,s)\lambda\rho(s)q(s)-\frac{\delta(s)}{H(t,s)}\varphi^2(t,s)\right]\Delta s \quad \leq H(t,T_2)w(T_2)$$

and this implies that

$$\limsup_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\left[H(t,s)\lambda\rho(s)q(s)-\frac{\delta(s)\varphi^2(t,s)}{H(t,s)}\right]\Delta s \quad \leq \quad w(T_2)$$

which contradicts (2.13). This contradiction completes the proof. ■

Corollary 2.6 *Suppose that the assumptions of Theorem 2.5 hold. If*

$$\limsup_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}H(t,s)\left[\lambda q(s)-\delta(s)\left(\frac{H_s^\Delta(t,s)}{H(t,s)}-\frac{p(s)}{r(s)}\right)^2\right]\Delta s=\infty,$$

then (1.1) is oscillatory.

Corollary 2.7 *Let assumption (2.13) in Theorem 2.5 be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\lambda\rho(s)q(s)=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[\frac{r(s)}{\rho(s)H(t,s)}\left(H(t,s)\xi(s)+\rho^\sigma(s)H_s^\Delta(t,s)\right)^2\right]\Delta s<\infty.$$

Then, (1) is oscillatory.

Lemma 2.8 ([3, Remark 2.3]) *Let $H(t,s)=(t-s)^n$, $(t,s)\in\mathbb{D}$ with $n>1$, we see that* H *belongs to the class $\mathfrak{R}$. Hence,*

$$((t-s)^n)^\Delta\leq -n(t-\sigma(s))^{n-1}.$$

Corollary 2.9 *Assume that $(H_1)-(H_4)$ hold. Let $\rho(t)$ be a positive real rd-continuous differentiable function and let $H:\mathbb{D}\to\mathbb{R}$ be an rd-continuous function such that H belongs to the class $\Re$. If*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n\lambda\rho(s)q(s)-\frac{\delta(s)\phi^2(t,s)}{(t-s)^n}\right]\Delta s=\infty,\quad for\quad n>1,$$

where

$$\phi(t,s)=(t-s)^n\xi(s)+n\rho^\sigma(t)(t-\sigma(s))^{n-1}, t\geq s\geq t_0,$$

then equation (1.1) is oscillatory on $[t_0,\infty)$.

3 The oscillation in case of $p(t)=0$

We will give some sufficient conditions for oscillation of equation (1.1) with $p(t)=0$.

Theorem 3.1 *Assume that $(H_1)-(H_4)$ hold and there exists a positive real rd-continuous function $\rho(t)$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\lambda\rho(s)q(s)-\delta(s)(\rho^\Delta(s))^2\right]\Delta s=\infty,$$

where $\delta(t)=\frac{\kappa r(t)}{4\rho(t)}$. Then, (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of (1.1) for $t\geq T_1>t_0$. From the proof of Theorem 2.1, we see that $x^\Delta(t)$ is positive or negative sign. If $x^\Delta(t)$ is eventually negative, from the second case of Theorem 2.1 we get a contradiction. If $x^\Delta(t)$ is eventually positive, then there exists $T_2\geq T_1$ such that $x^\Delta(t)\geq 0$. From the proof of first part of Theorem 2.1, we get (2.10). By (2.10), we have

$$w^\Delta(t)\leq-\lambda\rho(t)q(t)+\rho^\Delta(t)\frac{w^\sigma(t)}{\rho^\sigma(t)}-\rho(t)\frac{1}{\kappa(\rho^\sigma(t))^2r(t)}(w^\sigma(t))^2.$$

The proof is similar to that of Theorem 2.1 and hence is omitted. ■

Corollary 3.2 *Assume that $(H_1)-(H_4)$ hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\lambda q(s)\Delta s=\infty,$$

equation (1.1) is oscillatory.

Theorem 3.3 *Assume that* $(H_1) - (H_4)$ *hold. Let* $\rho(t)$ *be a positive real rd-continuous differentiable function and let* $H : \mathbb{D} \to \mathbb{R}$ *be rd-continuous function such that* H *belongs to the class* $\mathfrak{R}$. *If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\lambda\rho(s)q(s) - \frac{\delta(s)C^2(t,s)}{H(t,s)}, \right] \Delta s = \infty,$$

where

$$\delta(t) = \frac{\kappa r(t)}{4\rho(t)},\ C(t,s) = \rho^{\sigma}(s)H_s^{\Delta}(t,s) + H(t,s)\rho^{\Delta}(s),$$

equation (1.1) is oscillatory.

Corollary 3.4 *Assume that* $(H_1) - (H_4)$ *hold. Let* $\rho(t) = 1$. *If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(\lambda H(t,s)q(s) - \kappa r(s)(H_s^{\Delta}(t,s))^2 \right) \Delta s = \infty,$$

(1.1) is oscillatory.

4 The oscillation in case of $f'(u) \geq k > 0$

In this section, we assume that $f : \mathbb{R} \to \mathbb{R}$ is such that $f'(u) \geq k$ for $u \neq 0$ and some $k > 0$.

Theorem 4.1 *Assume* $(H_1) - (H_4)$ *hold and there exists a positive real rd-continuous function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)q(s) - \frac{\delta(s)\xi^2(s)}{\upsilon} \right] \Delta s = \infty,$$

where $\delta(t)$, $\xi(t)$ *are as defined in Theorem 2.1. Then, (1.1) is oscillatory.*

Proof. Let x be an eventually positive solution of (1.1) for $t \geq T_1 > t_0$. From the proof of Theorem 2.1, we see that $x^{\Delta}(t)$ is positive or negative sign. If $x^{\Delta}(t)$ is eventually negative, we get a contradiction. If $x^{\Delta}(t)$ is eventually positive, there exists $T_2 \geq T_1$ such that $x^{\Delta}(t) \geq 0$. We now define

$$w(t) := \rho(t)\frac{r(t)\Psi(x^{\Delta}(t))}{f(x(t))}, t \geq T_2.$$

Then, $w(t)$ satisfies

$$w^{\Delta}(t) = (r(t)\Psi(x^{\Delta}(t)))^{\sigma} \left[\frac{\rho(t)}{f(x(t))} \right]^{\Delta} + \frac{\rho(t)}{f(x(t))}(r(t)\Psi(x^{\Delta}(t)))^{\Delta}.$$

In view of (1.1) and (2.5), we have

$$\begin{aligned} w^{\Delta}(t) &= \frac{\rho^{\Delta}(t)f(x(t))-\rho(t)f^{\Delta}(x(t))}{f(x(t))f(x^{\sigma}(t))}(r(t)\psi(x^{\Delta}(t))^{\sigma} \\ &\quad +\frac{\rho(t)}{f(x(t)}[-p(t)x^{\Delta}(t)-q(t)f(x^{\sigma}(t))] \\ &= \frac{\rho^{\Delta}(t)}{f(x^{\sigma}(t))}(r(t)\Psi(x^{\Delta}(t)))^{\sigma}-\frac{\rho(t)f^{\Delta}(x(t))}{f(x(t))f(x^{\sigma}(t))}(r(t)\Psi(x^{\Delta}(t)))^{\sigma} \\ &\quad -\rho(t)\frac{p(t)\Psi(x^{\Delta}(t))}{f(x(t))}-\rho(t)q(t)\frac{f(x^{\sigma}(t))}{f(x(t))}. \end{aligned}$$

Since f is nondecreasing, we have $f(x^{\sigma})\geq f(x)$. Using chain rule [4]

$$f^{\Delta}(x(t))=f'(x(\tau))x^{\Delta}(t)\geq \upsilon x^{\Delta}(t),\tau\in[t,\sigma(t)],$$

we have

$$\begin{aligned} w^{\Delta}(t)\leq -\rho(t)q(t)+\frac{\rho^{\Delta}(t)}{\rho^{\sigma}(t)}w^{\sigma}(t) &- \frac{\rho(t)\upsilon x^{\Delta}(t)}{f^{2}(x^{\sigma}(t))}(r(t)\Psi(x^{\Delta}(t)))^{\sigma} \\ &- \rho(t)\frac{p(t)r(t)\Psi(x^{\Delta}(t))}{r(t)f(x^{\sigma}(t))} \\ \leq -\rho(t)q(t)+\frac{\rho^{\Delta}(t)}{\rho^{\sigma}(t)}w^{\sigma}(t) &- \frac{\rho(t)\upsilon x^{\Delta}(t)(w^{\sigma}(t))^{2}}{(\rho^{\sigma}(t))^{2}(r(t)(\Psi(x^{\Delta}(t)))^{\sigma}} \\ &- \rho(t)\frac{p(t)(r(t)\Psi(x^{\Delta}(t)))\sigma}{r(t)f(x^{\sigma}(t))} \\ \leq -\rho(t)q(t)+\frac{\rho^{\Delta}(t)}{\rho^{\sigma}(t)}w^{\sigma}(t) &- \frac{\rho(t)\upsilon x^{\Delta}(t)(w^{\sigma}(t))^{2}}{(\rho^{\sigma}(t))^{2}(r(t)\Psi(x^{\Delta}(t)))} \\ &- \rho(t)\frac{p(t)w^{\sigma}(t)}{r(t)\rho^{\sigma}(t)}. \end{aligned}$$

From (H_2) it follows that

$$\begin{aligned} w^{\Delta}(t) &\leq -\rho(t)q(t)+\left[\rho^{\Delta}(t)-\frac{\rho(t)p(t)}{r(t)}\right]\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)}-\frac{\rho(t)\upsilon}{\kappa(\rho^{\sigma}(t))^{2}r(t)}(w^{\sigma}(t))^{2} \\ &\leq -\rho(t)q(t)+\xi(t)\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)}-\frac{\rho(t)\upsilon}{\kappa(\rho^{\sigma}(t))^{2}r(t)}(w^{\sigma}(t))^{2}, \end{aligned}$$

where

$$\xi(t)=\rho^{\Delta}(t)-\frac{\rho(t)p(t)}{r(t)}.$$

Then, we obtain

$$w^{\Delta}(t)\leq -\rho(t)q(t)+\frac{\kappa r(t)\xi^{2}(t)}{4\upsilon\rho(t)}-\left[\sqrt{\frac{\upsilon\rho(t)}{\kappa r(t)}}\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)}-\frac{1}{2}\sqrt{\frac{\kappa r(t)}{\upsilon\rho(t)}}\xi(t)\right]^{2}$$

$$\leq -\rho(t)q(t) + \frac{\delta(t)\xi^2(t)}{\upsilon}.$$

Integrating from T_2 to t, we get contradiction for all large t. The proof is complete. ■

Corollary 4.2 *Assume that* $(H_1) - (H_4)$ *hold. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[q(s) - \frac{\kappa p^2(s)}{4\upsilon r(s)} \right] \Delta s = \infty,$$

then every solution of (1.1) is oscillatory.

Theorem 4.3 *Assume that* $(H_1) - (H_4)$ *hold. Let* $\rho(t)$ *be positive real differentiable function and let* $H : \mathbb{D} \to \mathbb{R}$ *be an rd-continuous function such that* H *belongs to the class* $\mathfrak{R}$ *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{\delta(s)}{\upsilon H(t,s)} B^2(t,s) \right] \Delta(s) = \infty,$$

where

$$B(t,s) = H(t,s)\xi(t) + H_s^{\Delta}(t,s)$$

and $\delta(t)$, $\xi(t)$ *are same as Theorem 4.1. Then, (1.1) is oscillatory.*

The proof is similar to that of Theorem 2.5 and hence is omitted.

As an immediate consequence of Theorem 4.3 using $\rho(t) = 1$, $H(t,s) = (t-s)^m$ and $m = n-1$, we get the following results respectively.

Corollary 4.4 *Assume that* $(H_1) - (H_5)$ *hold. If for* $n > 2$

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s) \Delta s = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{\kappa r(s) C^2(t,s)}{4\upsilon(t-s)^{n-1}} \Delta s < \infty,$$

where

$$C(t,s) = (t-s)^{n-1} \left(\frac{p(s)}{r(s)} \right) + (n-1)(t-\sigma(s))^{n-2}, t \geq s \geq t_0,$$

then (1.1) is oscillatory on $[t_0, \infty)$.

Corollary 4.5 *Suppose that* $(H_1) - (H_4)$ *hold. Let be* $\rho(t) = 1$ *. If the condition in Theorem 4.3 is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) q(s) \Delta s = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{(r(s)H_s^{\Delta}(t,s) - H(t,s)p(s))^2}{H(t,s)r(s)} \Delta s < \infty,$$

then every solution of (1.1) is oscillatory on $[t_0, \infty)$.

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